Mean field type control systems

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A mean field type control system is a mathematical model of a system of many interacting similar agents trying to achieve a common goal. The mean field approach implies the study in the limit case when the number of agents tends to infinity while the agents interact via some external media. This leads to the study of evolution of the probability measure describing the distribution of agents. It can be regarded as a phase variable for the mean field type control system. In the talk we will restrict our attention to the case when the dynamics of each agent is given by an ordinary differential equation

$$\frac{d}{dt}x(t) = f(t, x(t), m(t), u(t)). \tag{1}$$

Here t stands for the time, $x(t)$ describes the state of the agent at time t, $u(t)$ is the control of the agent, while $m(t)$ is a current distribution of agents.

Notice that the mean field type control theory finds applications in the control of systems of autonomous vehicles and drones. Recently, the mean field type control theory was used for the analysis of machine learning.

The talk will deal with the survey of modern approaches to the mean field type control theory. Primarily, we will consider only mathematical aspects of this field of science.

The first point we will discuss in the talk is the formal definition of open-loop strategies and the corresponding evolution of distribution of agents. It can be performed in three ways.

First, one can choose a probability space $(\Omega, \mathcal{F}, P)$ with non-atomic measure P and label the agents within the elements of Ω. Assuming that the motion of each agent obeys

$$\frac{d}{dt}x(t,\omega) = f(t, x(t,\omega), m(t), u(t,\omega)),$$

we claim that, for each t and every measurable $Y \subset \mathbb{R}^d$,

$$m(t, Y) = P\{\omega \in \Omega \colon\ x(t,\omega) \in Y\}.$$

This definition refers to the so called Lagrangian approach. In many cases it is convenient to choose Ω equal to the product of the space of initial agents' states and the space of controls.

Finally, let us mention the Eulerian approach, which implies that the decision maker chooses a strategy $u(t,x)$, while $m(t)$ satisfies in the sense of distribution the continuity equation

$$\frac{\partial}{\partial t}m(t) + \operatorname{div}\big(f(t, x, m(t), u(t, x(t)))m(t)\big) = 0.$$

Based on [5] and [7], we will discuss the equivalence between the aforementioned definitions.

The second point is the feedback formalization. In this case, we assume that the decision maker has information on the current distribution of agents and uses it to form a control. Additionally, we will adopt the Krasovskii–Subbotin approach which assumes stepwise constant controls. The results of [3] show that this feedback formalization is equivalent to the open-loop one.

One of the key questions within control theory is the optimal control problems and characterization of their solutions via dynamic programming. For simplicity, we will consider the case when all agents try to minimize the terminal payoff given by $\sigma(m(T))$, where T is a final time and $m(T)$ stands for the final distribution of agents. Certainly the dynamics is given by (1). We describe the dynamic programming for this mean field type optimal control problem. Additionally, we will discuss a link between the mean field type optimal control problem and the Hamilton–Jacobi equation in the space of probability measures that takes a form

$$\frac{\partial \varphi}{\partial t} + H(t, m, \nabla_m \varphi) = 0, \qquad \varphi(T, m) = \sigma(m).$$

Here φ is a value function, i.e., it assigns to an initial time and a probability corresponding to the initial distribution of agents the optimal outcome. Further, $\nabla_m \varphi$ stands for the derivative w.r.t. the measure variable. Recall [8] that one should understand the solution of the Hamilton–Jacobi equation in the minimax/viscosity sense and define it via tools of nonsmooth analysis such as directional derivatives or sub/superdifferentials. We will use the approaches of [2, 4, 6, 7] to introduce the nonsmooth analysis in the space of probability measures, define a minimax/viscosity solution to the Hamilton–Jacobi equation in the space of probabilities and provide the link between this concept and the solution of the mean field type optimal control problems.

The last part of the talk will deal with the approaches to the numerical scheme for the mean field type optimal control problem. The most promising way relies on the lattice approximations, which reduces the original control problem in the space of probability measures to the finite-dimensional control problem. We will give a short introduction to this technique and discuss the corresponding approximation rate.

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