A functional analog of the Thue-Siegel-Roth theorem

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"Algebra, algebraic geometry, and number theory"

Memorial conference for academician
Igor Rostislavovich Shafarevich

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Room 104, 8 Gubkina, Moscow

Program of the talk

1. Diophantine approximants and functional analogues

1.1. Rational Approximants (RA) of algebraic numbers;
1.2. RA of algebraic functions;

2. Asymptotics of RA for functions with branch points

2.1. Analytic functions with branch points;
2.2. Geometry of the problem – Weak asymptotics
2.3. Strong asymptotics and Statement of the Result;
2.4 Corollary of the Result

3. About proofs and new directions

3.1. Matrix Riemann-Hilbert Problem (MR-HP);
3.2. Local MR-HPs and G-Ch-S $\varepsilon = 0$ conjecture

1. Diophantine approximants and Functional analogues

Rational approximants of algebraic numbers

Continued fractions for numbers

Notations for numbers:

Natural $\mathbb{N}$; Integer $\mathbb{Z}$; Rational $\mathbb{Q}$; Algebraic $\mathcal{A}$; Real $\mathbb{R}$:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathcal{A} \subset \mathbb{R}$$

Eucledian Algorithm : $\alpha \in \mathbb{R}_+ \ \rightarrow$ Continued Fraction

$\{a_l\}$ in $\mathbb{N}$ (incomplete quotients):

$$\alpha = a_0 + \frac{1}{a_1 +} \, \frac{1}{a_2 +} \, \frac{1}{a_3 + \ldots} \, .$$

Convergents – Rational Approximants

$$a_0 + \frac{1}{a_1 + \ldots} \, \frac{1}{a_n} := \frac{p_n}{q_n} \ \in \ \mathbb{Q} \, .$$

Degree of approximation, bounds from above and bellow

- Convergents C.F. are best approximants.

- Bound from above:
  Theorem (Hurwitz-Markov). $\alpha \notin \mathbb{Q} \quad \Rightarrow \quad \exists$ i.m.s.

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{\sqrt{5}}\, q^{-2}, \quad p/q \in \{\text{conver. c. f.}\}\,(\alpha)$$

- Bound from bellow:
  Theorem (Liouville). $\alpha \in \mathcal{A}_k,\ k \geqslant 2 \quad \Rightarrow$

$$\exists\, C(\alpha) \ : \quad \left|\alpha - \frac{p}{q}\right| \geqslant C(\alpha)\, q^{-k}, \quad \forall p,\, q \in \mathbb{Q}.$$

Thue–Siegel–Roth Theorem and S.A.I.

▲ Theorem (Thue-1909, Siegel-21, Dyson-47, Gelfond-48, Roth-55). $\alpha \in \mathcal{A} \quad \Rightarrow$

$$\forall \varepsilon \;\; \exists\, C(\varepsilon, \alpha) \; : \quad \left| \alpha - \frac{p}{q} \right| \geqslant \frac{C}{q^{2+\varepsilon}}\,, \quad \forall p,\, q \in \mathbb{Q}.$$

▲ Slowly Approximated Irrationalities $\tilde{\mathcal{A}}$:
$\alpha \in \tilde{\mathcal{A}} \;\; \Leftrightarrow \;\; \varepsilon = 0 \;\; \Leftrightarrow \;\; \exists C(\alpha) \; : \;\; a_n < C, \; \forall n \in \mathcal{A}$

$$\alpha = a_0 + \frac{1}{a_1 +} \, \frac{1}{a_2 +} \, \frac{1}{a_3 + \ldots} \, .$$

▲ We have (Euler–Lagrange) $\mathcal{A}_2 \equiv \tilde{\mathcal{A}}_2$,

$$\alpha \in \mathcal{A}_k, \;\; k > 2 \qquad ???$$

Example $\alpha := \sqrt[3]{2}$

Numer. computation (A.Bruno 1964, S.Lang-72, A.A.-75)

$$\alpha = 1 + \frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+\dots}\,.$$

$a_{10} = 14;$

$a_{36} = 543;$

$a_{572} = 7451;$

$a_{620} = 4941;$

Rational approximants of algebraic functions

Rational approximants – continued fraction

▲ Given a germ

$$f(z) = \sum_{k=0}^{\infty} \frac{f_k}{z^{k+1}} \in \mathbb{C}((z^{-1})) \qquad \leftrightarrow \qquad \alpha \in (0, 1) \subset \mathbb{R}_+.$$

▲ Eucledian Algorithm :

$f \ \to$ Continued Fraction $\{t_l(z)\}_{l=1}^{\infty}$,:

$$f(z) = \frac{1}{t_1(z)+}\ \frac{1}{t_2(z)+}\ \frac{1}{t_3(z)+\dots}\ , \quad t_l \in \mathcal{P} \ \leftrightarrow \ a_l \in \mathbb{N}$$

▲ Convergents – Rational Approximants

$$\frac{1}{t_1(z)+\dots}\ \frac{1}{t_n(z)} := \frac{p_n(z)}{q_n(z)} = \pi_n(z) \in \mathcal{R} \ \leftrightarrow \ \mathbb{Q}\ .$$

Kolchin's conjecture (Functional TSR conjecture)

notations: $Z(f) :=$ ord zero f at ∞;

$$\nu_n(f) = \sup_{r\in\mathcal{R}_n} Z(f-r);$$

$$\|f\|_a := a^{-Z(f)} \text{ for fixed } a > 1.$$

Kolchin's conjecture (1959):
$f \in \mathcal{A}(z)$ or solution of alg. D.E. $\Rightarrow \forall\varepsilon > 0 \quad \exists C(\varepsilon, f)$:

$$(\,2n \leqslant,\,) \quad \nu_n(f) < (2+\varepsilon)n + C(f), \qquad n \in \mathbb{N}\,.$$

S. Uchiyama (1961), Osgood (1984), Chudnovskies (1983, 1984), H.Stahl (1985)

$$\lim_{n\to\infty} \frac{\nu_n(f)}{n} = 2\,.$$

Chudnovskies:

"In the functional case, the solution of Kolchins problem, is equivalent to the normality and the almost normality of Pade approximations. ..."

The Wronskian Formalism for Linear Differential Equations and Pade Approximants, ADVANCES IN MATHEMATICS 53, 28-54 (1984)

Gonchar – Chudnovskies conjecture $\varepsilon = 0$

▲ Gonchar (1978), Chudnovskies(1984) conjecture:
$f \in \mathcal{A}(z) \Rightarrow \exists C(f)$:

$$\nu_n(f) \leqslant 2n + C(f), \qquad n \in \mathbb{N}\,.$$

▲ Chudnovskies: "We want to note that our conjecture that $\varepsilon = 0$ in Kolchin's problem is the feature of the rational approximation problem in the functional case only. For numbers it seems highly implausible that one can have $\varepsilon = 0$."

▲ for algebraic functions (branch points are in a generic position) validity of the G-Ch cojecture follows from Ap-Ya result.

Joint project with
Maxim Yattselev (IUPUI and MCFAM)

Paper:

A. I. Aptekarev and M. L. Yattselev,

"Pade approximants for functions with branch points – strong asymptotics of Nuttall-Stahl polynomials",

Acta Math., 215:2 (2015), 217 -280,   (ArXiv 1109.0332)

+ preprints and papers in progress

2. Asymptotics of RA for functions with branch points

Definition 1. RA – Pade Approximants

Diagonal Pade Approximants (PA)

$$\pi_n(z) = p_n/q_n \qquad \text{to} \qquad f = \sum_{k=0}^{\infty} \frac{f_k}{z^{k+1}},$$

is defined as the best local approximants, i.e.

$$f(z) - \pi(z) \quad --> \quad \text{max order of zero at} \quad z = \infty.$$

To determine q_n we have the linear system

$$R_n(z) := q_n(z)f(z) - p_n(z) = O\left(1/z^{n+1}\right) \quad \text{as} \quad z \to \infty$$

Definition 2. RA and Orthogonal Polynomials

▲ Cauchy theorem ⇒

$$\int_{\partial D : D \in \mathcal{D}_f} q_n(z)\, z^j\, f(z)\, dz \;=\; 0\,, \qquad j = 0, \ldots, n-1,$$

▲ $\Delta = \partial D^*$ – a finite union of analytic Jordan arcs ⇒

$$\rho(t) = (f^+ - f^-)(t), \qquad t \in \Delta,$$

▲ which turns q_n – non-hermitian orthogonal polynomials:

$$\int_\Delta q_n(t)\, t^j\, \rho(t)\, dt \;=\; 0. \qquad j = 0, \ldots, n-1,$$

Definition 3. RA and MR-H problem

▲ Riemann- Hilbert problem:

$$R_n^+ - R_n^- = q_n \, \rho \quad \text{on} \quad \Delta,$$

▲ Reminder function or second kind func.

$$R_n(z) = \int_\Delta \frac{q_n(t)\rho(t)}{t-z} \frac{dt}{2\pi i}, \quad R_n(z) = O\left(\frac{1}{z^{n+1}}\right) \quad \text{as } n \to \infty.$$

▲ Fokas, Its, Kitaev

$$Y = \begin{pmatrix} q_n & R_n \\ m q_{n-1} & m R_{n-1} \end{pmatrix}$$

▲ Matrix Riemann-Hilbert problem:

$$Y := \begin{cases} Y^{2\times 2} \in H(\mathbb{C} \setminus \Delta), \\ Y(z) = (I + O(\frac{1}{z}))\,\mathrm{diag}(z^n, z^{-n})\,, \quad \text{as} \quad z \mapsto \infty, \\ Y_+ = Y_- \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix} \quad \text{on} \quad \Delta. \end{cases}$$

Definition 4. RA – continued fraction

▲ Given a germ

$$f(z) = \sum_{k=0}^{\infty} \frac{f_k}{z^{k+1}},$$

▲ Eucledian Algorithm :

$f \;\; \to$ Continued Fraction $\{t_l(z)\}_{l=1}^{\infty}$:

$$f(z) = \frac{1}{t_1(z)+} \, \frac{1}{t_2(z)+} \, \frac{1}{t_3(z) + \dots} \, .$$

▲ Convergents – Rational Approximants

$$\frac{1}{t_1(z) + \dots} \, \frac{1}{t_n(z)} := \frac{p_n(z)}{q_n(z)} = \pi_n(z) \, ,$$

Class of Functions – Functions with branch points

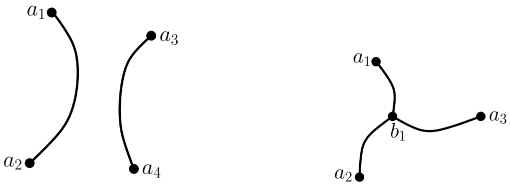
- f be an analytic (and multi-valued)

$$f \in \mathcal{A}(\overline{\mathbb{C}} \setminus A)\,, \qquad \sharp A < \infty,$$

A - branch points.

- Behavior of the poles of the diagonal PA ?

- Why it is interesting?



Weak asymptotics. Stahl's Theorem

H. Stahl 1985-1986. $f \in \mathcal{A}(\overline{\mathbb{C}} \setminus A)$:

$$\operatorname{cap}(A) = 0$$

▲ Why it is interesting? the existence of a domain $D^* \in \mathcal{D}_f$;

▲ weak (n-th root) asymptotics for the denominators q_n

$$\lim_{n\to\infty} \frac{1}{n} \log|q_n(z)| = -V^\lambda(z), \qquad z \in D^*,$$

where $V^\lambda := -\int \log|z-t|\, d\lambda(t)$
equilibrium measure λ – minimizer $I(\mu) := \int V^\mu(z)\, d\mu(z)$:

$$I(\lambda) := \min_{\mu(\Delta)=1} I(\mu);$$

▲ convergence in capacity.

Uniform convergence
Strong Asymptotics

Uniform convergence. Our goal:

the strong (or Szegő type) asymptotics

$$\lim_{n\to\infty} q_n(z) = ? \quad z \in D^*,$$

of the Nuttall-Stahl polynomials q_n, i.e., of the polynomials that are the denominators of the diagonal Padé approximants for

$$f \in \mathcal{A}(\overline{\mathbb{C}} \setminus A)\,, \qquad A - \text{branch points}, \sharp A < \infty.$$

Motivation: Uniform convergence; Spurious poles; .....

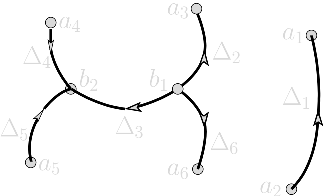
Class of functions:

$$f \in \mathcal{A}(\overline{\mathbb{C}} \setminus A), \qquad A := \{a_j\}, \quad 2 \le \sharp A < \infty.$$

- Restrictions:
  (I) Character of singularities at $\{a_j\}$
  (II) Disposition of the branch points $\{a_j\}$

- (I) Algebro-logarithmic branching condition:

- (II) General position condition:
  $\Delta = \overline{\mathbb{C}} \setminus D^* = E \cup \bigcup \Delta_k$,
  $E = \{a_1, \dots a_p, b_1, \dots b_q, \}$



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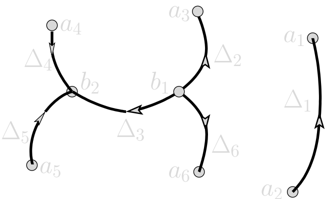

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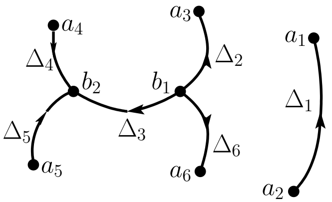


Class of functions. Restrictions:

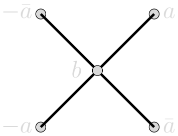
- Algebro-logarithmic branching

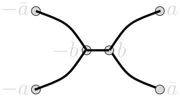
$$f(z) := \int_{\Delta} \frac{\rho(t)}{t-z} \frac{dt}{2\pi i}, \quad z \in D^*.$$

$$\rho(z) := w_k(z)(z-e_{k,1})^{\alpha_{k,1}}(z-e_{k,2}))^{\alpha_{k,2}},$$

$$z \in \Delta_k,\ w_k,\ 1/w_k \in \mathcal{H}(\Delta_k).$$

- GP "constellations"





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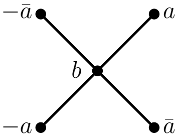

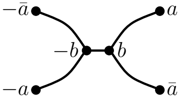

Geometry. Green Function for D^*

- g_Δ the Green function for D^*; $\qquad G := g + i\tilde{g}$
- $h(z) := G'(z) = \frac{1}{z} + \cdots = \sqrt{\frac{B(z)}{A(z)}}$,

$$A(z) := \prod_{k=1}^{p}(z - a_k), \quad \{a_1, \ldots, a_p\} := A \cap E,$$

$$B(z) = \prod_{j=1}^{p-2m}(z - b_j) \prod_{j=p-2m+1}^{g}(z - b_j)^2.$$

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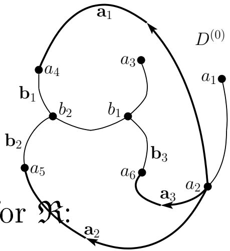
Geometry. $\mathfrak{R}$ the Riemann Surface for h



- ▲ $\mathfrak{R}$ – two copies of $\overline{\mathbb{C}}$ are cut along Δ_k.

- ▲ $\{\mathbf{a}_k\}_{k=1}^{g}$ and $\{\mathbf{b}_k\}_{k=1}^{g}$ –> homology basis for $\mathfrak{R}$:

- ▲ $\widetilde{\mathfrak{R}} := \mathfrak{R} \setminus \bigcup_{k=1}^{g}(\mathbf{a}_k \cup \mathbf{b}_k)$ and $\widehat{\mathfrak{R}} := \mathfrak{R} \setminus \bigcup_{k=1}^{g} \mathbf{a}_k$

- ▲ $\oint_{\mathbf{a}_j} d\Omega_k = \delta_{jk}$ and $\oint_{\mathbf{b}_j} d\Omega_k =: B_{jk}, \quad j,k \in \{1,\ldots,g\}$,

- ▲ $\Omega_k(\mathbf{z}) := \int_{a_1}^{\mathbf{z}} d\Omega_k, \quad \mathbf{z} \in \widetilde{\mathfrak{R}}$.

Geometry. "Mapping" Function for D^*

- $\Phi(\mathbf{z}) := \exp\left\{\int_{a_1}^{\mathbf{z}} h(t)dt\right\}, \quad \mathbf{z} \in \widetilde{\mathfrak{R}}.$

- $$\frac{\Phi^+}{\Phi^-} = \begin{cases} \exp\{2\pi i \omega_k\} & \text{on} \quad \mathbf{a}_k \\ \exp\{2\pi i \tau_k\} & \text{on} \quad \mathbf{b}_k, \end{cases}$$

$$\omega_k := -\frac{1}{2\pi i}\oint_{\mathbf{b}_k} h(t)dt \quad \text{and} \quad \tau_k := \frac{1}{2\pi i}\oint_{\mathbf{a}_k} h(t)dt,$$

- $$\Phi(z) = \frac{z}{\xi \operatorname{cap}(\Delta)} + O(1) \quad \text{as} \quad z \to \infty, \quad |\xi| = 1.$$

Jacobi Inversion Problem $\{\mathbf{t}_{n,j}\}_{j=1}^{g}$

$$\sum_{j=1}^{g} \int_{b_j^{(1)}}^{\mathbf{t}_{n,j}} d\vec{\Omega} \equiv \vec{c}_\rho + n(\vec{\omega} + \mathcal{B}_\Omega \vec{\tau}), \quad \left(\text{mod periods } d\vec{\Omega}\right),$$

where $\{b_j^{(1)}\}_{j=1}^{g}$ zeros of $B(z)$,

$$\vec{c}_\rho := \frac{1}{2\pi i} \oint_L \log(\rho/h^+)\, d\vec{\Omega},$$

and

$$\begin{cases} \vec{\omega} & := & \left(\omega_1, \ldots, \omega_g\right)^T, \\ \vec{\tau} & := & \left(\tau_1, \ldots, \tau_g\right)^T, \end{cases}$$

Szegő Functions $S_n(z)$

BVP in $\mathfrak{R} \setminus L$, $\quad \pi(L) = \Delta_+ \bigcup \Delta_-$:

(I)

$$(S_n\Phi^n) \in \mathfrak{M}(\mathfrak{R} \setminus L)\,, \qquad \exists\; S_n^{\pm} \in C(L \cup \bigcup_{k=1}^{g} a_k)\,;$$

(II)

$$(S_n\Phi^n)^- = (\rho/h^+)(S_n\Phi^n)^+ \qquad \text{on} \quad L \setminus E\,;$$

(III)

$$S(t) = 0\,, \quad t \in \{\mathbf{t}_{n,j}\}_{j=1}^{g} \setminus \left(\{a_j\}_{j=1}^{p} \cup \{b_j^{(1)}\}_{j=1}^{g}\right);$$

(IV) Condition at $\{a_j\} \bigcup \{b_j\}$

The result

- In $D\backslash$ zeros S_n

$$\begin{cases} q_n = \left(1+O\left(\frac{1}{n}\right)\right)\gamma_n\, S_n\, \Phi^n \,, \\ R_n = \left(1+O\left(\frac{1}{n}\right)\right)\gamma_n\, \frac{h\, S_n^{(1)}}{\Phi^n} \,, \end{cases}$$

- In Δ

$$\begin{cases} q_n = \left(1+O\left(\frac{1}{n}\right)\right)\gamma_n\left((S_n\Phi^n)^+ + (S_n\Phi^n)^-\right), \\ R_n^{\pm} = \left(1+O\left(\frac{1}{n}\right)\right)\gamma_n\left(\frac{hS_n^{(1)}}{\Phi^n}\right)^{\pm} , \end{cases}$$

The result

- In $D\backslash$ zeros S_n

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- In Δ

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The result

$$f - \pi_n = \frac{R_n}{q_n}. \; ==>$$

$$f - \pi_n = \left[1 + O\left(\frac{1}{n}\right)\right] \frac{S_n^{(1)}}{S_n} \frac{h}{\Phi^{2n}}$$

Gonchar – Chudnovskies – Stahl conjecture

Riemann surface $\mathfrak{R}$ – double of the extremal domain of holomorphicity D^*.

COROLLARY of the result for generic case:

$f \in \mathcal{A}(z)$ + GP condition $\Rightarrow$ $\exists C(f)$:

$$\nu_n(f) < 2n + C(f), \quad n \in \mathbb{N}\,, \quad C(f) \leqslant \operatorname{gen}\mathfrak{R}.$$

CONJECTURE for general case:

$f \in \mathcal{A}(z) \Rightarrow \exists C(f) \leqslant \operatorname{gen}\mathfrak{R} + \operatorname{degDiscrim}(f)$:

Block struture of the Pade table

and

Block structure of the Jacobi Inv. Pr.

Block structure of the Pade table

$\pi_n := \dfrac{p_n}{q_n}$ is P.A. of index n for $f(z) = \sum_{k=0}^{\infty} \dfrac{f_k}{z^{k+1}}$

$$f - \pi_n = O\left(\frac{1}{z^{2n+1}}\right)$$

$n \in \mathbb{N}_{ni}$ is normal index
$\exists\ (p_n\ :\ deg\, p_n = n - 1,\ q_n\ :\ deg\, q_n = n)$
do not have common factors.

BLOCK THEOREM: n and $n + k$ are consecutive n.i.

$$\pi_m = \pi_n\,, \quad \forall\, m \in [n, n+k)\ .$$

Thus IF $n \in \mathbb{N}_i$ AND

$$(f - \pi_n)(z) = \frac{A}{z^{n+k}} + \ldots\,, \quad A \neq 0\,,$$

THEN k is size of the block.

Block struture of the Jacobi Inv. Pr.

$$q_n \approx \gamma_n S_n \Phi_n \,, \quad R_n \approx \gamma_n \, h \, S_n^{(1)} \Phi^{(-n)}$$

$$\sum_{j=1}^{g} \int_{b_j^{(1)}}^{t_{n,j}} d\Omega \equiv \vec{c}_{\varphi} + n(\vec{\omega} + B_\Omega \vec{\tau}) \,, \quad (\text{mod periods } d\vec{\Omega}) \,.$$

$$(\mathbb{N}_0) \quad n \in \mathbb{N}_0 \;:\; \infty^{(0)} \notin \{t_{n,j}\}_{j=1}^{g} \,.$$

$$\int_{\infty^{(1)}}^{\infty^{(0)}} d\vec{\Omega} = \vec{\omega} + B_\Omega \vec{\tau} \,.$$

$$\infty^{(1)} \in \{t_{n-1,j}\}_{j=1}^{g} \Rightarrow \infty^{(0)} \in \{t_{n,j}\}_{j=1}^{g}$$

Block structure of the Jacobi Inv. Pr.

$$n,\ n+k \in \mathbb{N}_0\ ,\ n+l \notin \mathbb{N}_0\ ,\ l = 1, \ldots, k-1\ ,$$

then

$$\vec{t}_n = \{t_{n,j}\}_{j=1}^{g-k+1} \cup \{(\infty^{(1)})^{k-1}\}\ , \quad \pi(t_{n,j}) < \infty\ ,$$

and

$$\vec{t}_{n+l} = \{t_{n,j}\}_{j=1}^{g-k+1} \cup \{(\infty^{(1)})^{k-1-l}\} \cup \{(\infty^{(0)})^{l}\}\ ,\ l = 1, \ldots, k-1\ ,$$

moreover

$$\vec{t}_{n+k} \notin \infty^{(0)}\ ,$$

3. About proofs and new directions

Geometry. Riemann surface $\mathfrak{R}$

▲ Stahl theorem ($\sharp A < \infty$) -> ∃xistence, properties of Δ

▲ Szego function and BVP on $\mathfrak{R}$

E.I. Zverovich. Boundary value problems in the theory of analytic functions in Hölder classes on Riemann surfaces. Russian Math. Surveys, 26(1):117–192, 1971.

▲ Theta functions

D. Mumford, Tata Lectures on Theta I, volume 28 of Progress in Mathematics. Birkhäuser, Boston, 1983.

MRHP. Statement

Let $\mathcal{Y}$ be a 2×2 matrix function:

(a) $\mathcal{Y}$ is analytic in $\mathbb{C} \setminus \Delta$ and $\lim_{z\to\infty} \mathcal{Y}(z) z^{-n\sigma_3} = \mathcal{I}$, where $\mathcal{I}$ is the identity matrix;

(b) $\mathcal{Y}$ has continuous traces on each Δ_k that satisfy $\mathcal{Y}_+ = \mathcal{Y}_- \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}$;

(c) $\mathcal{Y}$ is bounded near each $e \in E \setminus A$ and the behavior of $\mathcal{Y}$ near each $e \in A$ is described by

$$\begin{cases} O\begin{pmatrix} 1 & |z-e|^{\alpha_e} \\ 1 & |z-e|^{\alpha_e} \end{pmatrix}, & \text{if} \quad \alpha_e < 0, \\ O\begin{pmatrix} 1 & \log|z-e| \\ 1 & \log|z-e| \end{pmatrix}, & \text{if} \quad \alpha_e = 0, \qquad \text{as} \quad D^* \ni z \to e. \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if} \quad \alpha_e > 0, \end{cases}$$

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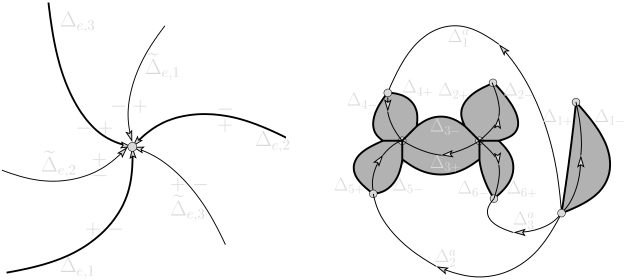
MRHP. Transformations

- Normalization by Φ^n;
- Opening global lenses:



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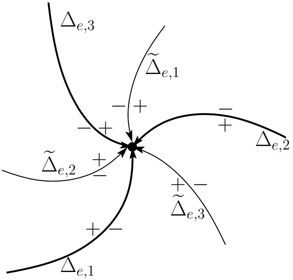

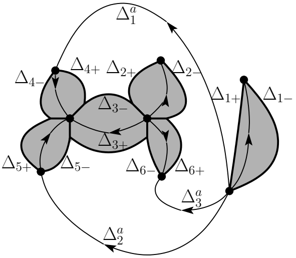

MRHP. Global BVP

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  (a) $\mathcal{N}$ is analytic in D_a and $\mathcal{N}(\infty) = \mathcal{I}$;
  (b) $\mathcal{N}$ has continuous traces on $\bigcup \Delta_k \cup \bigcup \Delta_k^a$ that satisfy:
  (1) $\mathcal{N}_+ = \mathcal{N}_- e^{-2\pi i n \omega_k \sigma_3}$ on each Δ_k^a;
  (2) $\mathcal{N}_+ = \mathcal{N}_- \begin{pmatrix} 0 & e^{2\pi i n \delta_k} \rho \\ -e^{-2\pi i n \delta_k}/\rho & 0 \end{pmatrix}$ on each Δ_k;
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$$\mathcal{N} = Const(\infty) \begin{pmatrix} T_n / S_\rho S_\tau^n & h S_\rho S_\tau^n T_n^* \\ * & * \end{pmatrix}.$$

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MRHP. Local BVPs

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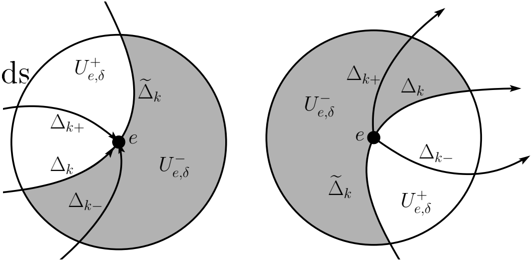
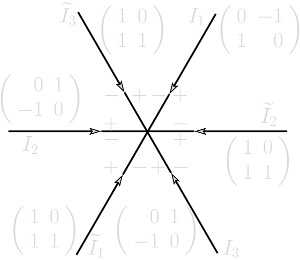


- Trivalent ends



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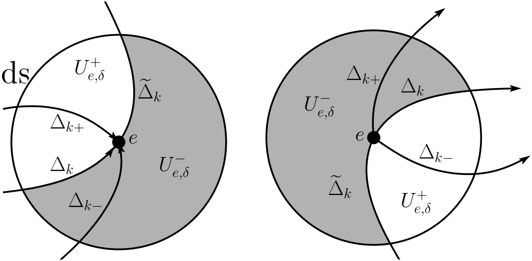

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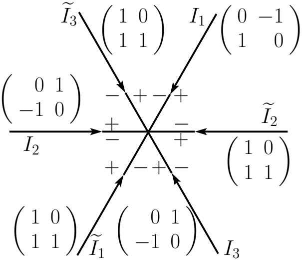

MRHP. Final BVPs