Parametric smooth variational principles and probabilistic dynamic optimization problems

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Let $\mathbb{X}$ be a Banach space and $(\Omega, \mathcal{F}, \mu)$ a probability space with sample space Ω, σ-algebra of events $\mathcal{F}$ and probability measure μ.

In this talk we discuss a generalized differentiation [1, 3] of the probabilistic functionals

$$F(x) := \mu\{\omega \in \Omega\colon\ f(x, \omega) \leq 0\} \tag{1}$$

for smooth Banach spaces $\mathbb{X}$ and apply them to the study of dynamic optimization problems with probabilistic functionals.

Optimization problems with objective and constraint functionals in the form (1) were studied intensively during last decades for the case of finite-dimensional $\mathbb{X}$ in the framework of stochastic programming [2, 4, 5].

Here we develop an approach for finding sub(super)-gradients of the functional F given by (1) in a more general setting than [5, 6] by using an integral representation of F,

$$F(x) = \int_\Omega \theta(f(x, \omega))\, \mu(d\omega), \tag{2}$$

where $\theta\colon \mathbb{R} \to \mathbb{R}$ is a Heavyside-like function: $\theta(t) = 1$ for $t \leq 0$ and $\theta(t) = 0$ for $t > 0$.

In view of (2), to evaluate the subgradients of the probabilistic functional F (1), we prove the following result for a representation of subgradients of the parametric integral

$$H(x) := \int_\Omega h(x, \omega)\, \mu(d\omega), \tag{3}$$

which is more general than the existing ones [5, 6].

Theorem 1. *Let $\mathbb{X}$ be a smooth Banach space, a function $h\colon \mathbb{X} \times \Omega \to \mathbb{R}$ be lower continuous in x for μ-a.e. ω and Borel measurable in ω for all x, and there exist an integrable function such that*

$$|h(x, \omega)| < \ell(\omega).$$

Then for any subgradient $\zeta \in \partial_F H(x)$ for any $\varepsilon > 0$ there exist $x(\omega)$ and $\xi(\omega) \in \partial_F h(x(\omega), \omega)$ such that

$$\|x(\omega) - x\| < \varepsilon, \qquad \|\zeta - \int_\Omega \xi(\omega)\mu(d\omega)\| < \varepsilon.$$

The proof of this important result is based on the following new parametric smooth variational principle.

Theorem 2. *Let $\mathbb{X}$ be a separable smooth Banach space and a function $h\colon \mathbb{X} \times \Omega \to \mathbb{R}$ satisfy the assumptions of Theorem 1. Then there exists a constant a such* *that for any $\varepsilon > 0$ and any $x_0(\omega)$ satisfying*

$$h(x_0(\omega), \omega) < \inf_{x \in \mathbb{X}} h(x, \omega) + a\varepsilon^2$$

there exists a function $g\colon \mathbb{X} \times \Omega \to \mathbb{R}$ which is smooth in x and Borel measurable in ω, and a Borel measurable $x(\omega)$ such that the function $x \to h(x, \omega) + g(x, \omega)$ attains a unique minimum at $x(\omega)$ and for all ω

$$\|g(\cdot, \omega)\|_{\mathbb{X}} \le \varepsilon, \qquad \|g'(\cdot, \omega)\|_{\mathbb{X}} \le \varepsilon.$$

Finally, we consider the following dynamic optimization problem.

For a function $L\colon \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ and a constant $\alpha \in (0, 1)$ consider the problem

$$\text{minimize} \quad c$$

subject to the probabilistic constraint

$$\mu\left\{\omega \in \Omega\colon \int_0^1 L(x(t), \dot{x}(t), \omega)\, dt \le c\right\} \ge \alpha \tag{4}$$

over the set of all absolutely continuous functions $x\colon [0.1] \to \mathbb{R}^n$ satisfying the initial condition $x(0) = x_0$ and all c satisfying the probabilistic constraint (4).

We derive optimality conditions for this dynamic optimization problem with probabilistic constraint.

References

1. *Clarke F.H., Ledyaev Yu.S., Stern R.J., Wolenski P.R.* Nonsmooth analysis and control theory. Springer, 1998.
2. *Prékopa A.* Stochastic programming. Kluwer, 1995.
3. *Ledyaev Yu.S., Treiman J.* Sub- and supergradients of envelopes, semicontinuous closures, and limits of sequences of functions // Russ. Math. Surv. 2012. V. 67, No. 2. P. 345–373.
4. *Kibzun A., Uryasev S.* Differentiability of probability function // Stoch. Anal. Appl. 1998. V. 16, No. 6. P. 1101–1128.
5. *Van Ackooij W., Henrion R., Pérez-Aros P.* Generalized gradients for probabilistic/robust (probust) constraints // Optimization. 2020. V. 69, No. 7–8. P. 1451–1479.
6. *Correa R., Hantoute A., Pérez-Aros P.* Subdifferential calculus rules for possibly nonconvex integral functions // SIAM J. Control Optim. 2020. V. 58, No. 1. P. 462–484.