q-cut-and-join operators related to Reflection Equation algebras

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Plan

Let N^2 elements m_i^j, $1 \le i, j \le N$, be generators of the coordinate ring of the commutative algebra $Sym(gl_N)$. We denote by $\partial_j^i = \partial / \partial m_i^j$ the partial derivative in the indeterminate m_i^j.

Thus, in our notations $\partial_j^i(m_k^l) = \delta_j^l\, \delta_k^i$.

Consider the matrix $\hat{L} = MD$, where $N \times N$ matrices $M = \|m_i^j\|$ and $D = \|\partial_i^j\|$ are composed from m_i^j and ∂_i^j respectively. Hereafter, the lower index enumerates the rows of a matrix, while the upper one — the columns. Note that the entries $\hat{l}_i^j$ of the matrix $\hat{L}$ meet the commutation relations:

$$[\hat{l}_i^j, \hat{l}_k^r] = \delta_k^j \hat{l}_i^r - \delta_i^r \hat{l}_k^j$$

and consequently generate the universal enveloping algebra $U(gl_N)$. It is well known, the elements $Tr\hat{L}^k$, $1 \le k \le N$, generate the center $Z(U(gl_N))$ of the enveloping algebra.

Let $\lambda = (\lambda_1, \dots, \lambda_N)$ be a partition and V_λ be the corresponding irreducible finite dimensional $U(gl_N)$-module. The Schur lemma implies that the central elements of $U(gl_N)$ are represented by scalar operators on the spaces V_λ (we call them the Casimir operators). The eigenvalues related to the Casimir operators $\mathrm{Tr}\hat{L}^k$ were computed by Perelomov-Popov.

The following operators, which differ from product of the above Casimir operators

$$W^{\Delta} =: \mathrm{Tr}\,\hat{L}^{\Delta_1} \dots \mathrm{Tr}\,\hat{L}^{\Delta_k} : \qquad (1)$$

by the normal ordering, are called cut-and-join ones. Here $\Delta = (\Delta_1, \dots, \Delta_k)$ is a partition. Also, we omit normalizing factors. These operators play an important role in the Hurwitz-Kontsevich theory. In another form the simplest cut-and-join operator was introduced by Goulden.

Our purpose is to introduce q-analogs of these operators. The role of the commutative algebra will be attributed to the so-called Reflection Equation (RE) algebra, analogs of derivatives will be introduced via the so-called Quantum Doubles (QD) and the role of the enveloping algebra $U(gl_N)$ will be attributed to the so-called modified RE algebra.

All algebras in question are introduced by means of special braidings which are called Hecke symmetries. Let us define them.

We call an invertible linear operator $R : V^{\otimes 2} \to V^{\otimes 2}$ *braiding* if it satisfies the so-called *braid relation*

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23}, \quad R_{12} = R \otimes I, \;\; R_{23} = I \otimes R.$$

Then the operator $\mathcal{R} = R\, P$ where P is the usual flip is subject to the QYBE

$$\mathcal{R}_{12}\, \mathcal{R}_{13}\, \mathcal{R}_{23} = \mathcal{R}_{23}\, \mathcal{R}_{13}\, \mathcal{R}_{12}.$$

A braiding R is called *involutive symmetry* if $R^2 = I$.
A braiding is called *Hecke symmetry* if it is subject to the Hecke condition

$$(q\, I - R)(q^{-1}\, I + R) = 0,\;\; q \in \mathbb{C},\;\; q \neq 0,\;\; q \neq \pm 1.$$

In particular, such a symmetry comes from the QG $U_q(sl(N))$.

We assume q to be generic. This means that

$$k_q = \frac{q^k - q^{-k}}{q - q^{-1}} \neq 0 \text{ for any integer } k.$$

As for the braidings coming from the QG of other series B_n, C_n, D_n, each of them has 3 eigenvalues and it is called BMW symmetry.

Examples.

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

Note that for $q \to 1$ we respectively recover the usual flip P and the super-flip $P_{1|1}$.

Let us discuss a possible form of the Hecke symmetries.

In order to classify Hecke symmetries, consider "R-symmetric" and "R-skew-symmetric" algebras

$$Sym_R(V) = T(V)/\langle Im(qI - R)\rangle, \ \bigwedge\nolimits_R(V) = T(V)/\langle Im(q^{-1}I + R)\rangle,$$

where $T(V)$ is the free tensor algebra. Also, consider the corresponding Poincaré-Hilbert series

$$P_+(t) = \sum_k dim\, Sym_R^{(k)}(V)t^k, \ P_-(t) = \sum_k dim \bigwedge\nolimits_R^{(k)}(V)t^k,$$

where the upper index (k) labels homogenous components of these quadratic algebras.

If R is involutive, we put $q = 1$ in these formulae.

The following holds $P_-(-t)P_+(t) = 1$.

Proposition. (Phung Ho Hai)

The HP series $P_-(t)$ (and hence $P_+(t)$) is a rational function:

$$P_-(t) = \frac{N(t)}{D(t)} = \frac{1 + a_1\, t + \ldots + a_r\, t^r}{1 - b_1\, t + \ldots + (-1)^s\, b_s\, t^s} = \frac{\prod_{i=1}^r (1 + x_i t)}{\prod_{j=1}^s (1 - y_j t)},$$

where a_i and b_i are positive integers, the polynomials $N(t)$ and $D(t)$ are coprime, and all the numbers x_i and y_i are real positive.

We call the couple $(r|s)$ bi-rank. In this sense all involutive and Hecke symmetries are similar to super-flips, for which the role of the bi-rank is played by the super-dimension $(m|n)$.

Examples. If R comes from the QG $U_q(sl(m))$, then

$$P_-(t) = (1+t)^m.$$

If R is a deformation of the super-flip $P_{m|n}$, then

$$P_-(t) = \frac{(1+t)^m}{(1-t)^n}.$$

Also, there exist "exotic" examples: for any $N \geq 2$ there exits a Hecke symmetry such that

$$P_-(t) = 1 + Nt + t^2.$$

Here $dim V = N$, the bi-rank is $(2|0)$.
If $P_-(t)$ is a polynomial, i.e. the bi-rank of R is $(m|0)$, R is called *even*. This polynomial is assumed to be monique.

Thus, for a given involutive or Hecke symmetry R, we consider 3 quantum matrix algebras. If $T=(t_i^j)$ is subject to the system

$$RT_1T_2 - T_1T_2R,\ \ T=(t_i^j), 1\le i,j\le m,$$

is called RTT algebra. If L is subject to

$$RL_1RL_1 - L_1RL_1R = 0,\ \ L=(l_i^j),\ \ 1\le i,j\le m,$$

the algebra is called RE one. And if the system is

$$RL_1RL_1 - L_1RL_1R = RL_1 - L_1R,\ \ L=(l_i^j),\ \ 1\le i,j\le m$$

the algebra is called modified RE algebra.

Introduce the following notations

$$L_{\overline{1}} = L_1,\ L_{\overline{2}} = R_{12}\, L_{\overline{1}}\, R_{12}^{-1},\ L_{\overline{3}} = R_{23}\, L_{\overline{2}}\, R_{23}^{-1} = R_{23}\, R_{12}\, L_{\overline{1}}\, R_{12}^{-1}\, R_{23}^{-1}, \dots$$

In this notation the defining relations of the RE algebra become similar to the RTT ones

$$RL_{\overline{1}}L_{\overline{2}} = L_{\overline{1}}L_{\overline{2}}R.$$

Let us observe remind that in the RTT algebra the defining relations are

$$R\,T_1\,T_2 = T_1\,T_2\,R,$$

where

$$T_2 = P\,T_1\,P.$$

Similarly to the classical case, in the RE algebra it is possible to define analogs of the power sums

$$p_k(L) = Tr_R\, L^k.$$

Here, Tr_R is the so-called quantum or R-trace, which is defined by the formula

$$Tr_R L = Tr\, C_R\, L.$$

The matrix C_R is a matrix which is completely defined by the initially given involutive or Hecke symmetry R and can be constructed for almost any such a symmetry (or a braiding). Also, analogs $e_k(L)$ of elementary symmetric polynomials can be introduced as follows

$$e_k(L) = Tr_{R(1\dots k)}\, A^{(k)}\, L_1\, L_{\overline{2}} \dots L_{\overline{k}}.$$

Here, $A^{(k)} : V^{\otimes k} \to \bigwedge_R^{(k)}(V)$ are the skew-symmetrizers.

Note that if R is of bi-rank $(m|0)$, the highest elementary symmetric polynomial $e_m(L)$ is the "quantum determinant".

Observe that the elementary symmetric polynomials $e_k(L)$ and power sums $p_k(L)$ are central in the RE algebra. Moreover, as shown in [GPS, IOP], in the RE algebra there is a version of the Newton identities

$$p_k - qp_{k-1}\, e_1 + (-q)^2 p_{k-2}\, e_2 + \ldots + (-q)^{k-1} p_1\, e_k + (-1)^k k_q e_k = 0.$$

Also, in this algebra there exists a quantum analog of the Cayley-Hamilton identity similar to the classical one

$$L^m - q\, L^{m-1}\, e_1 + (-q)^2 L^{m-2}\, e_2 + \ldots + (-q)^{m-1} L\, e_{m-1} + (-q)^m\, I\, e_m = 0.$$

Let A and B be two associative unital algebras equipped with a map (called permutation map)

$$\sigma : A \otimes B \to B \otimes A, \ \ (a \otimes b) \to \sigma(a \otimes b), \ \ a \in A, \ \ b \in B \qquad (2)$$

Definition.

By a quantum double (QD) we mean the data (A, B, σ), where the map σ is defined by means of a braiding R.

Also, we'll speak about the permutation relations

$$a \otimes b = \sigma(a \otimes b).$$

Example. Introduce the following QD

$$R\,M_1\,R\,M_1 = M_1\,R\,M_1\,R,$$

$$R^{-1}\,D_1\,R^{-1}\,D_1 = D_1\,R^{-1}\,D_1\,R^{-1},$$

$$\sigma : D_1\,R\,M_1\,R \to R\,M_1\,R^{-1}\,D_1 + R.$$

The corresponding permutation relation is

$$D_1\,R\,M_1\,R = R\,M_1\,R^{-1}\,D_1 + R.$$

If $R \to P$, this QD tends to the usual Weil-Heisenberg algebra. Thus, the above QD $(A = \mathcal{D}(\mathcal{R}), B = \mathcal{M}(R))$ is a q-Weil-Heisenberg algebra.

Note that the entries of the matrix $D=\|\partial_i^j\|$ are treated to be quantum analogs of the patrial derivatives. In order to get an action of the elements ∂_k^l onto any monomial p in m_i^j, we permute these elements

$$\partial_k^l\otimes p\to\sigma(\partial_k^l\otimes p)$$

and kill all partial derivatives located on the right hand side. For instance, in the classical case and if $p=m_i^j$ we have

$$\partial_k^l\triangleright m_i^j=\delta_k^j\,\delta_i^l.$$

Theorem.

Let $M = \|m_i^j\|_{1\le i,j\le N}$ and $D = \|\partial_i^j\|_{1\le i,j\le N}$ be matrices entering the above QD. Then the matrix

$$\hat{L} = M\,D$$

generates the modified RE algebra.

This theorem says that in our quantum setting, the situation is similar to the classical one. Recall that in the classical setting the matrix $\hat{L} = M\,D$, where M is a matrix with commutative entries m_i^j and D is the matrix composed from the partial derivatives $\partial_i^j = \partial_{m_j^i}$, generates the algebra $U(gl(N))$.

However, in the quantum case there is a property, which is absent in the classical situation. If the matrix $\hat{L}$ is subject to the modified RE algebra, then the matrix

$$L = I - (q - q^{-1})\hat{L}$$

meets the non-modified RE algebra. Thus, the RE algebra and its modified version are in fact one algebra but written in different bases.

Note that this isomorphism fails if $q \to 1$. Since the RE algebra tends to $Sym(gl(N))$ and the modified RE algebra tends to $U(gl(N))$, we get two algebras for which we have no isomorphism.

Now, introduce q-cut-and-join operators

$$W_q^{\Delta} =: Tr_R\, \hat{L}^{\Delta_1} \dots Tr_R\, \hat{L}^{\Delta_k} : .$$

They are similar to these above but with R-traces and the matrix $\hat{L}$ generating the corresponding modified RE algebra.
If we cancel the normal ordering, these operators are called q-Casimir operators. Our next objective is to perform spectral analysis of all these operators.
We'll consider the normal ordering below. Now, we deal with the q-Casimir operators.
However, first we have to define algebras which will play the role of the function algebras.

Now, consider a QD $(A = \mathcal{L}(R), B = \mathcal{M}(R))$ composed from two copies of the RE algebra and equipped with the permutation relation

$$R\, L_1\, R\, M_1 = M_1\, R\, L_1\, R^{-1}.$$

This QD is a q-version of the algebra $U(gl(N))$, acting onto the algebra $Sym(gl(N))$ by left vector fields. In order to see this, we pass to the generating matrix $\hat{L}$, by using the relation $L = I - (q - q^{-1})\, \hat{L}$. Then we have the QD $(A = \hat{\mathcal{L}}(R), B = \mathcal{M}(R))$ equipped with the permutation relation

$$R\, \hat{L}_1\, R\, M_1 = M_1\, R\, \hat{L}_1\, R^{-1} + R\, M_1.$$

By setting $R = P$, we get the relation meaning that the algebra $U(gl(N))$ acts on the algebra $Sym(gl(N))$ by left vector fields.

Now, let us consider the problem of spectral analysis of the Casimir operators from the algebra $\hat{\mathcal{L}}(R)$ acting on the algebra $\mathcal{M}(R)$. Let $\lambda = (\lambda_1 \geq ... \geq \lambda_N)$ be a partition of an integer k and T be a standard table, corresponding to it. Then there exists an idempotent $P_{\lambda,T}$, which is en element of the group algebra $\mathbb{C}[S_N]$ of the symmetric group S_N.

In the Hecke algebra $H_N(q)$ there exists an analogous idempotent, which are a deformation of that above. By means of these idempotents it is possible to construct eigenspaces of the operator $Tr_R L$ acting in the algebra $\mathcal{M}(R)$.

Proposition.

Let $\lambda \vdash k$ be a partition and T be a standard Young table corresponding to the Young diagram of the partition λ. Then the following relation holds

$$Tr_R L \triangleright P_{(\lambda,T)}(R) M_1 M_{\overline{2}} \dots M_{\overline{k}} = \chi_\lambda(Tr_R L) P_{(\lambda,T)}(R) M_1 M_{\overline{2}} \dots M_{\overline{k}},$$

where

$$\chi_\lambda(Tr_R L) = \frac{N_q}{q^N} - \frac{q - q^{-1}}{q^{2N}} \sum_{i=1}^{k} q^{-2c(i)},$$

the sum is taken over all boxes of the table (λ, T) and $c(i)$ is the content $c = n - m$ of the box in which the integer i is located. (Here n (resp. m) is the number of the column (resp., row) where the box is located.)

The quantity $\chi_\lambda(Tr_R L)$ is called λ-character of the Casimir operator $Tr_R L$. In a similar manner λ-characters of other Casimir operators are defined. We would like to compute λ-characters of these $Tr_R L^k$. Let us remember the CH identity

$$L^k - qL^{k-1}\, e_1 + (-q)^2 L^{k-2}\, e_2 + \ldots + (-q)^{k-1} L\, e_k + (-1)^k k_q\, I\, e_k = 0.$$

Let us denote by $\mu_i,\ i = 1 \ldots N$ the roots of the corresponding characteristic polynomial. Thus, we have

$$q^k e_k(L) = \sum_{1 \le i_1 < \cdots < i_k \le m} \mu_{i_1} \ldots \mu_{i_k}.$$

In particular, $e_1(L) = q^{-1} \sum_{k=1}^m \mu_k$.

Since the coefficients of the above polynomial are central in the algebra $\mathcal{L}(R)$, μ_i are elements of its central extension.
Let us assign to the eigenvalues μ_i λ-characters as follows $\chi_\lambda(\mu_i) = q^{-2(\lambda_i+m-i)}$. We have conjectured that this assignment is consistent with the above relations, that is

$$\chi_\lambda(e_k(L)) = q^{-k} \sum_{1\le i_1<\cdots<i_k\le m} \chi_\lambda(\mu_{i_1})\ldots\chi_\lambda(\mu_{i_k}).$$

As follows from above Proposition this conjecture is valid for $k=1$. We have also checked it for $k=2$.

By assuming this conjecture to be true, we become able to compute the λ-characters of the elementary symmetric polynomials. In virtue of the Newton relations we can compute λ-characters of all power sums:

$$Tr_R\, L^k = \sum \mu_i^k\, d_i, \;\text{ where }\; d_i = q^{-1} \prod_{p=1}^{N} \frac{\mu_i - q^{-2}\mu_p}{\mu_i - \mu_p}. \qquad (3)$$

Also, by passing to the generating matrix $\hat{L}$ we can compute λ-characters of the Casimir operators $Tr_R \hat{L}^k$.

Now, we pass to q-cut-and-join operators. First, we have to define the quantum normal ordering rule. For this purpose we replace each matrix $\hat{L}$ in the definition of the operators W_q^{Δ} by $M\,D$ and then we push all the matrices D through theses M to the most right position by means of the following permutation relations:

$$D_1 RM_1 R = RM_1 R^{-1} D_1 \quad \Leftrightarrow \quad D_1 RM_1 = RM_1 R^{-1} D_1 R^{-1}.$$

These permutation relations are obtained from the QD $(\mathcal{D}(R^{-1}), \hat{\mathcal{L}}(R))$ by omitting the last (constant) term from the permutation relation.

Similarly to the classical case, the cut-and-join operators can be expressed via the quantum Casimir ones.

Example

For $\Delta = (2, 0, \dots, 0)$ the cut-and-join operator reads $W_q^{(2)} = :\mathrm{Tr_R}\hat{\mathrm{L}}^2:$. In order to get the explicit expression for the normal ordered form, we use the identity

$$\mathrm{Tr_R}\hat{\mathrm{L}}^2 = \mathrm{Tr}_{\mathrm{R}(12)}(\hat{\mathrm{L}}_1\hat{\mathrm{L}}_{\overline{2}}\mathrm{R}_1) = \mathrm{Tr}_{\mathrm{R}(12)}(\hat{\mathrm{L}}_1\mathrm{R}_1\hat{\mathrm{L}}_1) =$$

$$\mathrm{Tr}_{\mathrm{R}(12)}(\mathrm{M}_1\mathrm{D}_1\mathrm{R}_1\mathrm{M}_1\mathrm{D}_1)$$

and apply the mentioned permutation relation. By straightforward computation we get

$$W_q^{(2)} = \mathrm{Tr_R}\hat{\mathrm{L}}^2 - \mathrm{q}^{-\mathrm{N}}\mathrm{N_q}\mathrm{Tr_R}\hat{\mathrm{L}}.$$

Now, exhibit the classical Capelli identity. Let $\hat{L} = L\,D$. Then we have

$$rDet(\hat{L}+K) = detL\,detD,$$

where K is the diagonal matrix $diag(0,1,...,n-1)$ and $rDet$ is the so-called row-determinant.

Observe that the term $rDet(\hat{L}+K)$ in the l.h.s. can be written in the following form

$$cDet(\hat{L}+K) = detL\,detD,$$

where K is the diagonal matrix $diag(n-1,...1,0)$ and $cDet$ is the so-called column-determinant. Also, the following form

$$Tr_{1..N}A^{(N)}\hat{L}_1\,(\hat{L}+I)_2\,(\hat{L}+2I)_3...(\hat{L}+(N-1)I)_N$$

is possible.

Proposition.

In the RE algebra the following holds

$$Tr_{R(1\dots m)}\, A^{(m)}\, \hat{L}_1\, (\hat{L}_{\overline{2}} + q\, I)(\hat{L}_{\overline{3}} + q^2\, 2_q\, I)\dots(\hat{L}_{\overline{m}} + q^{m-1}\, (m-1)_q\, I) =$$

$$q^{-m}\, det_R\, L\, det_{R^{-1}}\, D.$$

Here m is the rank of R. (Note that in the classical case $m = N$.)

Whereas the determinants in the r.h.s. are defined by the formulas

$$det_R\, L = \langle v|\, L_1\, L_{\overline{2}}\dots L_{\overline{m}}\, |u\rangle,$$

$$det_{R^{-1}}\, D = \langle v|\, D_{\overline{m}}\, D_{\overline{m-1}}\dots D_1\, |u\rangle,$$

where u and v are some tensor such that

$$A^{(m)} = u\rangle\langle v \text{ and } \langle v,\, u\rangle = 1.$$

Observe that there are known numerous attempts to generalize the classical Capelli identity.
I want to only mention the paper by Noumi, Umeda, Wakayama (1994). Their construction is related to the RTT algebra. Their R is the standard Hecke symmetry, i.e., it comes from the QG $U_q(sl(N))$. Whereas ours is valid in general situation.

In 1996, A.Okounkov introduced the notion of quantum immanants. Our technique enables us to introduce q-analogs of these objects.