Multiplicative Bessel Kernels: from addition laws to CY periods

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Object of study and motivation

$$\boxed{\Psi_\lambda(x)\Psi_\lambda(y) = \int_\gamma K(x,y|z)\Psi_\lambda(z)dz}$$

Motivation :

- Apèry-Beukers-Zagier equations, Arithmetic, Mirror Symmetry
- Non-Abelian Abel theorem
- Analytic Langlands, Hecke operators
- Separation of Variables, Integrability

Papers :

- P.Etingof, E.Frenkel and D.Kazhdan *Hecke operators and analytic Langlands correspondence for curves over local fields* (2021)
- V.Golyshev, A.Mellit, V.Rubtsov and D.van Straten *Non-abelian Abel's theorems and quaternionic rotation* (2021).
- M.Kontsevich and A.Odesskii *Multiplication kernels* (2021)
- I.Gaiur, V.Rubtsov and D.van Straten *Bessel kernels and beyond : geometry, combinatorics and number theory*, *In progress*(2022)

Plan of talk

1. Non-abelian Abel's theorem
2. Multiplication kernels for spectral problems
3. Kernels for Bessel type equations : experimental mathematics
4. Multiple Accessory parameters (Sklyanin SoV)

Calabi-Yau DO

Motivation 1 : Find an analogue of Apèry-Beukers-Zagier list for **Calabi-Yau DO** - monic operators of the type :

$$L := \theta^n + tP_1(\theta) + t^2 P_2(\theta) + \ldots + t^r P_r(\theta), \quad \theta = t\frac{d}{dt}, P_i -$$

degree n polynomial in θ with **MUM - maximally unipotent monodromy** at $t = 0$.
Let M_t- be a family of CY $n-$ folds, $t \in \mathbb{P}^1$ with $\omega_t \in \Omega^n(M_t)-$unique (up to a scalar) holomorphic differential $n-$form.
Gauss-Manin connection theory : $\Longrightarrow$ **periods** $f = \int_{\gamma_t} \omega_t$ of M_t satisfy certain DE (**Picard-Fuchs**.) Here γ_t are $r-$ cycles on M_t.

Reminder : MUM point of DE

Let $\mathcal{L}$ be a differential operator with singularity at $t = 0$. Monodromy matrix is

$$M = T^{-1}\left(\mathrm{I} + \sum_i E_{i,i+1}\right)T$$

Locally fundamental solution writes

$$\phi_k(t) = \sum_{j=0}^{k} \frac{\ln(t)^j}{j!} f_{k-j}, \quad \text{i. e. } \phi_0(t) = f_0(t).$$

where

$$f_0(t) \in 1 + t\mathbb{C}[t], \quad f_1, \ldots f_{n-1} \in t\mathbb{C}[t]$$

In particular, $\phi_0(t)$ is an **analytic solution** at the singular point

Examples of CY DO

- $n=1, M_t-$ 1D CY - **elliptic curve** $\mathcal{E}_t : y^2 = x(x-1)(x-t)$.
  Period $f(t) = \int_1^\infty \frac{dx}{y}$ satisfied the Picard-Fuchs :

$$\theta^2 f - \frac{t}{1-t}\theta f - \frac{t}{4(1-t)} f = 0.$$

- $n=2, M_t-$ 2D CY - **K3 surface**, say, the family of quartics in $\mathbb{P}^3 : x_1^4 + x_2^4 + x_3^4 + x_4^4 - t^{-1} x_1 x_2 x_3 x_4 = 0$. Picard-Fuchs :

$$\theta^3 f + 4\theta(4\theta+1)(4\theta+2)(4\theta+3) f = 0.$$

- $n=3, M_t-$ CY - threefold has the Hodge number $h^{2,1} = 1 \Longrightarrow$ Picard -Fuchs has order 4. For the **Dwork quintic** in $\mathbb{P}^4 : x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - t^{-1} x_1 x_2 x_3 x_4 x_5 = 0$, Picard -Fuchs :

$$\theta^4 f - 5\theta(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4) f = 0.$$

Motivation 2 : Non-Abelian Abel Theorem

Group law as an integral identity

Multiplication theorem in the abelian setup are known as addition theorems, but this is a no-go beyond geometric class field theory.

- $(\mathbb{R}^\star, \cdot)$ is given by

$$\begin{array}{ccccc} \int_1^x \frac{dt}{t} & + & \int_1^y \frac{dt}{t} & = & \int_1^{z=xy} \frac{dt}{t} \\ \log(x) & + & \log(y) & = & \log(xy) \\ x & \text{``}+\text{''} & y & = & xy \end{array}$$

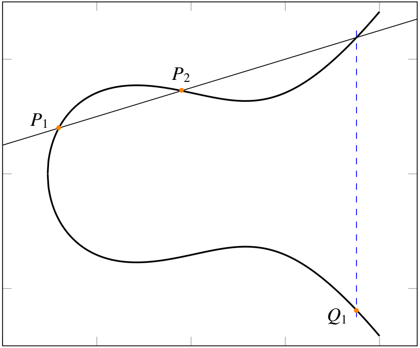
- S^1

$$\begin{array}{ccccc} \int_1^x \frac{dt}{\sqrt{1-t^2}} & + & \int_1^y \frac{dt}{\sqrt{1-t^2}} & = & \int_1^z \frac{dt}{\sqrt{1+t^2}} \\ \arcsin(x) & + & \arcsin(y) & = & \arcsin(z) \\ x & \text{``}+\text{''} & y & = & x\sqrt{1-y^2}+y\sqrt{1-x^2} \end{array}$$

- Euler (1750) (Landen transform for elliptic functions for lemniscate's arc length)

$$\begin{array}{ccccc} \int_1^x \frac{dt}{\sqrt{1-t^4}} & + & \int_1^y \frac{dt}{\sqrt{1-t^4}} & = & \int_1^z \frac{dt}{\sqrt{1-t^4}} \\ x & \text{``}+\text{''} & y & = & \frac{x\sqrt{1-y^4}+y\sqrt{1-x^4}}{1+x^2y^2} \end{array}$$

Elliptic curve as an Abelian variety



$$\int_0^{P_1} \omega + \int_0^{P_2} \omega = \int_0^{Q_1} \omega \bmod \Lambda,$$

$$\omega = \frac{dx}{\sqrt{x^3 + g_1 x + g_2}}.$$

$$\mathcal{O}(P_1 - 0) \otimes \mathcal{O}(P_2 - 0) \simeq \mathcal{O}(Q_1 - 0)$$
$$\Updownarrow$$
$$P \quad \text{“}+\text{”} \quad P_2 = Q_1$$

"Abelian" Abel theorem

Instead of studying integral identities, consider line bundle with connection

For $g = 1, \mathrm{d}\,\Psi = \Psi\omega, \quad \omega \in H^{1,0}(X_1, \mathbb{C}) \Longrightarrow \Psi(P) = \exp\int\limits_0^P \omega$. the addtion rewrites as ("modulo of lattices)

$$\Psi(P_1)\Psi(P_2) = \Psi(Q_1) \quad \Longleftrightarrow \quad \int\limits_0^{P_1}\omega + \int\limits_0^{P_2}\omega = \int\limits_0^{Q_1}\omega \quad \Longleftrightarrow \quad P_1 \text{``}+\text{''} P_2 = Q_1.$$

For a hyperelliptic $X_g := \{F(x,y) = 0\}$ Abel (1828) : $\omega = R(x,y)dx-$ rational differential, then there exist a number $p : \quad \forall N \gg p$

$$\int_0^{x_1}\omega + \ldots + \int_0^{x_N}\omega = \int_0^{y_1}\omega + \ldots + \int_0^{y_p}\omega + E(x,y)$$

"Abelian" Abel theorem. Multiplicative version

An obvious multiplicative reformulation of Abel's theorem is that for any DE on a Riemann surface X_g

$$d\Psi(x) = \Psi(x)\omega$$

there exists a simple "kernel" $K(\mathbf{x}|\mathbf{y})$ such that

$$\Psi(x_1)\dots\Psi(x_N) = \int K(\mathbf{x}|\mathbf{y})\Psi(y_1)\dots\Psi(y_p)dy_1\dots dy_p$$

One can fill in all the details ; K, obviously, is a delta-kernel supported on the graph of the relation $\sum(x-0) = \sum(y-0)$ in $\mathrm{Jac}^0(X_g)$.

Non-Abelian Abel theorem

Abelian		**Non-Abelian**
line bundles	$\to$	vector bundles
1-st order ODE		N-th order ODE

Study multiplication formulas for the equations

$$\mathcal{L}_x \Psi = \lambda \Psi$$

where λ is an **accessory** parameter (the local monodromy is independent on λ).
Study formulas

$$\phi_0(\lambda; x)\phi_0(\lambda; y) = \int K(x, y|z)\phi_0(\lambda; z)dz$$

where ϕ_0 is solution of the form

$$\phi_0(x) = \sum_{i=0}^{\infty} P_i(\lambda)x^i, \quad P_i \in \mathbb{C}[\lambda]$$

Examples

- Bessel function of zero kind. **Sonine-Gegenbauer formula**

$$J_0(x)J_0(y) = \int_{|x-y|}^{|x+y|} \frac{2}{\pi\sqrt{S(x,y,z)}} J_0(z)dz$$

- ***D*2-equation (small Apèry)**. Algebraic kernel

$$[\partial_t \circ f(t) \circ \partial_t + t]\,\psi = \lambda\psi, \quad f(t) = t^3 + At^2 + t$$

$$\phi_\lambda(x)\phi_\lambda(y) = \int \frac{\phi_\lambda(z)}{\sqrt{P(x,y,z)}} dz, \quad P(x,y,z) = \mathrm{discr}_t\,[f(t) - (t-x)(t-y)(t-z)]$$

An appearance of **Kontsevich polynomials**

Kernel and structure constants

$P_0(\lambda), P_1(\lambda), \dots \in \mathbb{C}[\lambda]$ and $P_0(\lambda) = 1$, $\deg(P_i) = i$, then

$$P_i(\lambda)P_j(\lambda) = \sum_{k=0}^{\infty} C_{ij}^k P_k(\lambda)$$

Consider generating functions

$$f_\lambda(x) = \sum_{i=0}^{\infty} P_i(\lambda)x^i = 1 + O(x), \quad K(x,y|z) = \sum_{i,j,k} C_{i,j}^k \frac{x^i y^j}{z^{k+1}} dz$$

Theorem (M.Kontsevich, A.Odesskii 2021)

$$f_\lambda(x)f_\lambda(y) = \frac{1}{2\pi i} \oint K(x,y|z) f_\lambda(z)$$

Remark Can be generalized for $\mathbb{C}[\lambda_1, \lambda_2 \dots \lambda_g]$, instead of binary operation - $g+1$ to g.

Master formula. Spectral problems

Consider spectral problems

$$\mathcal{L}_x \Psi = \lambda \Psi$$

- λ - accessory and spectral parameter
- Always true for $\mathfrak{gl}_2$ case, i.e. $\mathcal{L}$ is second order differential operator
- Assume that there is an analytic solution at $x = 0$ (MUM/regular)

Master formula. Spectral theorem

Choose solution $\phi_0(\lambda;x) = 1 + \sum_{i=1}^{\infty} P_i(\lambda)x^i$, where P_i are polynomials in λ of degree i.

$$\mathcal{L}_y \phi_0(\lambda;y) = \lambda \phi_0(\lambda;y)$$

Spectral theorem $\Rightarrow$ $\boxed{P_i(\mathcal{L}_y)\phi_0(\lambda;y) = P_i(\lambda)\phi_0(\lambda;y)}$

Master formula

$$\phi_0(\mathcal{L}_y;x)\phi_0(\lambda;y) = \sum_{i=0}^{\infty} x^i P_i(\mathcal{L}_y)\phi_0(\lambda;y) = \sum_{i=0}^{\infty} x^i P_i(\lambda)\phi_0(\lambda;y) = \phi_0(\lambda;x)\phi_0(\lambda;y)$$

Motivation 3 : Langlands correspondence– from Bessel to Kontsevich

Exponential. 1-Bessel operator

Consider spectral problem for $\frac{d}{dx}$ operator

$$\theta\psi = \lambda x\psi, \quad \theta = x\frac{d}{dx}.$$

Normalized solution is exponential function

$$\psi(x;\lambda) = e^{\lambda x} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}x^n.$$

The multiplication formula is straightforward

$$e^{\lambda x}e^{\lambda y} = e^{\lambda(x+y)} = \int \delta(x+y-z)e^{\lambda z}dz = \frac{1}{2\pi i}\oint_{x+y} \frac{1}{z-(x+y)}e^{\lambda z}dz.$$

Applying master formula for this problem we get the same kernel :

$$K(x,y;z) = \frac{1}{2\pi i}\frac{1}{z-(x+y)},$$

Weil algebra and integral operators

Use Cauchy formula

$$f(y) = \frac{1}{2\pi i}\oint \frac{dz}{z-y} f(z), \quad f^{(k)} = \frac{1}{2\pi i}\oint \frac{k!dz}{(z-y)^{k+1}} f(z)$$

Set

$$y^j \frac{d^k}{dy^k}\circ = \frac{1}{2\pi i}\oint y^j \frac{k!dz}{(z-y)^{k+1}}\circ, \quad K\left(y^j \frac{d^k}{dy^k}\right) = \frac{y^j}{2\pi i}\frac{k!}{(z-y)^{k+1}}$$

Lemma

$$K\left(y^j \frac{d^k}{dy^k}\circ y^r \frac{d^l}{dy^l}\right) = K\left(y^j \frac{d^k}{dy^k}\right) \star K\left(y^r \frac{d^l}{dy^l}\right) = y^j \frac{d^k}{dy^k} K\left(y^r \frac{d^l}{dy^l}\right)$$

$$K(\mathcal{L}_y) = K(\mathcal{L}_y \circ 1) = \mathcal{L}_y K(1) = \mathcal{L}_y\left[\frac{1}{z-y}\right]$$

We pass from the Weyl algebra $\mathbb{C}[y, \partial_y]/\langle[\partial_y, y] = 1\rangle$ to the formal algebra of elementary kernels.

"Proof" of Master formula concept

$$K \exp\left(x\frac{d}{dy}\right) := \sum_{n=0}^{\infty} \frac{K(\frac{d^n}{dy^n})}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{(z-y)^{n+1}} = \frac{1}{z-y-x}$$

2-Bessel operator-1

Consider the following spectral problem

$$\theta^2 \psi = \lambda x \psi, \quad \theta = x\frac{d}{dx}.$$

Solution of these equation may be written in terms of the Bessel function of the 0 kind. Normalized solution takes form

$$J_0(x;\lambda) = \sum_{i=0}^{\infty} \frac{\lambda^n}{(n!)^2} x^n$$

The multiplication formula is given by a version of Sonine-Gegenbauer formula and takes form

$$J_0(x;\lambda)J_0(y;\lambda) = \frac{1}{2\pi i}\int \frac{J_0(z;\lambda)dz}{\sqrt{x^2+y^2+z^2-2xy-2yz-2xz}}. \quad (1)$$

Applying master formula in that case we obtain a power series with coefficients

We can also easily find an equation for $K(x,y;z)$ using **GFUN**. Equation is

$$\left(x^2+y^2+z^2-2xy-2xz-2yz\right)\frac{d}{dx}K + (x-z-y)\,K = 0, \quad K(0,y,z) = \frac{1}{z-y}$$

which solution provides a Sonine kernel.

N-Bessel

Consider the $N-$ Bessel and its generating function $N-$ Bessel equation

$$\left(x\frac{d}{dx}\right)^N \Psi - x\lambda\Psi = 0$$

Solution is generalized Bessel function

$$J^{(N)}(\lambda; x) = \sum_{k=0}^{\infty} \frac{\lambda^k}{(k!)^N} x^k$$

Clearly $(\lambda = 1)$

$$J^{(N)}(x)J^{(N)}(y) = \sum_{n,m} \left(\frac{1}{n!m!}\right)^N x^n y^m = \sum_{n,m} \binom{n+m}{n}^N x^n y^m \left(\frac{1}{(n+m)!}\right)^N$$

N-Bessel-2

When we multiply this by $z^{-(m+n)}$, multiply with $J^{(N)}(z)$, expand in powers of z we find :

$$J^{(N)}(x)J^{(N)}(y) = \frac{1}{2\pi i}\oint K(x,y,z)J^{(N)}(z)\frac{dz}{z} = \frac{1}{2\pi i}\oint H\left(\frac{x}{z},\frac{y}{z}\right)J^{(N)}(z)\frac{dz}{z},$$

where

$$H(X,Y) = \sum_{n,m}\binom{n+m}{n}^N X^n Y^m.$$

Try to understand the singular locus of the $D-$ module generated by H. For $N=2$ we have the algebraic kernel $H(X,Y) = \frac{1}{\sqrt{\Delta(X,Y,1)}}$ and $\Delta(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$. This defines a circle, tangent the coordinate lines $xyz=0$ at the mid-points $(0:1:1),(1:0:1),(1:1:0)$ in $\mathbb{P}^2$.

$$\Delta(u^2,v^2,w^2) = (u+v+w)(-u+v+w)(u-v+w)(u+v-w) = 16S_{\Delta}^2-$$

(**"Heron configuration"**)

3- and 4-Bessel equation

For $N = 3$ kernel is

$$K_3(x, y|z) = -\frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; -\frac{27zxy}{(x-z+y)^3}\right)}{x - z + y}$$

Singularity locus is

$$x(2x + z - y)((x - z + y)^3 + 27xyz) = 0$$

Conjecture

The singular locus of the $D-$ module of H_N consists of the coordinate triangle $xyz = 0$, together with a rational curve R_N of degree N with $\frac{(N-1)(N-2)}{2}$ double points, which intersects the coordinate triangle only in the three mid-points $(0:1:1), (1:0:1), (1:1:0)$, defined by a symmetric polynomial

$$\Delta(x, y, z) = x^N + y^N + z^N + \dots,$$

determined by the property

$$\Delta(u^N, v^N, w^N) = \prod_{\omega,\eta} (x + \omega y + \eta z),$$

where in the product ω and η run over the $N-$th roots of unity.

from Sonine kernel to Kontsevich polynomials

The polynomial $\Delta(x, y, z)$ is a very special case to a more general class of (generalized) Kontsevich polynomials :

$$P_{a,b,c}(x, y, z) = (xy + yz + xz - b)^2 - 4(xyz + c)(x + y + z + a),$$
$$= (x - y)^2 z^2 - 2((xy + b)(x + y) + 2axy + c)z$$
$$+ (xy - b)^2 - 4c(x + y + a,$$

which is a discriminant of the quadratic trinomial :

$$t^3 + at^2 + bt + c - (t - x)(t - y)(t - z).$$

Two-valued formal group laws

Let R be a commutative ring with unit. ($R = \mathbb{Z}, \mathbb{R}, \mathbb{C}$, or the polynomial rings over $\mathbb{Z}, \mathbb{R}, \mathbb{C}$. The equation $w = \Phi(u, v)$ where $\Phi(u, v)$ is a formal series, defines a classical formal group law over the ring R. This means that :

- $\Phi(u, 0) = u$;
- $\Phi(u, \Phi(v, w)) = \Phi(\Phi(u, v), w)$.

Buchstaber family of polynomials-1

Consider the two-valued formal group with multiplication defined by the relation

$$B(x,y,z) := z^2 - \Theta_1(x,y)z + \Theta_2(x,y) = 0,$$

where

$$\Theta_1(x,y) = Z_+ + Z; \quad \Theta_2(x,y) = Z_+ Z,$$

with

$$Z_+ = \Phi(u,v)\Phi(\bar{u},\bar{v}) = |\Phi(u,v)|^2; \quad Z = \Phi(\bar{u},v)\Phi(u,\bar{v}) = |\Phi(\bar{u},v|^2.$$

Such two-valued group is called the **"square modulus"** of the original formal group.

Buchstaber family of polynomials-2

For the elementary formal group structure with $\Phi(u, v) = u + v$ connected with cohomology theory the two-valued group determined by

$$B(x, y, z) := x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = 0.$$

This was the very first example of a two-valued group of Buchstaber–Novikov (1971).
In K-theory $\Phi(u, v) = u + v - quv$ (Buchstaber-Mischenko-Novikov) and

$$B(x, y, z) := x^2 + y^2 + z^2 - 2xy - 2xz - 2yz - q^2xyz = 0-$$

Cayley nodal cubic surface

Buchstaber family of polynomials-3

Buchstaber (1990) had classified the two-valued algebraic groups coming from the square modulus construction for formal groups with the multiplication law suggested by the addition theorem for Baker–Akhiezer elliptic functions by a zero locus of the following discriminant family depending in 3 parameters (a_1, a_2, a_3) :

$$B_{a_1,a_2,a_3}(x, y, z) := (x + y + z - a_2xyz)^2 - 4(1 + a_3xyz)(xy + yz + zx + a_1xyz).$$

The parameters are expressed via the standard Weierstrass elliptic parameters g_2, g_3 and a point α on the corresponding elliptic curve $v^2 = 4u^3 - g_2u - g_3$ by

$$a_1 = 3\wp(\alpha), a_2 = 3\wp(\alpha)^2 - g_2/4, a_3 = 1/4(4\wp(\alpha)^3 - g_2\wp(\alpha) - g_3).$$

Buchstaber - Veselov theorem

Theorem (Buchstaber - Veselov , 2019)
The discriminant locus multiplication law $B_{a_1,a_2,a_3}(x,y,z)=0$ with the "elliptic parametrization" can be reduced to the addition law $X\pm Y\pm Z=0$ via the change of variables :

$$x=\frac{1}{\wp(X)+\wp(\alpha)}, y=\frac{1}{\wp(Y)+\wp(\alpha)}, z=\frac{1}{\wp(Z)+\wp(\alpha)}.$$

The proof is based on the addition formula of Burnside discriminant (1873) (Weierstrasse $\wp-$ function addition) :

$$P_{(0,\frac{-g_2}{4},\frac{-g_3}{4})}(x,y,z):=(xy+yz+zx+\frac{g_2}{4})^2-4(1-\frac{g_3}{4}xyz)(x+y+z)$$

which is nothing but the Kontsevich polynomial for $f(t)=t^3-\frac{g_2}{4}t-\frac{g_3}{4}$. Zero locus is a beautiful K3 quartic in $\mathbb{P}^3$ with six A_1 and three D_4 singularities...

Numerology for k-Bessel. Clausen duplication formula

Consider

$$\phi_k(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!^k} = \sum_{n=0}^{\infty} a_n^{(k)} x^n$$

and its square

$$\phi_k^2(x) = \sum_{n=0}^{\infty} b_n^{(k)} x^n.$$

Statement : The following series

$$\pi^{(k)}(t) = \sum_{n=0}^{\infty} \frac{b_n^{(k)}}{a_n^{(k)}} t^n \in \mathbb{Z}[t]$$

is a period function, for the Landau-Ginzburg potential

$$W_k = \prod_{i=1}^{k-1}(1+x_i) + \prod_{i=1}^{k-1}(1+x_i^{-1})$$

i.e.

$$\pi^{(k)}(t) = \frac{1}{(2\pi i)^{k-1}} \oint \frac{1}{1-tW_k} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \dots \frac{dx_{k-1}}{x_{k-1}}, \qquad \frac{b_n^{(k)}}{a_n^{(k)}} = [(W_k)^n]_0.$$

7-Bessel function

a_n	1	1/128	1/279936	1/4586471424	1/358318080000000
b_n	2	65/64	547/34992	156353/2293235712	5039063/895795200000000
b_n/a_n	2	130	4376	312706	20156252

Master-Formula numerology

Consider again kernel for 2-Bessel

$$K(x,y;z) \simeq \frac{1}{z-y} + \frac{(y+z)}{(z-y)^3}x + \frac{(y^2+4yz+z^2)}{(z-y)^5}x^2 +$$
$$+ \frac{(y^3+9y^2z+9yz^2+z^3)}{(z-y)^7}x^3 + \frac{(y^4+16y^3z+36y^2z^2+16yz^3+z^4)}{(z-y)^9}x^4 + O(x^5).$$

Rewrite $K(x,y|z)$ as

$$K(x,y|z) = \sum_{i=0}^{\infty} P_i(y,z)\frac{x^i}{(z-y)^{2i+1}}, \quad P_i(y,z) \in \mathbb{Z}[y,z]$$

Statement : $P_i(x,y)$ are monic palindromic

$$P_i(y,z) = \sum_{k+n=i} T^{(i)}_{k,n} y^k z^n, \quad \mathbf{T^{(i)}_{k,n} = T^{(i)}_{n,k} \quad T^{(i)}_{i,0} = T^{(i)}_{0,i} = 1}$$

Generating function for the "square" of Pascal triangle

Master-Formula numerology

$$
\begin{array}{ccccc}
 & & T^{(0)}_{0,0} & & \\
 & T^{(1)}_{1,0} & & T^{(1)}_{1,0} & \\
T^{(2)}_{2,0} & & T^{(2)}_{1,1} & & T^{(2)}_{0,2} \\
 & & \cdots & &
\end{array}
=
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 4={\color{magenta}2^2} & & 1 & & & & \\
 & & & 1 & & 9{\color{magenta}=3^2} & & 9 & & 1 & & & \\
 & & 1 & & 16 & & 36 & & 16{\color{magenta}=4^2} & & 1 & & \\
 & 1 & & 25 & & 100 & & 100 & & 25 & & 1 & \\
1 & & 36 & & 225 & & 400 & & 225 & & 36 & & 1 \\
 & & \cdots & & & & \cdots & & & & \cdots & &
\end{array}
$$

$$
T^{(m)}_{k,m-k} = \frac{1}{k!}\frac{d^k}{dt^k}\left[(1-t)^{2m+1}\sum_{j=0}^{k}\binom{m+j}{j}^2 t^j\right] :=
$$

$$
= k\text{-th coefficient of } (1-t)^{2m+1}\sum_{j=0}^{k}\binom{m+j}{j}^2 t^j
$$

The last sum may be rewritten via hypergeometric series

$$
(1-t)^{2n+1}\sum_{j=0}^{k}\binom{n+j}{j}^2 t^j = (1-t)^{2n+1}\left[{}_2F_1(n+1,n+1|1|t) - \right.
$$

$$
\left.\binom{n+k+1}{k+1}^2 t^{k+1}{}_3F_2(1,n+2+k,n+2+k|k+2,k+2|t)\right]
$$

Master-Formula numerology. N-Bessel

Kernel is

$$K^{(N)}(x,y|z) = \sum_{m=0}^{\infty} P_m^{(N)}(y,z)\frac{x^m}{(z-y)^{mN+1}}, \quad \deg P_m^{(N)} = m(N-1).$$

Statement : $P_i(x,y)$ are monic palindromic

$$P_i^{(N)}(y,z) = \sum_{k+n=i(N-1)} {}^{(N)}T_{k,n}^{(i)} y^k z^n, \quad \mathbf{T_{k,n}^{(i)}} = \mathbf{T_{n,k}^{(i)}} \quad \mathbf{T_{i,0}^{(i)}} = \mathbf{T_{0,i}^{(i)}} = \mathbf{1}$$

Generating function for the "square" of Pascal triangle
Coefficients are

$${}^{(N)}T_{k,m(N-1)-k}^{(m)} = \frac{1}{k!}\frac{d^k}{dt^k}\left[(1-t)^{Nm+1}\sum_{j=0}^{k}\binom{m+j}{j}^N t^j\right] :=$$

$$= k\text{-th coefficient of } (1-t)^{Nm+1}\sum_{j=0}^{k}\binom{m+j}{j}^N t^j$$

SOME OF THESE NUMBERS ARE KNOWN
https ://oeis.org/A181544 for $N = 3$
https ://oeis.org/A262014 for $N = 4$

Periods again

Take a triangle of ${}^{(N)}T^{(m)}_{k,m(N-1)-k}$ and sum over rows, i.e.

$$a_m^{(N)} = \sum_{k=0}^{m(N-1)} {}^{(N)}T^{(m)}_{k,m(N-1)-k} = \frac{(mN)!}{(m!)^N} \in \mathbb{Z}$$

Gives a period for Dwork family of hypersurfaces

$$X_1^N + X_2^N + \ldots X_N^N = tX_1X_2\ldots X_N.$$

Landau-Ginzburg potential is

$$W_N = \left(\sum_{i=1}^{N-1} x_i + \prod_{i=1}^{N-1} x_i^{-1}\right)^N$$

There is a limit for kernel, s.t. we get Dwork period

More numbers and integrality

$(j)_n := j(j+1)\dots j+n-1)$ - Pochammer symbol.
Pochammer "perturbation" :

$$(\theta^n - x)\phi = 0 \quad \to \quad (\theta(\theta+1)\dots(\theta+n-1) - x)\phi$$

Solution

$$\phi = \sum \frac{x^k}{(k!)^n} \quad \to \quad \phi = \sum_{k=0}^{\infty} \frac{x^k}{\prod_{j=1}^{k} (j)_n}$$

Then kernel

$$K^{(N)}(x, y|z) = \sum_{m=0}^{\infty} P_m^{(N)}(y, z) \frac{x^m}{(z-y)^{mN+1}}, \quad \deg P_m^{(N)} = m(N-1).$$

N-Narayana numbers are coefficients of the polynomials $P_m^{(N)}(y, z)$:

$$N(n, k) := \frac{1}{n}\binom{n}{k-1}\binom{n}{k}, \quad \sum_{k=1}^{n} N(n, k) = C_n = \frac{\binom{2n}{n}}{n+1} = \frac{(2n)!}{(n!(n+1)!)}$$

C_n- - Catalan numbers.

Clausen formula also gives integer-coefficient power series

Periods ?

Motivation 4 : Sklyanin SoV $\Longrightarrow$ Multiple Accessory Parameters

Genus g case

Consider smooth genus g curve X_g, then $h^{1,0} = \dim H^{1,0}(X_g) = g$. Abel theorem rewrites as

$$\int_{P_0}^{P_1} \omega_i + \int_{P_0}^{P_2} \omega_i + \dots \int_{P_0}^{P_{g+1}} \omega_i = \int_{P_0}^{Q_1} \omega_i + \dots \int_{P_0}^{Q_g} \omega_i \mod \Lambda, \quad \forall i = 1 \dots g,$$

$$H^{1,0}(X_g) = \mathrm{span}\langle \omega_1, \omega_2 \dots, \omega_g \rangle$$

To write down multiplicative version consider line bundle

$$\mathrm{d}\,\Psi = \Psi \sum_{i=1}^{g} \omega_i$$

Then on $g+1$-th symmetric power of X_g

$$\Psi(P_1)\Psi(P_2)\dots\Psi(P_g)\Psi(P_{g+1}) = \Psi(Q_1)\Psi(Q_2)\dots\Psi(Q_g)$$

Non-Abelian Abel theorem

Abelian		Non-Abelian
line bundles	$\rightarrow$	vector bundles
1-st order ODE		N-th order ODE

Study multiplication formulas for the equations

$$\mathcal{L}_{\lambda_1,\lambda_2,\ldots\lambda_g}\Psi = 0$$

If $\mathcal{L}$ has g accessory parameters then

$$\phi_0(x_0)\phi_0(x_1)\ldots\phi_0(x_g) = \int K(x_0,x_1,\ldots x_g|y_1,y_2\ldots y_g)\phi_0(y_1)\ldots\phi_0(y_g)$$

where ϕ_0 is solution of the form

$$\phi_0(x) = \sum_{i=0}^{n} P_i(\lambda_1,\lambda_2\ldots\lambda_g)x^i, \quad P_i \in \mathbb{C}[\lambda_1,\lambda_2\ldots\lambda_{g-1},\lambda_g]$$

SoV and Master formula

$g+3$ punctures $\mathfrak{gl}_2$ case

$$\left[\mathcal{L}_t - \sum_{i=1}^{g} \lambda_i t^{i-1}\right]\Psi = 0, \quad \mathcal{L}_t = f(t)\partial^2 + f'(t)\partial + \frac{(g+1)^2}{4} t^g$$

$$f(t) = t(t-1)(t-u_1)(t-u_2)\dots(t-u_g)$$

Let $\phi(t)$ will be a solution, consider function

$$\Phi(\lambda_1, \lambda_2, \dots \lambda_g; x_1, x_2 \dots x_g) = \phi(x_1)\phi(x_2)\dots\phi(x_g)$$

Theorem

(Enriquez, R.) *The set of operators*

$$\mathcal{M}_i = \sum_{j=1}^{g} \frac{\partial_t^{i-1} \prod^{k\neq j}(t-x_k)|_{t=0}}{\prod_{k\neq j}(x_j - x_k)} \mathcal{L}_{x_j}$$

then

$$[\mathcal{M}_i, \mathcal{M}_j] = 0, \quad \mathcal{M}_i \Phi(x_1, x_2, \dots x_g) = \lambda_i \Phi.$$

5 singularities example

$$\mathcal{L}_t = f\partial^2 + f'\partial + \frac{9}{4}t^2, \quad f = t(t-1)(t-u_1)(t-u_2)$$

Commutative family is

$$\mathcal{M}_1 = \frac{y}{y-x}\mathcal{L}_x + \frac{x}{x-y}\mathcal{L}_y$$

$$\mathcal{M}_2 = \frac{1}{x-y}\mathcal{L}_x + \frac{1}{y-x}\mathcal{L}_y$$

SoV and Master formula

$g+3$ punctures $\mathfrak{gl}_2$ case

$$\left[\mathcal{L}_t - \sum_{i=1}^{g} \lambda_i t^{i-1}\right]\Psi = 0, \quad \mathcal{L}_t = f(t)\partial^2 + f'(t)\partial + \frac{(g+1)^2}{4}t^g$$

$$f(t) = t(t-1)(t-u_1)(t-u_2)\dots(t-u_g)$$

Choose analytic at $t=0$ solution $\phi(\lambda_1,\dots\lambda_g,x_0)$. Master formula

$$\phi(\mathcal{M}_1,\mathcal{M}_2,\dots\mathcal{M}_g,x_0)\Phi(x_1,\dots x_g) = \phi(\mathcal{M}_1,\mathcal{M}_2,\dots\mathcal{M}_g,x_0)\phi(x_1)\phi(x_2)\dots\phi(x_g) = \\ = \phi(x_0)\phi(x_1)\phi(x_2)\dots\phi(x_g)$$

So the kernel is

$$K(x_0,x_1,\dots x_g|y_1\dots y_g) \in \mathbb{C}[[x_0,x_1,\dots x_g,(x_1-y_1)^{-1},(x_2-y_2)^{-1}\dots(x_g-y_g)^{-1}]]$$

Future plans : Multi-parameter computations, connection with Hitchin systems/integrability/isomonodromy

Lenya and Nikita – 60+60 = 120 to each !

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