The most non-algebraic complex tori - an algebraic construction

Yuri Zarhin (Penn State, MPIM Bonn)

based on a joint work with Tatiana Bandman (Bar-Ilan)

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Meaning: there is no good moduli space of tori of dimension $g > 1$.

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Meaning: there is no good moduli space of tori of dimension $g > 1$. But there is a "versal family", i.e., there is a flat morphism $\tau : \mathcal{X}_g \to \mathcal{B}_g$ of irreeducible complex spaces $\mathcal{X}_g, \mathcal{B}_g$ such that

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(Will be discussed later)

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Construction is based on the following Theorem that mimicks the construction of abelian varieties of CM type (Shimura-Taniyama).

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and a $\mathbb{Z}$-lattice Λ of rank $2g$ in $E \subset E_{\mathbb{R}}$.
The isomorphism Ψ provides $E_{\mathbb{R}}$ with the structure of a g-dimensional complex vector space.

Then the complex torus $T = T_{E,\Psi,\Lambda} := E_{\mathbb{R}}/\Lambda$ is special and its endomorphism algebra $\mathrm{End}^0(T)$ is isomorphic to E.

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$\{T_{E,\Psi,\Lambda}\}$ is precisely the isogeny class of T_0 (up to an isomorphism).

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The subset of all g-dimensional *special* tori is dense in the "moduli space."

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- By the Theorem,

$$T(f) := \mathbb{C}^g / \Phi(\Lambda) = T_{E,\Phi,\Lambda}$$

  is a *special torus*.

Below we present such polynomials for every even degree $2g \geq 4$.

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Thus $T(\exp_{2g})$ is special g-dimensional complex torus with endomorphism algebra (field) $K_g = \mathbb{Q}[x]/\exp_{2g}(x)\mathbb{Q}[x]$.

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(iii) The polynomial $f_g(x)$ has no real roots if and only if

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That gives us a sequence of g-dimensional **special non-isogenous tori** T_{g,l,p,b_n,c_n}.

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