Wormholes/Boltzmann Fields Duality

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Outlook

- Boltzmann Field Theory
- Boltzmann and Matrix Models
- Random Matrices, Topological Recursion, Quantum Curves
- Moduli of Riemann Surfaces
- Generating Functional for Weil-Petersson Volumes
- From Gas of Baby Universes to Wormholes

Boltzmann QFT

- Zero dimensional case

$$\phi = a + a^{+}$$

where a and a^{+} satisfy the following relation

$$aa^{+} = 1. \qquad\qquad O.W.Greenberg'91$$

- This algebra has a realization in the free (or Boltzmannian) Fock space generated by the vacuum $|0\rangle$, $a|0\rangle = 0$, and n-particle states

$$|n\rangle = (a^{+})^{n}|0\rangle.$$

- A free n-point Green's function is defined as the vacuum expectation of n-th power of master field

$$G_n^{(0)} = \langle 0|\phi^n|0\rangle \qquad (*)$$

- The Green's function (*) is given by a n-th moment of Wigner's distribution D.Voiculescu et al'92 *Free random variables*

Boltzmann QFT

$$G_{2n}^{(0)} = \int_{-2}^{2} \frac{d\lambda}{2\pi} \lambda^{2n} \sqrt{4-\lambda^2} = \frac{(2n)!}{n!(n+1)!} \qquad (**)$$

- Representation (**) can be also obtained as a solution of the Schwinger-Dyson equations

$$G_{2n}^{(0)} = \sum_{m=1}^{n} G_{2m-2}^{(0)} G_{2n-2m}^{(0)} \qquad (***)$$

- Interacting Green's functions are defined as IA,Zubarev'95

$$G_n = \langle 0|\phi^n (1 + S_{int}(\phi))^{-1}|0\rangle \qquad (****)$$

In contrast to the ordinary QFT where one deals with the exponent of an interaction, here we deal with the rational function of an interaction. Under natural assumptions the form (****) is unique one which admits Schwinger-Dyson-like equations.

Boltzmann QFT

- Let us consider an algebra generated by operators $A(p)$ and $A^+(p)$ satisfying the relations

$$A(p)A^+(q) = \delta^{(D)}(p-q)$$

- One can realized this algebra in a space which is an analogue of the usual Fock space. This space is generated by the vacuum vector $|0>$, $A(p)|0>=0$ and n-particle states of n non-identical particles,

$$|p_1, ...p_n >= A^+(p_1)...A^+(p_n)|0>$$

There is no symmetization or antisymmetrization as in the Bose or Fermi cases. We shall call this Fock space the Boltzmannian Fock space (it is also called the free Fock space).

- One defines

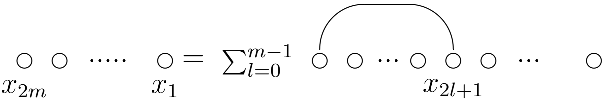
$$\phi(x) = \phi^+(x) + \phi^-(x) = \frac{1}{(2\pi)^{D/2}} \int \frac{d^D p}{\sqrt{p^2+m^2}} (A^+(p)e^{ipx} + A(p)e^{-ipx})$$

and therefore

$$< 0|\phi(x)\phi(y)|0 >= D(x-y) = \frac{1}{(2\pi)^D} \int \frac{d^D p}{p^2+m^2} e^{ip(x-y)}$$

Boltzmann QFT

To calculate the n-point correlation function one has to apply a Boltzmannian Fock space analog of the ordinary Wick theorem. The specific feature of the Wick theorem in this case is that for a given diagram one has not additional symmetry factors related with that an annihilation operator can be contracted with any creation operator on the right. In the Boltzmannian Fock space an annihilation operator can been contracted only with a nearest creation operator on the right.

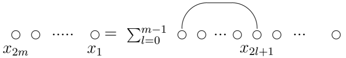


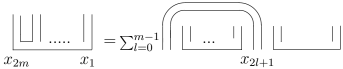
Rainbow diagrams

Boltzmann QFT and Matrix QFT

Let us compare correlation functions

$$< 0|\phi(x_{2m})...\phi(x_1)|0 > \quad \textit{and} \quad \lim_{N\to\infty}\frac{1}{N^{1+m}} < \mathrm{tr}(M(x_{2m})...M(x_1)) >^{(0)}$$





Wick theorem in the Bolzmannian Fock space

Wick theorem for rainbow graphs in free matrix theory

We get

$$\lim_{N\to\infty}\frac{1}{N^{1+m}} < \mathrm{tr}(M(x_{2m})...M(x_1)) >^{(0)} = < 0|\phi(x_{2m})...\phi(x_1)|0 >$$

Boltzmann Field as Master Field for Matrix QFT

IA, I.Volovich

- To construct the master field for interacting QFT we have to work in $\mathbb{M}^{1,d-1}$ and use the Yang-Feldman formalism.
- Let us consider a model of an Hermitian scalar matrix field $M(x) = (M_{ij}(x))$, $i, j = 1, ..., N$ with the field equations

$$(\Box + m^2)M(x) = J(x) \qquad J(x) = -\frac{g}{N}M^3(x)$$

- The Yang-Feldman equation

$$M(x) = M^{(in)}(x) + \int D^{ret}(x-y)J(y)dy$$

$$D^{ret}(x) = \frac{1}{(2\pi)^4}\int \frac{e^{-ikx}}{m^2 - k^2 - i\epsilon k^0}dk$$

is the retarded Green function and $M^{(in)}(x)$ is a free matrix Bose field.

Master Field for Interacting Matrix Scalar Field

- The $U(N)$-invariant Wightman functions are defined as

$$W(x_1, ..., x_k) = \frac{1}{N^{1+\frac{k}{2}}} < 0|\mathrm{tr}(M(x_1)...M(x_k))|0 > \qquad (*)$$

where $|0>$ is the Fock vacuum for the free field $M^{(in)}(x)$.

Theorem. *At every order of perturbation theory in the coupling constant one has the following relation*

$$\lim_{N\to\infty} W(x_1, ..., x_k) =< 0|\phi(x_1)...\phi(x_k)|0 >$$

where

$$\phi(x) = \phi^{(in)}(x) + \int D^{ret}(x-y)j(y)dy \qquad j(x) = -g\phi^3(x)$$

The master field $\phi(x)$ has no matrix indexes.

The proof of the theorem see in (IA+IV'95)

Few calculations in $D = 0$ for Boltzmann QFT

$$G_n = \langle 0|\phi^n(1+V_{int}(\phi))^{-1}|0\rangle \qquad (****)$$

Advantages — closed for of nonlinear equations. $V_{int} = g\phi^4$

$$-gG_0(g)G_4(g) + G_0(g) - gG_2(g)^2 - G_2(g) = 0$$

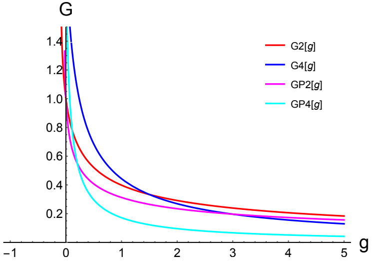
Solution:

$$G_0 = \frac{\sqrt{2}}{\sqrt{\sqrt{16g+1}+1}}$$

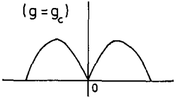
$$G_2 = \frac{1}{4g}\left(\sqrt{2}\sqrt{\sqrt{16g+1}+1} - 2\right)$$

$$G_4 = \frac{1}{g}\left(1 - \frac{\sqrt{2}}{\sqrt{\sqrt{16g+1}+1}}\right)$$

Comparison with the full matrix case

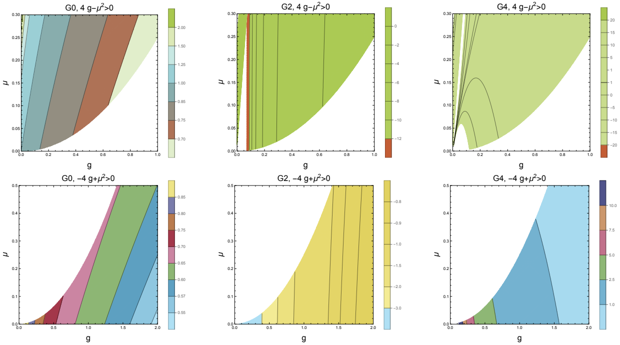


Few calculations in $D = 0$ for Boltzmann QFT



Planar theory $V_{int}(M) = \frac{g}{N}\mathrm{Tr}(M^4) - \mu \mathrm{Tr}(M^2)$

Correlations for Boltzmann Higgs model $V_{int}(\phi) = \frac{g}{4}\phi^4 - \mu^2\phi^2$



JT gravity and matrix models

- It has been shown by Saad, Shenker and Stanford (SSS, '19) that the genus expansion of a certain matrix integral generates the partition functions of JT, Jackiw, 1984,Teitelboim,1983, quantum gravity on Riemann surfaces of arbitrary genus with an arbitrary fixed number of boundaries.
- It is shown that an important part of JT quantum gravity is reduced to computation of the Weil-Petersson volumes of the moduli space of hyperbolic Riemann surfaces with various genus and number of boundaries, for which Mirzakhani'13 established recursion relations. Eynard and Orantin '07 proved that Mirzakhani's relations are a special case of random matrix recursion relations with the spectral curve $y = \sin(2\pi z)/4\pi$.
- This is a natural extension of results on topological gravity, Witten, 1991.
- Relation of random matrices and gravity, including black hole description, has a long history.

JT gravity and matrix models

- The results of Saad, Shenker and Stanford provide a nonperturbative approach to JT quantum gravity on Riemann surfaces of various genus and perturbative description of boundaries.
- We use an extension of this result for nonperturbative studying of gas of baby universes in JT gravity.
- To investigate the boundaries nonperturbatively we explore the generating functional of boundaries in the matrix model and in JT gravity.

JT quantum gravity/matrix models duality

- Let $Z^{grav}_{g,n}(\beta_1, ..., \beta_n)$ be the JT gravity path integral for Riemann surface of genus $g \geq 2$ with n boundaries with lengths $\beta_1, ..., \beta_n$.
- Consider a generating function for these functions

$$\mathcal{Z}^{grav}_n(\beta_1, ..., \beta_n; \gamma) \simeq \sum_{g=0}^{\infty} \gamma^{2g+n-2} \mathcal{Z}^{grav}_{g,n}(\beta_1, ..., \beta_n) \tag{1}$$

where γ is a constant.

- The following remarkable relation between correlation functions in MM and JT gravity holds (Saad, Shenker and Stanford, '19):

$$\mathcal{Z}^{matrix,d.s.}_n(\beta_1, ..., \beta_n; \gamma) \simeq Z^{grav}_n(\beta_1, ..., \beta_n; \gamma) \tag{2}$$

Here $\mathcal{Z}^{matrix,d.s.}_n(\beta_1, ..., \beta_n; \gamma)$ is the double scaling (d.s.) limit of the correlation function in a matrix model with the spectral curve mentioned above.

- This form of the curve was obtained in SSS by computing the JT path integral for the disc.

Generating functional

I.A., I.Volovich, '19

- Generating functional for the gravitational correlation functions $Z_n^{grav}(\beta_1, ..., \beta_n; \gamma)$

$$\mathcal{Z}^{grav}(J;\gamma) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\infty} d\beta_1 ... \int_0^{\infty} d\beta_n Z_n^{grav}(\beta_1, ..., \beta_n; \gamma) J(\beta_1)...J(\beta_n)$$

where $J(\beta)$ is a source function.

- An appropriate generating functional in matrix theory has the form

$$\begin{aligned} \mathfrak{Z}^{matrix}(J) &= < \mathrm{e}^{N\int Z(\beta)J(\beta)d\beta} > \\ &= \sum_{n=0}^{\infty} \frac{N^n}{n!} \int d\beta_1 ... \int d\beta_n \mathcal{Z}_n^{matrix}(\beta_1, ...\beta_n) J(\beta_1)...J(\beta_n) \end{aligned}$$

Here $Z(\beta) = \mathrm{Tr} e^{-\beta M}$ where M is a random $N \times N$ Hermitian matrix.

Shift of the potential

- This amounts to shifting the potential in the matrix model

$$V(x) \to V(x) - \tilde{J}(x)$$

  where $\tilde{J}(x)$ is the Laplace transform of $J(\beta)$.
- We define the generating functional for connected correlation functions

$$\mathfrak{G}^{matrix}(J) = -\frac{1}{N^2} \log \mathfrak{Z}^{matrix}(J) \tag{3}$$

- Take the double scaling limit introducing the parameter γ and obtain the relation between JT gravity and the matrix model in terms of the generating functionals:

$$d.s. \lim \mathfrak{G}^{matrix}(J) \simeq \mathfrak{Z}^{\text{grav}}(J;\gamma) \tag{4}$$

  The "$\simeq$" symbol indicates the equality in the sense of formal series.

Generating functional in matrix models

We consider ensemble of $N \times N$ Hermitian matrices with potential $V(M)$. Let

$$Z(\beta) = \mathrm{Tr} e^{-\beta M} = \sum_{i=1}^{N} \exp(-\beta\lambda_i), \;\; \beta > 0.$$

where λ_i are eigenvalues of the matrix M. The n-point correlation function of $Z(\beta)$ in the matrix model is given by

$$\begin{aligned} & Z_n^{matrix}(\beta_1, ..., \beta_n) \equiv < Z(\beta_1)...Z(\beta_n) > \\ = & \int Z(\beta_1)...Z(\beta_n) \exp(-N\mathrm{Tr} V(M))dM / \int \exp(-N\mathrm{Tr} V(M))dM \end{aligned}$$

Its generating functional can be presented as

$$\mathfrak{Z}^{matrix}(J) = < \mathrm{e}^{N\int Z(\beta)J(\beta)d\beta} >= \sum_{n=0}^{\infty} \frac{N^n}{n!} \int Z_n^{matrix}(\beta_1, ...\beta_n) \prod_i J(\beta_i) d\beta_i$$

Generating functional in matrix models

$$\mathfrak{Z}^{matrix}(J) = \int \exp\{N\sum_{j=1}^{N} \tilde{J}(\lambda_j)\}\, d\mu_N(\lambda_1, ...\lambda_N)$$

where

$$\begin{aligned} d\mu_N(\lambda_1, ...\lambda_N) &= \frac{1}{Z_N}\prod_{j>k}(\lambda_j - \lambda_k)^2 \prod_{j=1}^{N} e^{-NV(\lambda_j)}, \\ \tilde{J}(\lambda) &= \int d\beta J(\beta)\mathrm{e}^{-\beta\lambda} d\beta \end{aligned}$$

This amounts to shift the potential $V(x) \to V(x) - \tilde{J}(x)$.
One expands $\mathfrak{G}(J) = \log \mathfrak{Z}^{matrix}(J)$ to get the connected correlation functions

$$\mathfrak{G}^{matrix}(J) = \sum_{n=0}^{\infty} \frac{1}{n!} \int Z^{matrix}_{n,conn}(\beta_1, ...\beta_n) \prod_i J(\beta_1) d\beta_i$$

Generating functional in JT gravity.

- The Euclidean action in JT gravity

$$I_{JT}=-\frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}R+\int_{\partial\mathcal{M}}\sqrt{h}K\right]-\left[\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}\phi(R+2)+\int_{\partial\mathcal{M}}\sqrt{h}\phi(K-1)\right].$$

Here $g_{\mu\nu}$ is a metric on a two dimensional manifold $\mathcal{M}$,
ϕ is a scalar field (dilaton), S_0 is a constant

- The path integral for Riemann surface of genus $g\geq 2$ with n boundaries with lengths $\beta_1,...,\beta_n$ reads

$$Z^{grav}_{g,n}(\beta_1,...,\beta_n)=e^{-S_0\chi}\int\frac{\mathcal{D}g_{\mu\nu}\mathcal{D}\phi}{\mathrm{Vol(diff)}}\,e^{-\hat{I}_{JT}[g_{\mu\nu},\phi]}$$

χ is the Euler characteristic $\chi=2-2g-n$ and $\hat{I}_{JT}$ is the JT action with the first S_0 term left out.

- Relation between the matrix model and JT gravity (SSS):

$$\mathcal{Z}^{matrix,d.s.}_{g,n}(\beta_1,...,\beta_n)=Z^{grav}_{g,n}(\beta_1,...,\beta_n)$$

Partition functions in JT and the WP volume of the moduli space

$$\mathcal{Z}_{n}^{grav}(\beta_1,...,\beta_n) \simeq \sum_{g=0}^{\infty}(e^{-S_0})^{2g+n-2}\mathcal{Z}_{g,n}^{grav}(\beta_1,...,\beta_n) \tag{5}$$

According Saad, Shenker and Stanford, '19

$$Z_{g,n}^{grav}(\beta_1,...,\beta_n) = \int_{b_i>0} V_{g,n}(b_1,...,b_n)\prod Z_{\mathrm{Sch}}^{\mathrm{trumpet}}(\beta_i,b_i)b_i db_i$$

where $(g \geq 2)$, $V_{g,n}(b_1,...,b_n)$ is the Weil-Petersson (WP) volume of the moduli space of a genus g Riemann surface with n geodesic boundaries of lengths $b_1,...,b_n$ and

$$Z_{\mathrm{Sch}}^{\mathrm{trumpet}}(\beta,b) = \frac{1}{2\pi^{1/2}\beta^{1/2}}\mathrm{e}^{-\frac{b^2}{4\beta}}.$$

Partition functions in JT and the WP volume of the moduli space

$$\mathcal{Z}_g^{grav}(J) = \sum_{n=0}^{\infty} \frac{1}{n!}(e^{-S_0})^{2g+n-2} \int_{\beta_i>0} Z_{g,n}^{grav}(\beta_1, ..., \beta_n) \prod J(\beta_i) d\beta_i$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!}(e^{-S_0})^{2g+n-2} \int V_{g,n}(b_1, ..., b_n) \prod_i \hat{J}(b_i) b_i db_i,$$

where

$$\hat{J}(b) = \int_0^{\infty} d\beta \frac{1}{2\pi^{1/2}\beta^{1/2}} e^{-\frac{b^2}{4\beta}} J(\beta)$$

and the generating functional

$$\mathfrak{Z}^{grav}(J) \simeq \sum_{g=0}^{\infty} \mathcal{Z}_g^{grav}(J)$$

Finally,

$$d.s. \lim \log \mathfrak{Z}^{matrix}(J) \simeq \mathfrak{Z}^{\mathrm{grav}}(J)$$

Partition functions in MMT and the WP volume of the moduli space

Similarly, the correlation functions $W_{g,n}^{matrix}$ are related with volumes of the moduli spaces as

$$W_{g,n}^{matrix}(z_1,...,z_n) = \int_{b_i>0} V_{g,n}(b_1,...,b_n).\prod \mathrm{e}^{-b_i z_i}\, b_i db_i$$

The generating functional is

$$\mathcal{Z}_{R,g}(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \int W_{g,n}^{matrix}(z_1,...,z_n) \prod f(z_i) dz_i$$

$$= \quad \sum_{n=0}^{\infty} \frac{1}{n!} \int V_{g,n}(b_1,...,b_n) \prod_i \tilde{f}(b_1) b_i db_i$$

where

$$\tilde{f}(b) = \int dz e^{-bz} f(z). \tag{6}$$

Baby universes.

In cosmology, one usually deals with baby universes that branch off from, or join onto, the parent(s) Universe(s). In matrix theories one parent is a connected Riemann surface with arbitrary number of handles and at least one boundary.

Baby universes.

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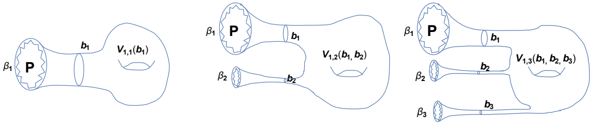
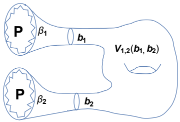


$+ \dots$

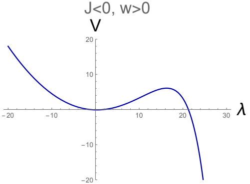
Gas of baby universes. Here $|\beta_1| >> |\beta_i|$ for $i \geq 2$ and $b_i \leq b_c$ for $i \geq 2$



Two parents connected by the wormhole

Deformation of MM by an exponential potential

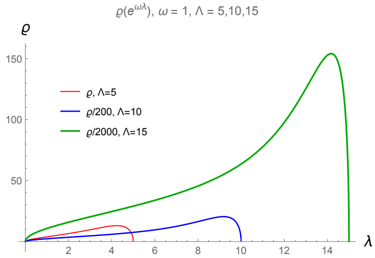
$$U(x) = V_0(x) + V_1(x) = \frac{m^2 x^2}{2} + Je^{\omega x}$$



Perturbations of the gaussian ensemble by the exp. potential $V_1 = Je^{\omega x}$

To shift $\lambda \to \lambda + \delta\lambda$ add linear term $V(x) = m^2\left(\frac{1}{2}x^2 + Cx + J\,\mathrm{e}^{\omega x}\right)$

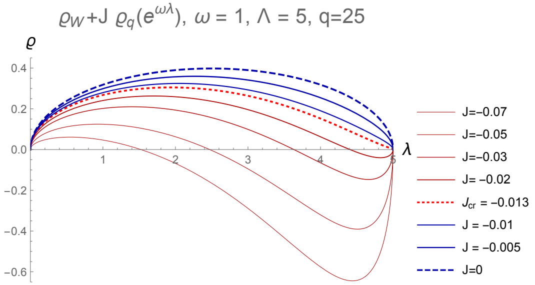
Deformation of MM by an exponential potential



Non–normalized density plot for the exponential potential with negative $\omega = -1$.

Deformation of Wigner density by an exponential potential

$$\rho_{nn}(\lambda) = \rho_W(\lambda, \Lambda) + J\rho_{e^{\omega\lambda}}(\lambda, \Lambda, \omega)$$



The plot of non–normalized density for the quadratic potential deformed by the exp. potential for different values of the regularization parameter J, $\Lambda = 5$, $\omega = 1$

Conclusion

- Relations between matrix model and Boltzmann QFT are considered
    - Boltzmann QFT is a part of the corresponding Matrix theory
- Relation between double scaling limit of MM and JT gravity is incorparated

Conjecture:

MM as full quantum theory
Gravity with baby universities as dual to MM

Boltzmann QFT as mesoscopic theory
Gravity with (part of) wormholes as dual to Boltzmann QFT