Hamiltonian homoclinic dynamics*

Lev Lerman

HSE University, Nizhny Novgorod, Russia

llerman@hse.ru

The goal of this talk is to present some results in the study of an orbit structure of a Hamiltonian system near homoclinic orbits to specific orbits of different types: equilibria, periodic orbits and quasi-periodic orbits (whose closures, as is known, form invariant tori). This is what I call *Hamiltonian homoclinic dynamics*. The results to be discussed have been obtained at different times; nevertheless, as far as I am aware, they are not very well known, so I hope this will be useful for the audience.

I recall the well-known fact that the discovery by H. Poincaré of homoclinic orbits gave rise to studying chaotic phenomena or complicated dynamics, though Poincaré himself did not explore this topic. The study of the homoclinic dynamics near a transverse homoclinic orbit to a saddle periodic orbit was started by Birkhoff for a symplectic diffeomorphism. Then after a long period, the first description of invariant subsets near a transverse homoclinic orbit of a saddle for any smooth two-dimensional diffeomorphism was done by Smale, and finally a complete description of a whole invariant set was performed by Shilnikov for an arbitrary differential system near a transversal homoclinic orbit to a saddle periodic orbit. Shilnikov also discovered a new phenomenon of chaotic orbit behavior near a homoclinic orbit of a saddle-focus with a positive saddle value in $\mathbb{R}^3$, the topic that gave origin to the so-called spiral chaotic dynamics.

All these results also occur in Hamiltonian systems with their own specifics, because of the existence of an integral, the Hamiltonian. This feature sometimes prevents the direct application of the results found for dissipative systems in the Hamiltonian dynamics. But it also helps by allowing the reduction to smaller dimensions.

Homoclinic orbits of equilibria. The first results here are due to Devaney, who, for the case of a homoclinic orbit of a saddle-focus in a two-DOF Hamiltonian system, selected an invariant hyperbolic subset described by means of a Bernoulli scheme with two symbols. A complete description of the whole invariant subset near a transversal homoclinic orbit of a saddle-focus is done on the singular level of the Hamiltonian (containing the saddle-focus) via symbolic dynamics with infinite number of symbols (Belyakov–Shilnikov, Lerman), but it is impossible to perform a complete study near a neighborhood of the homoclinic orbit, since varying the level of the Hamiltonian leads to an infinite set of bifurcations related to the formation of hyperbolic sets whose description involves more and more symbols (Lerman). The case of homoclinic dynamics near homoclinic orbits to a saddle is more tricky. In fact, the existence of a transversal homoclinic orbit of a saddle does not necessarily lead to complicated dynamics: this is possible in an integrable system (Devaney, Lerman–Umanskiy). But Shilnikov and Turaev indicated cases where the presence of transversal homoclinic orbits of a saddle does lead to complicated homoclinic dynamics.

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The case of dynamics near a homoclinic orbit to a nonhyperbolic equilibrium is much more interesting. The existence of such an orbit is not structurally stable within the class of Hamiltonian systems. Consider the case, initially indicated by Conley, of a homoclinic orbit of a saddle-center, the equilibrium p with a pair of nonzero real eigenvalues $\pm\lambda$ and a pair of imaginary ones $\pm i\omega$. Here the stable and unstable manifolds are one-dimensional (smooth curves); they generically do not intersect in the three-dimensional level $H = H(p)$. Thus, the formation of homoclinic orbits is generic in two-parametric families of Hamiltonians or in one-parameter families of reversible Hamiltonians. Here, to study the homoclinic dynamics, one needed to develop a scattering theory for the linear non-autonomous system being the linearization at the homoclinic orbit (Koltsova–Lerman). This allowed to prove

Theorem 1 (Lerman, Koltsova–Lerman)**.** *If the linearized system at the homoclinic orbit is non-degenerate, then all Lyapunov periodic orbits on the center manifold $W^c(p)$ possess four transverse homoclinic orbits on the related level of the Hamiltonian, which leads to complicated homoclinic dynamics.*

This result was extended in several directions. First, it was shown that on the singular level of the Hamiltonian there are countable families of periodic orbits of different types (hyperbolic and elliptic) accumulating to the homoclinic orbit, and a semi-global invariant was found responsible for these transitions (Koltsova–Lerman). Also results on four transverse homoclinic orbits were proved for the case of a Hamiltonian system with n degrees of freedom near a homoclinic orbit of a 1-elliptic equilibrium (with the only pair of simple pure imaginary eigenvalues) [1].

Homoclinic orbits to periodic orbits. The previous results were extended [2] to the case of homoclinic dynamics for a three-DOF system near a homoclinic orbit doubly asymptotic to a 1-elliptic periodic orbit (its multipliers are two eigenvalues on the unit circle and two outside the unit circle). The problem is reduced to the study of a four-dimensional symplectic diffeomorphism with a 1-elliptic fixed point possessing a homoclinic orbit to this point. Here the scattering theory was generalized to the case of a sequence of linear symplectic transformations obtained by the linearization of the symplectic diffeomorphism at the homoclinic orbit.

One more extension was made for the case of a homoclinic orbit to an equilibrium of an elliptic–hyperbolic type (with two pairs of pure imaginary eigenvalues and a pair of nonzero reals) in a Hamiltonian system being nearly integrable (Koltsova–Lerman–Delshams–Gutierrez). Here transversal one-round homoclinic orbits to invariant non-resonant 2-tori on the center manifold of the equilibrium were found (up to 16 of them).

Homoclinic orbits to an invariant torus. Hamiltonian homoclinic dynamics arises naturally in the study of a symplectic diffeomorphism f in a neighborhood of a homoclinic orbit to a smooth invariant curve γ being hyperbolic. For instance, one encounters this situation when studying the homoclinic dynamics of a symplectic four-dimensional diffeomorphism near a homoclinic orbit to a 1-elliptic fixed point p. There the smooth center two-dimensional manifold $W^c(p)$ contains many invariant curves with irrational rotation number.

The notion of a hyperbolic invariant curve for a symplectic 4-diffeomorphism can be defined in invariant terms; but here, for simplicity, we use the following setup. Let γ be a smooth invariant curve of a smooth symplectic diffeomorphism and Σ be a smooth symplectic two-dimensional cylinder containing γ. We assume the restriction of f on the cylinder is a twist map near γ. Also this cylinder is supposed to be a hyperbolic manifold of f; thus, there are two invariant three-dimensional submanifolds $W^{\mathrm{s}}(\Sigma)$ and $W^{\mathrm{u}}(\Sigma)$, which are foliated into smooth curves $l^{\mathrm{s}}(m)$ and $l^{\mathrm{u}}(m)$, $m \in \Sigma$, forming smooth invariant foliations of each of them (Fenichel). The union of $l^{\mathrm{s}}(m)$, $m \in \gamma$, makes up the smooth stable manifold $W^{\mathrm{s}}(\gamma)$. Similarly, $W^{\mathrm{u}}(\gamma)$ is defined. These two submanifolds can be extended by f, providing global stable and unstable manifolds of γ. Suppose $W^{\mathrm{s}}(\gamma)$ intersects $W^{\mathrm{u}}(\gamma)$ at some point $q \ni \gamma$ and this intersection is transverse (a transversal homoclinic point q to γ generating a homoclinic orbit Γ). What is the homoclinic dynamics near Γ? This was investigated by many authors, starting with Easton (Treschev, Bolotin, Cresson, Turaev–Gelfreich, and others).

Because of transversality, the intersection $W^{\mathrm{s}}(\gamma) \cap W^{\mathrm{u}}(\gamma)$ is transverse and is therefore along a two-dimensional symplectic disk D. Both $W^{\mathrm{s}}(\gamma)$ and $W^{\mathrm{u}}(\gamma)$ intersect this disk transversely within the related $W^{\mathrm{s}}(\Sigma)$ and $W^{\mathrm{u}}(\Sigma)$ along two transversely intersecting smooth curves in D near the point q. In fact, near q there is a Cantorian set of smooth curves from the related stable (and unstable) manifolds of smooth invariant closed curves on Σ near γ. Our goal in this section will be to describe other orbits passing near this Cantorian set.

References

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