Algebraic geometric properties of spectral varieties of quasi-elliptic rings

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Algebra and geometry: 70th birthday of Vik. S. Kulikov

Motivation

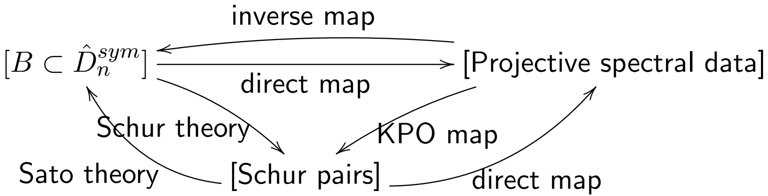
This work appeared as an attempt to classify a wide class of commutative rings of operators found in the theory of integrable systems, such as rings of commuting differential, difference, differential-difference, etc. operators.

As a result a notion of quasi-elliptic rings appeared (to be defined below).

All such rings are contained in a certain non-commutative "universal" ring - a purely algebraic analogue of the ring of pseudodifferential operators on a manifold, and admit (under certain mild restrictions) a convenient algebraic-geometric description.

This description is also one of steps toward the higher dimensional Krichever correspondence – a higher-dimensional analogue of the well-known and fruitful interrelation between KdV- or, more general, KP-equations, and algebraic curves with additional geometric data on the other side.

Generic scheme and notation



Notation:

K is a field of characteristic zero.

$\hat{R} := K[[x_1, \dots, x_n]]$

$D_n := \hat{R}[\partial_1, \dots, \partial_n]$ — the ring of differential operators.

$\hat{D}_n^{sym}$ – the "universe" ring (to be defined below)

B - a quasi-elliptic ring (with certain restrictions, to be defined below)

Schur pairs, etc. to be defined below.

Brief history

Case $n = 1$ (the classical Krichever correspondence) The theory is well known and rich (Wallenberg, Schur, Burchnall, Chaundy, Baker, Dixmier, Krichever, Novikov, Mumford, Drinfeld, Sato, Verdier, Manin, Mulase, Previato, Segal, Wilson, etc.)

Case $n = 2$ One-to one correspondences in the diagram are established, some examples are calculated (Parshin, Osipov, Kurke, Zheglov, Burban, Vik.S. Kulikov)

Case $n > 2$ Some partial constructions are known: the KPO map (Krichever- Parshin-Osipov map) for certain geometric data, (partial) direct maps, Schur-Sato theory.

Case $n = 1$: commuting ordinary differential operators

Definition

An ordinary differential operator $P = a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0 \in D_1$ of positive order n is called *(formally) elliptic* if $a_n \in K^*$. A ring $B \subset D_1$ containing an elliptic element is called *elliptic*.

- (reduction to the elliptic case)
  If $P = a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0 \in D_1$, where $a_n(0) \neq 0$, then there is a change of variables $\varphi \in \text{Aut}(D_1)$ such that

$$Q := \varphi(P) = \partial^n + b_{n-2}\partial^{n-2} + \cdots + b_0 \quad (1)$$

  for some $b_0, \ldots, b_{n-2} \in K[[x]]$.
- Let B be a commutative subalgebra of D_1 containing an elliptic element P. Then *all* elements of B are elliptic.

Krichever-Mumford classification in $n = 1$ case.

Theorem

There is a one-to-one correspondence

$$[B \subset D_1 \quad \textit{of rank } r] \longleftrightarrow [(C, p, \mathcal{F}, z, \phi) \quad \textit{of rank } r]/\simeq$$

$$[B \subset D_1 \quad \textit{of rank } 1]/\sim \longleftrightarrow [(C, p, \mathcal{F}) \quad \textit{of rank } 1]/\simeq$$

where

- $[B]$ *means a class of equivalent commutative elliptic subrings, where* $B \sim B'$ *iff* $B = f^{-1}B'f$, $f \in D_1^*$.
- $\sim$ *means "up to linear changes of variables"*
- $(C, p, \mathcal{F}, z, \phi)$ *means the algebraic-geometric spectral data of rank* r

Here the *rank* of B is

$$\mathrm{rk}(B) := GCD\{\mathrm{ord}(P),\ P \in B\}.$$

Spectral data

Definition

- C is an integral projective curve over K;
- $p \in C$ is a closed regular K-point;
- $\mathcal{F}$ is a coherent torsion free sheaf of rank r on C with

$$h^0(C, \mathcal{F}) = h^1(C, \mathcal{F}) = 0;$$

- z is a local coordinate at p;
- $\phi : \hat{\mathcal{F}}_p \simeq (K[[z]])^{\oplus r}$ is a trivialisation (i.e. an $\hat{\mathcal{O}}_p \simeq K[[z]]$-module isomorphism).

The ring $\hat{D}_n^{sym}$ and its order function

Consider the K–vector space:

$$\mathcal{M} := \hat{R}[[\partial_1, \ldots, \partial_n]] = \left\{ \sum_{\underline{k} \geq \underline{0}} a_{\underline{k}} \underline{\partial}^{\underline{k}} \;\middle|\; a_{\underline{k}} \in \hat{R} \text{ for all } \underline{k} \in \mathbb{N}_0^n \right\}.$$

Let $v : \hat{R} \to \mathbb{N}_0 \cup \infty$ be the discrete valuation defined by the unique maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ of $\hat{R}$. Denote by $|\underline{k}| = k_1 + \ldots + k_n$.

Definition

For any $0 \neq P := \sum_{\underline{k} \geq \underline{0}} a_{\underline{k}} \underline{\partial}^{\underline{k}} \in \mathcal{M}$ we define its *order* to be

$$\mathbf{ord}(P) := \sup\{ |\underline{k}| - v(a_{\underline{k}}) \ \} \in \mathbb{Z} \cup \{\infty\}.$$

Definition

$$\hat{D}_n^{sym} := \{Q \in \mathcal{M} \mid \mathbf{ord}(Q) < \infty\}.$$

Properties of $\hat{D}_n^{sym}$:

- $\hat{D}_n^{sym}$ is a ring (with natural operations $\cdot$, $+$ coming from D_n); $\hat{D}_n^{sym} \supset D_n$.
- $\hat{R}$ has a natural structure of a left $\hat{D}_n$-module, which extends its natural structure of a left D_n-module.
- $\hat{D}_n^{sym}$ is a suitable "universum" that contains many useful objects, e.g.: operators from $\hat{D}_n^{sym}$ can realize arbitrary *continuous endomorphisms* of the K-algebra $\hat{R}$, Dirac delta functions δ_i, operators of integration, difference operators, etc.

  Typical example: Define

  $$\hat{D}_n^{sym} \ni P_\alpha = \sum_{\underline{i} \geq 0} (\underline{i}!)^{-1} \underline{u}^{\underline{i}} \underline{\partial}^{\underline{i}} =: \exp(u_1 * \partial_1 + \ldots + u_n * \partial_n)$$

  Then for any $f \in \hat{R}$ we have

  $$P_\alpha \circ f(x_1, \ldots, x_n) = f(x_1 + u_1, \ldots, x_n + u_n);$$

The Abhyankar formula

Corollary

Assume $F = (F_1, \ldots, F_n)$, $F_i \in K[[x_1, \ldots, x_n]]$ *is of the form* $F = \underline{x} + H$, *where* H *involves only monomials of degree* ≥ 2 *in* x, *and* $j(F) = 1$. *Put* $H_i := x_i - F_i(\underline{x})$. *Then the inverse map is given by the formula*

$$G_i = \sum_{\underline{p} \in \mathbb{N}_0^n} \frac{\partial^{\underline{p}}}{\underline{p}!}(x_i H^{\underline{p}}). \quad (2)$$

Analogously, for any $U \in K[[x_1, \ldots, x_n]]$ *we have*

$$U = \sum_{\underline{p} \in \mathbb{N}_0^n} \frac{\partial^{\underline{p}}}{\underline{p}!}(U(F) H^{\underline{p}}). \quad (3)$$

Reason: F is given by conjugation on $S = \exp(-H_1 * \partial_1 - \ldots - H_n * \partial_n)$ and $S^{-1} = S^*$, where $* : D_n \to D_n$, $*(x_i) := x_i$, $*(\partial_i) := -\partial_i$ is the anti-involution on the ring of differential operators.

The notion of Γ-order

The Γ-order ord_Γ is defined on *some elements* of the algebra $\hat{D}_n^{sym}$, and its definition is recursive:

Notation: $\hat{D}_n^{i_1,\dots,i_q}$ — the subring in $\hat{D}_n^{sym}$ consisting of operators *not depending* on $\partial_{i_1},\dots,\partial_{i_q}$.

We say that $\mathrm{ord}_\Gamma(P)=k_1$, where $0\neq P\in\hat{D}_n^{2,3,\dots n}$, if

$$P=\sum_{s=0}^{k_1}p_s\partial_1^s,\quad \text{where } 0\neq p_{k_1}\in\hat{R}.$$

We say that $\mathrm{ord}_\Gamma(P)=(k_1,\dots,k_n)$, where $P\in\hat{D}_n^{sym}$, if

$$P=\sum_{s=0}^{k_n}p_s\partial_n^s,\quad \text{where } p_s\in\hat{D}_n^{n},$$

and $\mathrm{ord}_\Gamma(p_{k_n})=(k_1,\dots,k_{n-1})$.

In this situation we say that the operator P is *monic* if the highest coefficient $p_{k_1,\dots,k_n}$ (defined recursively in analogous way) is 1.

Quasi-elliptic rings

Definition

We define $\hat{D}_1 = D_1$ and define $\hat{D}_n = \hat{D}_n^n[\partial_n] \subset \hat{D}_n^{sym}$.
The subring $B \subset \hat{D}_n \subset \hat{D}_n^{sym}$ of commuting operators is called *quasi elliptic* if there are n operators $P_1, \ldots, P_n$ such that

- For $1 \le i < n$

$$\mathrm{ord}_\Gamma(P_i) = (0, \ldots 0, 1, 0 \ldots 0, l_i),$$

  where 1 stands at the i-th place and $l_i \in \mathbb{Z}_+$;
- $\mathrm{ord}_\Gamma(P_n) = (0, \ldots, 0, l_n)$, where $l_n > 0$;
- For $1 \le i \le n$ $\mathbf{ord}(P_i) = |\,\mathrm{ord}_\Gamma(P_i)|$.
- P_i are monic.

The operators $P_1, \ldots, P_n$ are called quasi-elliptic as well.

Special properties of quasi-elliptic rings

Definition

The B-module $F = \hat{D}_n^{sym}/\mathfrak{m}\hat{D}_n^{sym} \simeq K[\partial_1, \dots, \partial_n]$ is called *spectral module* of the ring B.

Proposition

Let B be a quasi elliptic commutative subring in $\hat{D}_n^{sym}$. Then

1. *B is integral and the function $-\mathbf{ord}$ induces a discrete valuation on B and on its field of fractions $\mathrm{Quot}(B)$;*
2. *the Γ-order is defined on all elements of B, in particular, the function $-\mathrm{ord}_\Gamma$ is a discrete valuation of rank n;*
3. *the spectral module F is torsion free;*
4. *for any $P \in B$ holds: $\mathbf{ord}(P) = |\mathrm{ord}_\Gamma(\sigma(P))|$.*
5. *$\mathrm{trdeg}_K(\mathrm{Quot}(B)) = n$, $\mathrm{Quot}(B)$ is finitely generated over K and $\mathrm{Quot}(B) \cdot F$ is a finitely generated $\mathrm{Quot}(B)$-module.*

Commutative subrings in $\hat{D}_n^{sym}$ and their spectral module

Proposition

Let $B \subset \hat{D}_n^{sym}$ be a finitely generated commutative subring such that the spectral module F is finitely generated.
For any character $\chi_q : B \to K_q$, where $q \subset B$ is a maximal ideal and $K_q = B/q$ is the residue field, consider the vector space

$$Sol(B, \chi_q) = \{f \in K_q[[x_1, \dots, x_n]] \,\big|\quad Q(f) = \chi_q(Q)f \quad \forall Q \in B\}.$$

Then there exists a canonical isomorphism of vector spaces

$$F|_{\chi_q} := (B/\ker\chi_q) \otimes_B F \simeq Sol(B, \chi_q)^*,$$

$$\overline{\partial^{\underline{p}}} \in F|_{\chi_q} \mapsto \{f \mapsto \frac{1}{\underline{p}!}\frac{\partial^{|\underline{p}|} f}{\partial x_1^{p_1} \dots \partial x_n^{p_n}}(0, \dots, 0)\}$$

In particular, $\dim_K \Big(\mathsf{Sol}(B, \chi_q)\Big) < \infty$ *for any* χ_q.

The notion of rank

Definition

The analytic rank of $B \subset \hat{D}_n$ is

$$An.rank(B) := \mathrm{rk}\, F = \dim(Q(B) \otimes_B F) =$$

$$\dim\{\psi| \quad P \circ \psi = \chi(P)\psi \quad \forall P \in B, \chi - \text{generic point}\}.$$

The algebraic rank is

$$Alg.rank(B) = GCD\{\mathbf{ord}(P)| \quad P \in B\}.$$

We'll say that $B \subset \hat{D}_n$ is of rank r if $An.rank(B) = Alg.rank(B) = r$ and F is finitely generated.

Fact: $An.rank(B) \geq Alg.rank(B)$. Moreover, if B is a finitely generated ring over K, then the following conditions are equivalent: 1) F is a finitely generated B-module and $An.rank(B) = Alg.rank(B) = r$, 2) $\dim_K B_{rn}/B_{rn-1} \sim rn$ for all $n \gg 0$.

Classification theorem ($n = 2$)

For $n > 1$ practically all known examples of commutative rings of *PDOs* can be made 1-quasi-elliptic after a change of coordinates and conjugation by a unity.

Theorem

There is a one-to-one correspondence

$$[B \subset \hat{D}_2 \quad \text{of rank } r] \longleftrightarrow [(X, C, p, \mathcal{F}, \pi, \phi) \quad \text{of rank } r]/ \simeq$$

$$[B \subset \hat{D}_2^{sym} \quad \text{of rank } 1]/ \sim \longleftrightarrow [(X, C, \mathcal{F}) \quad \text{of rank } 1]/ \simeq$$

where

- $[B]$ *means a class of equivalent commutative* 1*-quasi-elliptic subrings, where* $B \sim B'$ *iff* $B = f^{-1}B'f$, $f \in \hat{D}_2^*$.
- $\sim$ *means:* $B_1 \sim B_2$ *if there is a linear change of variables* φ *and a unity* $U \in \hat{D}_2^{sym}$, $\mathbf{ord}(U) = 0$ *such that* $B_1 = U^{-1}B_2U$.
- $(X, C, p, \mathcal{F}, \pi, \phi)$ *are algebro-geometric spectral data of rank* r*:*

Spectral data

Definition

- X is an integral projective algebraic surface over K;
- C is an integral ample Cartier divisor on X. Moreover, $C^2 = r$.
- $p \in C$ is a closed K-point, which is regular on C and on X;
- $\mathcal{F}$ is a coherent torsion free sheaf of rank r on X, which is Cohen-Macaulay along C, and for $n \geq 0$

$$h^0(X, \mathcal{F}(nC)) = \frac{(nr+1)(nr+2)}{2}$$

- $\pi : \hat{\mathcal{O}}_{X,p} \simeq K[[u,t]]$ and $\phi : \hat{\mathcal{F}}_p \simeq \hat{\mathcal{O}}_{X,p}^{\oplus r}$ are some trivialisations of the local ring and module correspondingly.

Typical example

Theorem (Kurke, Osipov, Zh.)

Let $P_1, \ldots, P_n \in D_n = K[[x_1, \ldots, x_n]][\partial_1, \ldots, \partial_n] \subset \hat{D}_n^{sym}$ *be any commuting operators of positive order. Let* B *be any commutative* K*-subalgebra in* D_n *which contains the operators* $P_1, \ldots, P_n$*. Assume that the intersection of the characteristic divisors of* $P_1, \ldots, P_n$ *is empty. Then*

1. *The rings* B *and* $gr(B)$ *are finitely generated integral* K*-algebras of Krull dimension* n*.*
2. C *is an unirational and ample* $\mathbb{Q}$*-Cartier divisor.*
3. *The* B*-module* F*, can be naturally extended to a torsion free coherent sheaf* $\mathcal{F}$ *on* X*. Moreover, the self-intersection index* (C^n) *on* X *is equal to* $\delta^n / \mathrm{rk}(\mathcal{F})$*, where*

$$\delta = \gcd \{n \mid B_n/B_{n-1} \neq 0,\, n \geq 1\}. \quad (4)$$

X – Cohen-Macaulay

Remark: We can additionally assume that X is *Cohen-Macaulay* because of the following result:

Proposition

If $B \subset \hat{D}_2$ is a commutative subring, then there exist a Cohen-Macaulay commutative subring $\tilde{B} \supset B$.
Moreover, if $B \subset D_2$, then $\tilde{B} \subset D_2$.

Analogy with $n = 1$ case: Isospectral deformations of rank one commutative rings of ODOs determine the KP flows on the *Jacobian* of the spectral curve. Isospectral deformations of rank one commutative rings of PDOs determine some flows on the *moduli space M_χ of torsion free sheaves with fixed Hilbert polynomial $\chi(n) = \frac{(n+1)(n+2)}{2}$*.
A dense open subset of this moduli space parametrises *Cohen-Macaulay sheaves*. Cohen-Macaulay sheaves on Cohen-Macaulay surfaces can be effectively described with the help of *matrix-problem approach* due to Burban and Drozd.

Example: Quantum Calogero-Moser systems

Consider the Calogero–Moser operator

$$H = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) - 2 \left(\frac{1}{(x_1 - \xi_1)^2} + \frac{1}{(x_2 - \xi_2)^2} \right),$$

where $(\xi_1, \xi_2) \in \mathbb{C}^2$ is such that $\xi_1 \xi_2 \neq 0$. In this case we have due to Chalykh, Veselov, Styrkas:

- There is a commutative subring $B_H \subset D_2$, $B_H \simeq A = \mathbb{C}[z_1^2, z_1^3, z_2^2, z_2^3]$, the isomorphism is given with the help of the Berest BA-function:

$$\Psi_{Be} = z_1 z_2 + \frac{z_1}{\xi_2 - x_2} + \frac{z_2}{\xi_1 - x_1} + \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)},$$

  s.t. for any $q \in A$ there exists a unique $L_q \in B_H$

$$L_q \Psi_{Be} = q \Psi_{Be}.$$

Example (Burban-Zh.)

Deformed BA-function:

$$\Psi(x_1, x_2, z_1, z_2) = \Psi_{Be} + \beta\overline{\Psi},$$

where Ψ_{Be} is the Berest function and

$$\overline{\Psi} = \frac{1 + \beta\left(\frac{z_1}{\xi_2} + \frac{z_2}{\xi_1}\right)}{(\xi_1\xi_2 - \beta)(\xi_1 - x_1)(\xi_2 - x_2)} +$$

$$\frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)\xi_2}\left(\exp(x_1 z_1)z_1 + (\xi_1 - x_1)\exp(x_1 z_1)z_1^2\right) +$$

$$\frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)\xi_2}\left(\exp(x_2 z_2)z_2 + (\xi_2 - x_2)\exp(x_2 z_2)z_2^2\right).$$

Example (Burban-Zh.)

The simplest deformations of differential operators from the B_H:
for any $q \in z_1^2 z_2^2 A$ denote $q'(z_1, z_2) := q/(z_1^2 z_2^2)$.

$$\hat{D}_2^{sym} \ni L_q = Sq'(\partial_1, \partial_2)(\partial_1 - \frac{1}{1-x_1})(\partial_2 - \frac{1}{1-x_2}), \quad \text{where}$$

$$S = S_0 + \beta T,$$

$$S_0 = \partial_1\partial_2 + \frac{1}{\xi_2 - x_2}\partial_1 + \frac{1}{\xi_1 - x_1}\partial_2 + \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)},$$

$$T = \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)}\left(\frac{1}{\xi_2}\left(\delta_2\partial_1 + (\xi_1 - x_1)\delta_2\partial_1^2\right) + \frac{1}{\xi_1}\left(\delta_1\partial_2 + (\xi_2 - x_2)\delta_1\partial_2^2\right)\right) + \frac{1}{(\xi_1\xi_2 - \beta)(\xi_1 - x_1)(\xi_2 - x_2)}\delta_1\delta_2\left(1 + \beta\left(\frac{\partial_1}{\xi_2} + \frac{\partial_2}{\xi_1}\right)\right)$$

Questions

In the matrix problem approach it is important to know what are the Cohen-Macaulay sheaves with special properties on the *normalisation* of the spectral surface. So, it is important to know what are the possible *normal* surfaces X such that a pre-spectral datum $(X, C, \mathcal{F})$ from classification theorem exists. I'll call such surfaces *normal forms*.

Question

What are the normal forms? Can they be smooth? Can they be classified?

Normal forms of commuting PDOs

Q: Which geometric data describe commutative subrings $B \subset D_2$ of *PDOs*?

Theorem (Kurke, Zh.)

If $B \subset D_2$ is 1-quasi-elliptic of rank 1, with constant highest symbols, then

- *The sheaf $\mathcal{F}$ is Cohen-Macaulay of rank 1;*
- *The divisor C is a rational curve;*
- *If $n : \mathbb{P}^1 \to C$ is the normalisation map, then $\mathcal{F}|_C = (n_*(\mathcal{O}_{\mathbb{P}^1}))$.*

Conjecture

The conditions from theorem are sufficient.

Proposition

If X is a smooth normal form of a commutative ring of PDOs, then $X \simeq \mathbb{P}^2$ (and then $C \simeq \mathbb{P}^1$, $\mathcal{F} \simeq \mathcal{O}_X$).

Smooth normal forms

Q: Are there smooth normal forms of commutative subrings from $\hat{D}_2$?

Question

Find a smooth surface X such that there is a curve C and a divisor D with the following properties:

1. *C is ample (i.e. the sheaf $\mathcal{O}_X(C)$ is ample), $C^2 = 1$ and $h^0(X, \mathcal{O}_X(C)) = 1$;*
2. *$(D, C)_X = g(C) - 1$;*
3. *$h^i(X, \mathcal{O}_X(D)) = 0$, $i = 0, 1, 2$ and $h^0(X, \mathcal{O}_X(D + C)) = 1$.*

Remark: The condition $h^0(X, \mathcal{O}_X(C)) = 1$ means that we are looking for normal forms of "non-trivial" commutative subrings.

Definition

The subring $B \subset \hat{D}_2$ is "trivial", if it contains the operator ∂_1 or the operator ∂_2, i.e. B consists of operators not depending on x_1 or x_2.

Smooth normal forms

The examples of such algebras naturally arise from examples of commuting *ordinary differential operators* just by adding one extra derivation.

Proposition

The subring $B \subset \hat{D}_2$ *is "trivial" iff* $h^0(X, \mathcal{O}_X(C)) \geq 2$.

Proposition

Let $(X, C, \mathcal{F})$ *be a pre-spectral data of rank one with a smooth surface* X *and* $g(C) \leq 1$. *Then* $h^0(X, \mathcal{O}_X(C)) \geq 2$.

Conjecture

If X is a smooth normal form, then it is either rational (and corresponds to a "trivial" subring) or of general type.

Smooth normal forms

Theorem (Kulikov)

There is an eight-dimensional family of pairwise non-isomorphic Godeaux surfaces X such that on each X from this family there are at least 840 different divisors D_j and four curves C_i satisfying the conditions from Question.

Each of these Godeaux surfaces is a factor of a quintic in $\mathbb{P}^3$ by the group $\mathbb{Z}^5$.

Conjecture

All normal forms have the property $q = H^1(X, \mathcal{O}_X) = 0$. There are no other smooth normal forms of general type corresponding to "non-trivial" subrings.

Amazingly, the commutative rings of operators corresponding to the smooth normal forms do not have isospectral deformations!