# Generalized solutions of boundary value problems for general quasi-linear PDEs

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Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded domain with a smooth boundary $\partial\Omega$, $\mathcal{L} = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$ be some differential operation with smooth complex $j \times k$ matrix coefficients $a_\alpha(x)$, $\mathcal{L}^+$ be the formally adjoint differential operation. Let $L_0$ and $L_0^+$ be the minimal operators. M. Vishik introduced the following conditions:

$$\text{The operator } L_0\colon D(L_0) \to L_2^j(\Omega) \text{ has a continuous left inverse.} \tag{1}$$

$$\text{The operator } L_0^+\colon D(L_0^+) \to L_2^j(\Omega) \text{ has a continuous left inverse.} \tag{2}$$

He proved that these conditions are necessary and sufficient for the existence of a solvable expansion $L_B\colon D(L_B) \to L_2^j(\Omega)$ (i.e., $D(L_0) \subset D(L_B)$, $\exists L_B^{-1}\colon L_2^j(\Omega) \to D(L_B)$). L. Hörmander proved these conditions for any scalar operator with constant coefficients in a bounded domain; for the case of scalar operators

$$L(x,D) = P^0(D) + \sum_i C_i(x)P^i(D), \qquad C_i \in C^\infty(\overline{\Omega}), \quad P^i \in \mathbb{C}[\xi], \tag{3}$$

of the real principal type with order $P^i \le m-1$, $|\nabla P^0(\xi)| \ne 0$ if $\xi \ne 0$, and also for the case of operators of constant strength of the form (3) with $P^i \prec P^0$ and analytical $C^i$, conditions (1), (2) follow from the results of G. Gudmundsdottir.

We shall consider the equation

$$\mathcal{L}^+ A\mathcal{L}u = f \tag{4}$$

with some continuous (linear or nonlinear) operator $A\colon L_2^j(\Omega) \to L_2^j(\Omega)$.

The function $u \in D(L_0)$ satisfying the integral identity $\langle AL_0u, \mathcal{L}v\rangle = \langle f, v\rangle$ for every function $v \in (C_0^\infty(\Omega))^k$ will be called a generalized solution of the Dirichlet problem in the domain $\Omega$ for equation (4) with $f \in D'(L_0)$. This is equivalent to the equation $L_0'AL_0u = f$. The Dirichlet problem will be called well-posed if there exists a continuous inverse operator $M\colon D'(L_0) \to D(L_0)$ to the operator $L_0'AL_0$. Let $P\colon L_2^j(\Omega) \to \operatorname{Im} L_0$ be the orthoprojector.

**Statement.** *The generalized Dirichlet problem for equation* (4) *is well-posed if and only if condition* (1) *is fulfilled and the operator* $PA\colon \operatorname{Im} L_0 \to \operatorname{Im} L_0$ *is a homeomorphism.*

**Example.** Let $\mathcal{L} = \Box = \partial^2/\partial x_1 \partial x_2$ and $A$ be the Nemytsky operator given by means of a function $(Au)(x) = \varphi(x, |u|)\, u$, where the function $\varphi$ is bounded, satisfies the Carathéodory condition and $\varphi(x,t)t - \varphi(x,s)s \ge m(t-s)$ for $t \ge s$, $m > 0$. Then condition (1) is fulfilled in every bounded domain and the Dirichlet problem for the equation $\Box A \Box u = f$ is well-posed in such a domain. If the smooth boundary does not contain segments of characteristics, then the inclusion $u \in D(L_0)$ means that $u|_{\partial\Omega} = u'_\nu|_{\partial\Omega} = 0$ almost everywhere on $\partial\Omega$. Instead of $\Box$ one could consider any other scalar operator with property (1) and obtain the same conclusion.

The statements of other boundary value problems are analogous. For references see [1].

## References

1. *Burskii V.P.* Investigation methods of boundary value problems for general differential equations. Kyiv: Naukova dumka, 2002 (in Russian).