Dynamics of a voting process based on the bounded confidence principle

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We study a nonlinear dynamical system modeling the following problem of opinion dynamics. A society consisting of N agents has to choose between two options, 1 and -1. Let $v_k^n \in [-1, 1]$ be the opinion of the agent with index $k \in \{1, \dots, N\}$ at time moment $n = 0, 1, \dots$, and let

$$V^n = (v_1^n, \dots, v_N^n) \tag{1}$$

be the array of opinions at time moment n.

Following the bounded confidence principle (in the process of electoral decision, voters are influenced by individuals sharing an opinion close to their own) formulated by Hegselmann and Krause [1], the dynamical system is generated by a mapping Φ defined in two steps.

Fix numbers $\varepsilon \in (0, 1)$ (confidence level) and $h \in (0, 1)$ and consider the sets

$$J(v_k^n) = \big\{l \in \{1, \dots, N\}\colon\ |v_l^n - v_k^n| \le \varepsilon\big\}, \qquad k \in \{1, \dots, N\}, \quad n \ge 0.$$

The set $J(v_k^n)$ is the set of indices of agents whose opinions influence agent k when he modifies his own opinion at time moment n.

Denote by $I(v_k^n)$ the cardinality of the (nonempty) set $J(v_k^n)$.

At the first step of definition of Φ, we fix an array V^n of the form (1) and define an auxiliary array

$$W(V^n) = (w_1(V^n), \dots, w_N(V^n)),$$

where

$$w_k(V^n) = v_k^n + \frac{h}{I(v_k^n)} \sum_{l \in J(v_k^n)} v_l^n, \qquad k = 1, \dots, N.$$

It is possible that, for some k, $w_k(V^n) \notin [-1, 1]$, so that a kind of "norming" must be applied as the second step of definition of Φ. We consider two "normings" which give us dynamical systems with substantially different dynamics.

In [2], a "cutting" procedure was suggested; the new opinion value v_k^{n+1} is obtained by replacing the value $w_k(V^n)$ less than -1 by $v_k^{n+1} = -1$, and the value $w_k(V^n)$ greater than 1 is replaced by $v_k^{n+1} = 1$. Finally, $v_k^{n+1} = w_k(V^n)$ if $|w_k(V^n)| \le 1$.

The dynamics of the appearing dynamical system is completely described in [2]. It is shown that if $\varepsilon \le 1/2$, then any trajectory tends to a fixed point as time goes to infinity. All possible fixed points are characterized. It is shown that any fixed point $P = (p_1, \dots, p_N)$ with $|p_k| = 1$, $k \in \{1, \dots, N\}$, is attracting, while all the remaining fixed points are Lyapunov unstable. Modifications of the model studied in [2] are considered in the recent papers [3] (where the average of values v_l^n is replaced by the average of values $i(v_l^n)$ for a wide class of influence functions i) and [4] (where the finite set of agents is replaced by a set indexed by the continuum $[0, 1]$).

We note an important property of the system: the results of the voting process (-1 wins or 1 wins) can be different for the same initial array of opinions V^0 for different values of the confidence level ε.

In [5], we study a model similar to that considered in [2] but with a different norming. Instead of "cutting" the values obtained at the first step, for $W(V^n) \neq 0$, we divide the values w_k^n by the maximum absolute value of w_l^n to get v_k^{n+1}.

The main results of [5] are as follows: we show that if

$$\varepsilon(N-1) < 1, \tag{2}$$

then any trajectory tends to a fixed point as time goes to infinity; all fixed points in this case are described; it is shown that both fixed points $(-1, \dots, -1)$ and $(1, \dots, 1)$ are attracting while all the remaining fixed points are Lyapunov stable but not attracting; an example of a system is given for which condition (2) is not satisfied and there exists an unstable fixed point.

References

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