Modified algebraic Bethe ansatz

for XXZ chain on the segment

Samuel Belliard

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Belliard, Nuclear Phys. B892 (2015) 1-20
Belliard, Pimenta, Nuclear Phys. B894 (2015) 527-552
Avan, Belliard, Pimenta, Nuclear Phys. B899 (2015) 229-246

Belliard, Pimenta, Slavnov, J. Phys. A: Math. Theor. **54** (2021) 344001

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10 years of advances in ABA with Nikita

- 2012 : ABA scalar products, form factor for periodic XXX sl3 models

Highest coefficient of scalar products in SU (3)-invariant integrable models
S Belliard, S Pakuliak, E Ragoucy, NA Slavnov
Journal of Statistical Mechanics: Theory and Experiment 2012 (09), P09003

The algebraic Bethe ansatz for scalar products in SU (3)-invariant integrable models
S Belliard, S Pakuliak, E Ragoucy, NA Slavnov
Journal of Statistical Mechanics: Theory and Experiment 2012 (10), P10017

Bethe vectors of GL (3)-invariant integrable models
S Belliard, S Pakuliak, E Ragoucy, NA Slavnov
Journal of Statistical Mechanics: Theory and Experiment 2013 (02), P02020

Form factors in SU (3)-invariant integrable models
S Belliard, S Pakuliak, E Ragoucy, NA Slavnov
Journal of Statistical Mechanics: Theory and Experiment 2013 (04), P04033

- 2018 : MABA closed XXX sl2 (Proofs)

A note on gl2-invariant Bethe vectors
S Belliard, NA Slavnov
Journal of High Energy Physics 2018 (4), 1-15

Modified algebraic Bethe ansatz: twisted XXX case
S Belliard, NA Slavnov, B Vallet
SIGMA. Symmetry, Integrability and Geometry: Methods and Applications 14, 054

Scalar product of twisted XXX modified Bethe vectors
S Belliard, NA Slavnov, B Vallet
Journal of Statistical Mechanics: Theory and Experiment 2018 (9), 093103

Scalar products in twisted XXX spin chain. Determinant representation
S Belliard, NA Slavnov
SIGMA. Symmetry, Integrability and Geometry: Methods and Applications 15, 066

- 2019 : Linear system for Slavnov's Formula closed XXX sl2

Why scalar products in the algebraic Bethe ansatz have determinant representation
S Belliard, NA Slavnov
Journal of High Energy Physics 2019 (10), 1-17

- 2021 : Slavnov's Formula closed XXZ sl2 and Mixed Scalar product XXX closed

Scalar product for the XXZ spin chain with general integrable boundaries
S Belliard, RA Pimenta, NA Slavnov
Journal of Physics A: Mathematical and Theoretical 54 (34), 344001

Overlap between usual and modified Bethe vectors
S Belliard, NA Slavnov
Theoretical and Mathematical Physics 209 (1), 1387-1402

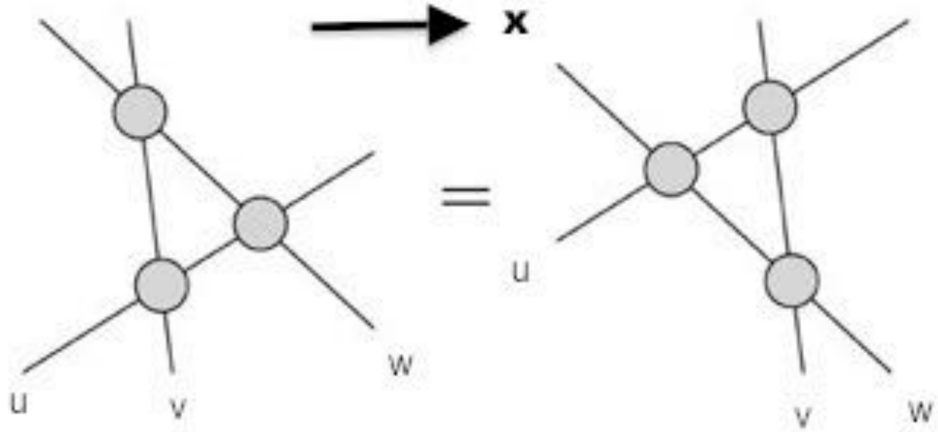
Plan of the talk

- Quantum integrability and the Yang-Baxter equation
- Quantum integrable models with Boundaries and Reflexion equation
- Quantum inverse scattering method (QISM) [Sklyanin 86]
- MABA : XXZ with generic boundaries [Avan, Belliard, Pimenta, Gros 15]
- Off-Shell action of the transfer matrix
- Slavnov's formula [Belliard, Pimenta, Slavnov 21]
- Discussion

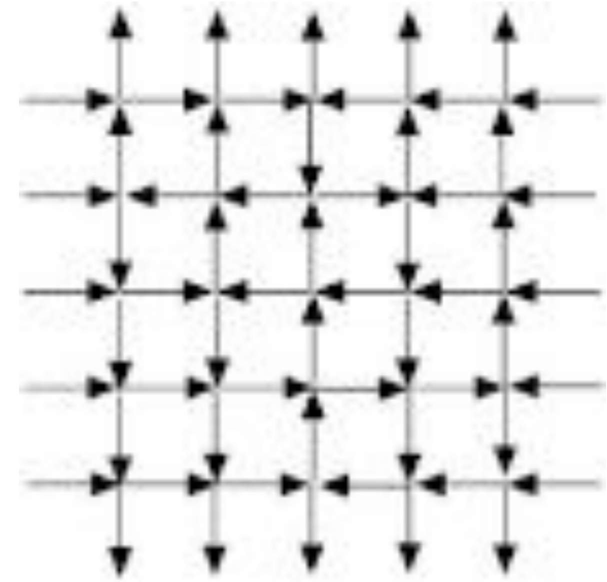
Quantum integrability

$$R_{ab}(u_a/u_b)R_{ac}(u_a/u_c)R_{bc}(u_b/u_c) = R_{bc}(u_b/u_c)R_{ac}(u_a/u_c)R_{ab}(u_a/u_b).$$

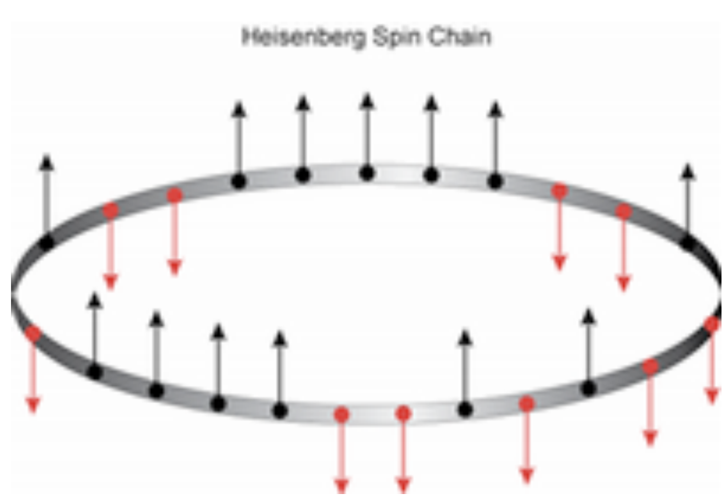
Star-triangle, Yang-Baxter equation [Yang 67][Baxter 72]



1+1 dimensional quantum field theory
[Zamolodchikov & Zamolodchikov 79],...



Two dimensional lattice models
[Onsager 44], [Baxter 72],...



One dimensional spin chains
[Heisenberg 28],[Bethe 31],...

- ABA [Faddeev et al. 79]: models on the circle
  - Models with U(1) symmetry !!!

    Results: Eigenvalues, Eigenvectors, Correlation functions (Slavnov), thermodynamic limit,...

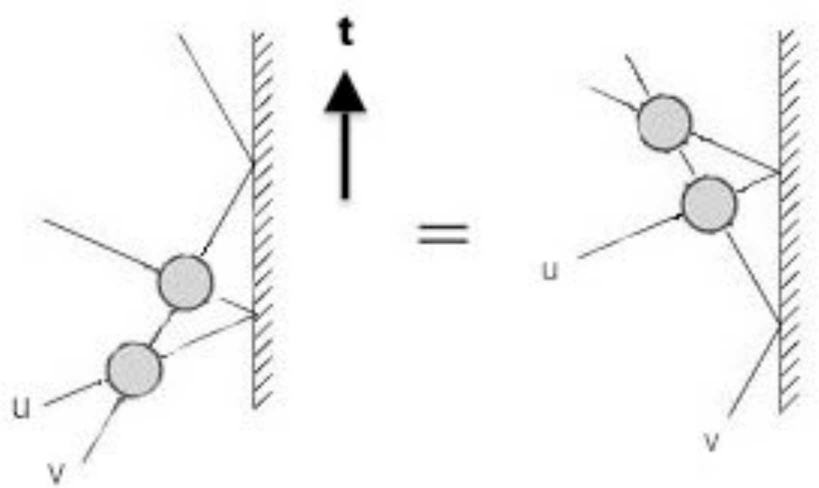
    Applications: AdS-CFT, ultra-cold atoms, quantum magnetism,....
  - Models without U(1) symmetry :

    ABA doesn't work !!!!

Quantum integrable models with Boundaries

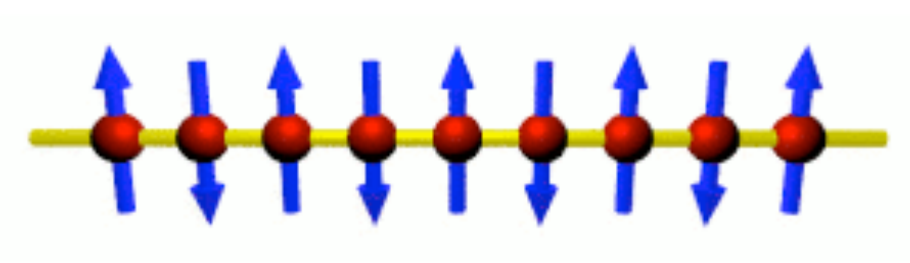
$$R_{ab}(u_1/u_2)K_a^-(u_1)R_{ab}(u_1u_2)K_b^-(u_2) = K_b^-(u_2)R_{ab}(u_1u_2)K_a^-(u_1)R_{ab}(u_1/u_2).$$

Reflexion equation, dual reflexion equation [Cherednik 84, Sklyanin 88]



quantum field theory on the half-line
[Cherednik 84, Ghoshal & Zamolodchikov 94,...]

Out of equilibrium models TASEP
[Crampé 14,...]



Spin chains on the segment
[Gaudin 83, Alcaraz et. al. 87,...]

- ABA [Sklyanin 88]: models on the segment
  - Models with U(1) symmetry !!!
    Results: Eigenvalues, Eigenvectors, Correlation functions (Slavnov), thermodynamic limit,...
    Applications: D-brane, out of equilibrium physics, quantum quench, ...
  - Models without U(1) symmetry :

  ABA doesn't work !!!!

Problematic for models without U(1) symmetry:

- No referent state for the transfer matrix !!! $\boxed{t(u)|\Omega\rangle \neq \Lambda(u)|\Omega\rangle}$

- No decomposition of the spectrum problem !!! $\boxed{\mathcal{H} \neq \oplus_{i=1}^{N} \mathcal{W}_i}$

State of the art from 88

- Constraints on parameters : -AnBA [Nepomechie 03], [Nepomechie Ravanini 03],...
  -ABA [Cao et al 03], [Martin Galleas 04],...
  -CBA [Crampé et al 10],...

- Alternatives : - q-Onsager [Baseilhac, Koizumi 07], Functional approach [Galleas 08]
  - SoV [Sklyanin 84; Frahm et al. 08; Niccoli et al. 12],
  - ODBA [Cao et al. 13] $\longrightarrow$ New term for the eigenvalues and BE!!!

New ABA : the modified ABA (MABA)
for models without U(1) symmetry
[Belliard, Crampé 13], [Belliard 14], [Belliard Pimenta 15],...

QISM for XXZ chain on the segment [Sklyanin 88]

$$H_{XXZ} = \epsilon\, \sigma_1^z + \kappa^-\, \sigma_1^- + \kappa^+\, \sigma_1^+ + \sum_{k=1}^{N-1} \Big(\sigma_k^x \otimes \sigma_{k+1}^x + \sigma_k^y \otimes \sigma_{k+1}^y + \Delta \sigma_k^z \otimes \sigma_{k+1}^z \Big) + \nu\, \sigma_N^z + \tau^-\, \sigma_N^- + \tau^+\, \sigma_N^+,$$

R matrix and K matrices (multiplicative formulation)

$$R_{ab}(u) = \begin{pmatrix} \frac{qu-q^{-1}u^{-1}}{q-q^{-1}} & 0 & 0 & 0 \\ 0 & \frac{u-u^{-1}}{q-q^{-1}} & 1 & 0 \\ 0 & 1 & \frac{u-u^{-1}}{q-q^{-1}} & 0 \\ 0 & 0 & 0 & \frac{qu-q^{-1}u^{-1}}{q-q^{-1}} \end{pmatrix}$$

$$K^-(u) = \begin{pmatrix} \nu_- u + \nu_+ u^{-1} & \tau\,(u^2 - u^{-2}) \\ \tilde\tau\,(u^2 - u^{-2}) & \nu_- u^{-1} + \nu_+ u \end{pmatrix}$$

$$K^+(u) = \begin{pmatrix} \epsilon_+ qu + \epsilon_- q^{-1}u^{-1} & \tilde\kappa(q^2u^2 - q^{-2}u^2) \\ \kappa\,(q^2u^2 - q^{-2}u^2) & \epsilon_+ q^{-1}u^{-1} + \epsilon_- qu \end{pmatrix}$$

XXZ model is not SU(2) invariant !

-constraints to have lower/upper triangular form $\tilde\kappa = \tilde\tau = 0$

-local gauge transformation [Baxter 73], [Cao et al. 03],...

Transfer matrix

$$t(u) = tr_a(K_a^+(u) K_a(u)) = \phi(u) k^+(u) \mathscr{A}(u) + k^+(q^{-1}u^{-1}) \mathscr{D}(u) + c(qu)\Big(\kappa\, \mathscr{B}(u) + \tilde\kappa\, \mathscr{C}(u)\Big)$$

with $K_a(u) = R_{a1}(u) \ldots R_{aN}(u) K_a^-(u) R_{aN}(u) \ldots R_{a1}(u) = \begin{pmatrix} \mathscr{A}(u) & \mathscr{B}(u) \\ \mathscr{C}(u) & \mathscr{D}(u) + \frac{1}{b(qu^2)} \mathscr{A}(u) \end{pmatrix}_a$

MABA : XXZ lower/upper boundaries

Transfer matrix

$$t_{lo/up}(u) = \phi(u)k^+(u)\mathscr{A}(u) + k^+(q^{-1}u^{-1})\mathscr{D}(u) + c(qu)\kappa\,\mathscr{B}(u)$$

$\tilde{\kappa} = \tilde{\tau} = 0$

Highest weight vector:

$$\mathscr{A}(u)|\Omega\rangle = \Lambda_1(u)|\Omega\rangle, \quad \mathscr{D}(u)|\Omega\rangle = \Lambda_2(u)|\Omega\rangle, \quad \mathscr{C}(u)|\Omega\rangle = 0$$

Bethe vector:

$$\Phi^N_{lo/up}(\bar{u}) = \mathscr{B}(u_1)\ldots\mathscr{B}(u_N)|\Omega\rangle$$

Off shell action of the transfer matrix :We use similar commutation relations than the XXX case

$$t_{lo/up}(u)\Phi^N_{lo/up}(\bar{u}) = \kappa\, c(qu)\mathscr{B}(u)\Phi^N_{lo/up}(\bar{u}) + \Lambda^N_d(u,\bar{u})\Phi^N_{lo/up}(\bar{u}) + \sum_{i=1}^{N} F(u,u_i)E^N_d(u_i,\bar{u}_i)\Phi^N_{lo/up}(\{u,\bar{u}_i\})$$

Off shell action of the creation operator for right upper triangular boundary

$$\kappa\, c(qu)\mathscr{B}(u)\Phi^N_{lo/up}(\bar{u}) = \Lambda^N_g(u,\bar{u})\Phi^N_{lo/up}(\bar{u}) + \sum_{i=1}^{N} F(u,u_i)E^N_g(u_i,\bar{u}_i)\Phi^N_{lo/up}(\{u,\bar{u}_i\})$$

$$\Lambda^N_g(u,\bar{u}) = -\tau\,\kappa\, c(u)c(q^{-1}u^{-1})\Lambda(u)\Lambda(q^{-1}u^{-1})m(u,\bar{u})$$

$$E^N_g(u_i,\bar{u}_i) = \tau\,\kappa\,\frac{c(u_i)c(q^{-1}u_i^{-1})}{b(qu_i^2)}\Lambda(u_i)\Lambda(q^{-1}u_i^{-1})m(u_i,\bar{u}_i)$$

MABA : XXZ with generic boundaries I

New parametrisation

$$\nu_+ = i\tilde{\tau}\tau\big(\mu/\tilde{\mu} + \tilde{\mu}/\mu\big) \quad \nu_- = i\tilde{\tau}\tau\big(\mu\tilde{\mu} + 1/(\mu\tilde{\mu})\big)$$
$$\epsilon_- = i\tilde{\kappa}\kappa\big(\xi/\tilde{\xi} + \tilde{\xi}/\xi\big) \quad \epsilon_+ = i\tilde{\kappa}\kappa\big(\xi\tilde{\xi} + 1/(\tilde{\xi}\xi)\big)$$

Gauge vectors [Cao et al. 03]

$$X(u,m) = \begin{pmatrix} \alpha q^{-m} u^{-1} \\ 1 \end{pmatrix}, \quad Y(u,m) = \begin{pmatrix} \beta q^{m} u^{-1} \\ 1 \end{pmatrix}$$

$$\tilde{X}(u,m) = \frac{qu}{\gamma_{m-1}} \begin{pmatrix} -1, & \alpha q^{-m} u^{-1} \end{pmatrix}, \quad \tilde{Y}(u,m) = \frac{qu}{\gamma_{m+1}} \begin{pmatrix} 1, & -\beta q^{m} u^{-1} \end{pmatrix}.$$

New «dynanical» operators [Cao et al. 03]

$$\mathscr{A}(u,m) = \tilde{Y}(u,m-2)K(u)X(u^{-1},m), \quad \hat{\mathscr{D}}(u,m) = \tilde{X}(u,m+2)K(u)Y(u^{-1},m),$$
$$\mathscr{C}(u,m) = \tilde{X}(u,m)K(u)X(u^{-1},m), \quad \mathscr{B}(u,m) = \tilde{Y}(u,m)K(u)Y(u^{-1},m)$$

$$\mathscr{D}(u,m) = q\Big(\frac{\gamma_{m+1}}{\gamma_m}\hat{\mathscr{D}}(u,m) - \frac{\gamma(u^{-2},m+1)}{b(qu^2)\gamma_m}\mathscr{A}(u,m)\Big).$$

MABA : XXZ with generic boundaries II

«Dynamic» transfer matrix:

$$t_d(u,m) = \tilde{a}(u,m)\mathscr{A}(u,m) + \tilde{d}(u,m)\mathscr{D}(u,m),$$

$$t(u) = t_d(u,m) + u^{-1}c(qu)\Big(\zeta_m \mathscr{B}(u,m) - \tilde{\zeta}_m \mathscr{C}(u,m)\Big),$$

$$\zeta_m = \frac{\kappa^2}{\gamma_m}\left(\alpha q^{-m-1} + i\frac{\tilde{\kappa}\xi}{\kappa\tilde{\xi}}\right)\left(\alpha q^{-m-1} + i\frac{\tilde{\kappa}\tilde{\xi}}{\kappa\xi}\right), \quad \tilde{\zeta}_m = \frac{\kappa^2}{\gamma_m}\left(\beta q^{m-1} + i\frac{\tilde{\kappa}\xi}{\kappa\tilde{\xi}}\right)\left(\beta q^{m-1} + i\frac{\tilde{\kappa}\tilde{\xi}}{\kappa\xi}\right),$$

Highest weight vector:

$$\alpha \Rightarrow \alpha_{dr} = iq^{m_0+N}\frac{\tau\mu}{\tilde{\tau}\tilde{\mu}}, \quad \beta \Rightarrow \beta_{dr} = iq^{-m_0-N}\frac{\tau\tilde{\mu}}{\tilde{\tau}\mu},$$

$$|\Omega^N_{m_0}\rangle_{dr} = \otimes_{i=1}^N |X(v_i, m_0+i)\rangle_{dr}.$$

$$\begin{aligned}
\mathscr{A}_{dr}(u,m_0)|\Omega^N_{m_0}\rangle_{dr} &= u\,\tilde{k}^-(u)\Lambda(u)|\Omega^N_{m_0}\rangle_{dr},\\
\mathscr{D}_{dr}(u,m_0)|\Omega^N_{m_0}\rangle_{dr} &= u\,\phi(q^{-1}u^{-1})\tilde{k}^-(q^{-1}u^{-1})\Lambda(q^{-1}u^{-1})|\Omega^N_{m_0}\rangle_{dr},\\
\mathscr{C}_{dr}(u,m_0)|\Omega^N_{m_0}\rangle_{dr} &= 0,
\end{aligned}$$

MABA : XXZ with generic boundaries III

Modified transfer matrix

$$\alpha \Rightarrow \alpha_{dl}(M) = -iq^{1+m_0+2M}\frac{\xi\tilde{\kappa}}{\tilde{\xi}\kappa}, \qquad \beta \Rightarrow \beta_{dl}(M) = -iq^{1-m_0-2M}\frac{\tilde{\xi}\tilde{\kappa}}{\xi\kappa},$$

$$t(u) = t_{dl}(u, m_0 + 2M) = \tilde{a}(u)\mathscr{A}_{dl}(u, m_0 + 2M) + \tilde{d}(u)\mathscr{D}_{dl}(u, m_0 + 2M).$$

Modified action on Highest weight vector:

$$\begin{aligned}
\mathscr{A}_{dl}(u, m_0)|\Omega^N_{m_0}\rangle_{dr} &= u\,\tilde{k}^-(u)\Lambda(u)|\Omega^N_{m_0}\rangle_{dr} + \eta_{m_0}\mathscr{B}_{dl}(u, m_0 - 2)|\Omega^N_{m_0}\rangle_{dr}, \\
\mathscr{D}_{dl}(u, m_0)|\Omega^N_{m_0}\rangle_{dr} &= u\,\phi(q^{-1}u^{-1})\tilde{k}^-(q^{-1}u^{-1})\Lambda(q^{-1}u^{-1})|\Omega^N_{m_0}\rangle_{dr} \\
&\quad -\phi(u)\eta_{m_0}\mathscr{B}_{dl}(u, m_0 - 2)|\Omega^N_{m_0}\rangle_{dr},
\end{aligned}$$

Modified Bethe vectors

$$B_{dl}(\bar{u}, m, M) = \mathscr{B}_{dl}(u_1, m + 2(M-1)) \ldots \mathscr{B}_{dl}(u_M, m),$$

$$|\Psi^M_{m_0}(\bar{u})\rangle = B_{dl}(\bar{u}, m_0, M)|\Omega^N_{m_0}\rangle_{dr}.$$

Off-shell action of the transfer matrix for arbitrary M

$$t_{dl}(u, m_0 + 2M)|\Psi^M_{m_0}(\bar{u})\rangle = \Lambda^M_d(u, \bar{u})|\Psi^M_{m_0}(\bar{u})\rangle + \sum_{i=1}^{M} \tilde{F}(u, u_i) E^M_d(u_i, \bar{u}_i)|\Psi^M_{m_0}(\{u, \bar{u}_i\})\rangle$$
$$+ \hat{\eta}_{m_0} u^{-1} c(qu) B_{dl}(\{u, \bar{u}\}, m_0 - 2, M+1)|\Omega^N_{m_0}\rangle_{dr},$$

Off-shell action of the modified operator for M=N

$$\hat{\eta}_{m_0} u^{-1} c(qu) B_{dl}(\{u, \bar{u}\}, m_0 - 2, N+1)|\Omega^N_{m_0}\rangle_{dr} = \Lambda^N_g(u, \bar{u})|\Psi^N_{m_0}(\bar{u})\rangle$$
$$+ \sum_{i=1}^{N} \tilde{F}(u, u_i) E^N_g(u_i, \bar{u}_i)|\Psi^N_{m_0}(\{u, \bar{u}_i\})\rangle$$

Off-shell action of the transfer matrix

$$t(u)|\Psi^N_{m_0}(\bar{u})\rangle = \Lambda^N(u,\bar{u})|\Psi^N_{m_0}(\bar{u})\rangle + \sum_{i=1}^{N} \tilde{F}(u,u_i)E^N(u_i,\bar{u}_i)|\Psi^N_{m_0}(\{u,\bar{u}_i\})\rangle$$

$$\Lambda^N(u,\bar{u}) = \Lambda^N_d(u,\bar{u}) + \Lambda^N_g(u,\bar{u}), \quad E^N(u_i,\bar{u}_i) = E^N_d(u_i,\bar{u}_i) + E^N_g(u_i,\bar{u}_i),$$

with

$$\Lambda^N_d(u,\bar{u}) = \psi(u)f(u,\bar{u}) + \psi(q^{-1}u^{-1})h(u,\bar{u}), \quad \psi(u) = \phi(u)\tilde{k}^+(u)\tilde{k}^-(u)\Lambda(u)$$
$$E^N_d(u_i,\bar{u}_i) = \phi(q^{-1}u_i^{-1})\psi(u_i)f(u_i,\bar{u}_i) - \phi(u_i)\psi(q^{-1}u_i^{-1})h(u_i,\bar{u}_i),$$

and

$$\Lambda^N_g(u,\bar{u}) = -\kappa\tilde{\kappa}\tau\tilde{\tau}\left(\frac{\kappa\tau}{\tilde{\kappa}\tilde{\tau}} + \frac{\tilde{\kappa}\tilde{\tau}}{\kappa\tau} + \frac{\xi\tilde{\mu}}{\tilde{\xi}\mu}q^{N+1} + \frac{\tilde{\xi}\mu}{\xi\tilde{\mu}}q^{-N-1}\right)c(u)c(q^{-1}u^{-1})\Lambda(u)\Lambda(q^{-1}u^{-1})G(u,\bar{u}),$$

$$E^N_g(u_i,\bar{u}_i) = \kappa\tilde{\kappa}\tau\tilde{\tau}\left(\frac{\kappa\tau}{\tilde{\kappa}\tilde{\tau}} + \frac{\tilde{\kappa}\tilde{\tau}}{\kappa\tau} + \frac{\xi\tilde{\mu}}{\tilde{\xi}\mu}q^{N+1} + \frac{\tilde{\xi}\mu}{\xi\tilde{\mu}}q^{-N-1}\right)\frac{c(u_i)c(q^{-1}u_i^{-1})}{b(qu_i^2)}\Lambda(u_i)\Lambda(q^{-1}u_i^{-1})G(u_i,\bar{u}_i)$$

Explicit rewriting for calculation of Slavnov's formula

« 2021 notations »

$$t(u)|\Psi(\bar u)\rangle = \Lambda(u|\bar u)|\Psi(\bar u)\rangle + \sum_{i=1}^{N} \frac{F(u)}{F(u_i)} \frac{\mathcal{Y}(u_i|\bar u)}{Q(u_i,\{u,\bar u_i\})}|\Psi(\{u,\bar u_i\})\rangle,$$

$$\langle\Psi(\bar v)|t(u) = \langle\Psi(\bar v)|\Lambda(u|\bar v) + \sum_{i=1}^{N} \frac{F(u)}{F(v_i)} \frac{\mathcal{Y}(v_i|\bar v)}{Q(v_i,\{u,\bar v_i\})}\langle\Psi(\{u,\bar v_i\})|\,.$$

$$F(u) = u^{-1}\frac{q^2u^2 - q^{-2}u^{-2}}{q-q^{-1}}\,, \quad Q(u,v) = U(u) - U(v)\,, \quad U(u) = \frac{qu^2 + q^{-1}u^{-2}}{(q-q^{-1})^2}\,,$$

$$\Lambda(u|\bar u) = \phi(u)\frac{Q(q^{-1}u,\bar u)}{Q(u,\bar u)} + \phi(q^{-1}u^{-1})\frac{Q(qu,\bar u)}{Q(u,\bar u)} + \frac{H(u)}{Q(u,\bar u)},$$

$$\mathcal{Y}(u|\bar u) = Q(u,\bar u)\Lambda(u|\bar u).$$

$$\phi(u) = -\kappa\tilde\kappa\tau\tilde\tau(\tilde\xi u + \tilde\xi^{-1}u^{-1})(\xi^{-1}u + \xi u^{-1})(\mu u + \mu^{-1}u^{-1})(\tilde\mu^{-1}u + \tilde\mu u^{-1})\frac{q^2u^2 - q^{-2}u^{-2}}{qu^2 - q^{-1}u^{-2}}V(u),$$

$$H(u) = \left((\kappa\tau)^2 + (\tilde\kappa\tilde\tau)^2 + \kappa\tilde\kappa\tau\tilde\tau\left(\frac{\xi\tilde\mu}{\tilde\xi\mu}q^{N+1} + \frac{\tilde\xi\mu}{\xi\tilde\mu}q^{-N-1}\right)\right)(u^2 - u^{-2})(q^2u^2 - q^{-2}u^{-2})V(u)V(q^{-1}u^{-1}),$$

$$V(u) = \prod_{i=1}^{N} Q(q^{1/2}u, q^{-1/2}x_i).$$

Linear system for Slavnov' formula

$LX = 0,$ $X_k = \langle \Psi(\bar{v}) | \Psi(\bar{u}_k) \rangle,$

N+1 variables

$$L_{kj} = \delta_{kj} \Lambda(u_j|\bar{v}) - \frac{F(u_k)}{F(u_j)} \frac{\mathcal{Y}(u_j|\bar{u}_k)}{Q(u_j, \bar{u}_j)}.$$

Left Bethe vector On-shell !

« Cauchy » Transformation

$\Omega = WL$

$$W_{ik} = \frac{Q(u_k, \bar{w}_i)}{F(u_k) Q(u_k, \bar{u}_k)},$$

$$\det_{N+1}(W) = \frac{1}{F(\bar{u})} \frac{\Delta(\bar{w})}{\Delta(\bar{u})},$$

$$\Omega_{ij} = \frac{1}{F(u_j) Q(u_j, \bar{u}_j)} \Big(\Lambda(u_j|\bar{v}) Q(u_j, \bar{w}_i) - \mathcal{Y}(u_j|\bar{w}_i) \Big).$$

For $\bar{w}_{N+1} = \bar{v}$ the last row vanish so $\det_{N+1}(L) = 0.$

Simple calculation allows to rewrite the entries into the form (N*N matrix), we get ride of the last free variable (w) by multiplication with a diagonal matrix!

$$\tilde{\Omega}_{ij} = \frac{M_{ij}}{\partial U(v_i) F(u_j) Q(u_j, \bar{u}_j)}, \quad M_{ij} = Q(u_j, \bar{v}) \partial_{v_i} \Lambda(u_j|\bar{v}),$$

Cramer's rule for homogenous system

$$X_l = \tilde{G}(\bar{u}|\bar{v})(-1)^{N+1+l} \det_N(\tilde{\Omega}_l)\,, \quad \Rightarrow \quad \frac{F(\bar{u}_l)\Delta(\bar{u}_l)X_l}{\det_N(M_l)} = \frac{F(\bar{u}_j)\Delta(\bar{u}_j)X_j}{\det_N(M_j)}$$

Intermediate result
Jacobian form !

$$X_l = G(\bar{v}) \frac{\det_N(M_l)}{\partial U(\bar{v}) F(\bar{u}_l) \Delta(\bar{u}_l)},$$

Remain to fix the free function of Bethe roots !

Fixing the free function →

R.H.S asymptotic

$u_i = ua^i$

$$\frac{\det_N(M_{N+1})}{\partial U(\bar v)\Delta(\bar u)F(\bar u)} = \frac{q^{N^2}\left(u^N a^{\frac{N(N+1)}{2}}\right)^{2N+3}\Delta'(\bar v)}{(q-q^{-1})^{2N^2-N}}\nu_N + \cdots,$$

↓

L.H.S asymptotic

$u_i = ua^i$

$$\langle\Psi(\bar v)|\Psi(\bar u)\rangle = \frac{(\bar u)^{2N+3}\langle\Psi(\bar v)|N\rangle}{(q-q^{-1})^{2N^2}}\left(i\frac{\kappa\tilde\xi}{\tilde\kappa\xi}\tau^2\right)^N \frac{\left(-\frac{\tilde\kappa\tilde\tau\tilde\xi\tilde\mu}{\kappa\tau\xi\mu}q^{1-3N};q^2\right)_N\left(-\frac{\tilde\kappa\tilde\tau\mu\tilde\xi}{\kappa\tau\xi\tilde\mu}q^{1-N};q^2\right)_N}{\left(\frac{\tilde\xi^2}{\xi^2}q^{2-4N};q^4\right)_N} + \cdots,$$

Final result

$$\langle\Psi(\bar v)|\Psi(\bar u)\rangle = \eta\langle\Psi(\bar v)|N\rangle\frac{\det_N\left(Q(u_j,\bar v)\partial_{v_i}\Lambda(u_j|\bar v)\right)}{\partial U(\bar v)\Delta'(\bar v)F(\bar u)\Delta(\bar u)},$$

$$\eta = \left(\frac{i\tilde\xi}{q^N(q-q^{-1})\tilde\kappa\kappa\xi}\right)^N \frac{\left(-\frac{\tilde\kappa\tilde\tau\tilde\mu\tilde\xi}{\kappa\tau\mu\xi}q^{1-3N};q^2\right)_N}{\left(\frac{\tilde\xi^2}{\xi^2}q^{2-4N};q^4\right)_N\left(-\frac{\tilde\kappa\tilde\tau\tilde\mu\xi}{\kappa\tau\mu\tilde\xi}q^{1-N};q^2\right)_N}.$$

Match with 2015 conjecture in the XXX limit !

Discussion

Results for the eigenvalues and the eigenvectors of

- XXX: MABA for general/general boundaries [Belliard Crampé 13]
- XXZ: MABA for generic boundaries [Avan Belliard Pimenta 15]
- XXX: MABA Slavnov's formula [Belliard Pimenta15]
- XXZ: MABA Slavnov's formula [Belliard Pimenta Slavnov 21]

Projects:

- Thermodynamic limit ?
- Vertex operator approach ?
- MABA for SL(3) cases ?