Singularity of Fuller type in a control problem of a rocket*

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Pontryagin's maximum principle (PMP) reduces the study of optimal control problems to the study of controlled Hamiltonian systems. If the Hamiltonian system is affine in control, then the typical phenomenon is the existence of a singular trajectory that is characterized by the fact that the Hamiltonian reaches its maximum over a finite time interval at more than one point. Thus, the optimal control cannot be determined directly from the PMP maximum condition. The singular second-order trajectories play an important role for optimal control problems, in particular, for applied problems. In [7] it was proved that if a mechanical system affine in control with a canonical Lagrangian is such that the kinetic energy depends only on generalized momenta, then its singular extremals are of the second order.

The behavior of optimal trajectories near a singular trajectory of the second order can be quite complex; for example, nonsingular trajectories can have an infinite number of control discontinuities near the matching point with the singular arc (this is called the chattering regime or Fuller's phenomenon). In the case of one-dimensional control, the problems with singular trajectories of the second order are well studied. In problems with one-dimensional control, it was proved that optimal chattering trajectories are typical for control systems with singular trajectories of the second order [2, 6]. A complete description of the structure of optimal solutions in a neighborhood of singular extremals of the second order was given in [6].

In problems with multidimensional control, the behavior near a singular trajectory can be much more complex and has been studied only in a few special cases [1, 6, 8].

In [4, 5] the behavior of solutions in a neighborhood of a singular extremal of the second order for a Hamiltonian system of the PMP that is affine in two-dimensional bounded control was studied. It was assumed that the dimension of the system is large (≥ 32) and controls take values in an ellipse. A family of solutions in the form of logarithmic spirals was found. These solutions reach the singular point in finite time and have a countable number of revolutions around this singular point. To obtain this result, the construction of the descending system of Poisson brackets, the Zelikin–Borisov method for the resolution of singularities in a neighborhood of a singular point and methods of dynamical systems to the resulting (after the resolution of singularities) system were applied.

The bound on the dimension of the Hamiltonian system in [4, 5] is a sufficient condition. In fact, the scheme used in [4, 5] also works for some systems of lower dimension. For example, in [3] a similar result was obtained for a problem in which

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the dimension of the Hamiltonian system is eight.

One more example with a Hamiltonian system of small dimension arises in the minimal time problem for the guidance of a rocket [9]. The equations of attitude motion of a rocket coupled with the orbit dynamics are

$$
\begin{aligned}
\dot{v}_x &= a \sin\theta \cos\psi - g, \\
\dot{v}_y &= -a \sin\psi, \\
\dot{v}_z &= a \cos\theta \cos\psi, \\
\dot{\theta} &= \frac{\omega_x \sin\phi + \omega_y \cos\phi}{\cos\psi}, \\
\dot{\psi} &= \omega_x \cos\phi - \omega_y \sin\phi, \\
\dot{\phi} &= (\omega_x \sin\phi + \omega_y \cos\phi) \tan\phi, \\
\dot{\omega}_x &= u_1, \\
\dot{\omega}_y &= u_2,
\end{aligned}
$$

where v_x, v_y, v_z are the components of the velocity vector, θ, ψ, ϕ are the Euler angles, ω_x, ω_y are the components of the angular velocity vector, g is the standard gravity, and $a > 0$. The controls u_1 and u_2 satisfy the constraint

$$u_1^2 + u_2^2 \le b^2, \qquad b > 0.$$

Initial conditions are fixed. On the target submanifold the desired final velocity is required to be parallel to the rocket axis.

In [9] the singular surface of the second order was found for this problem. The dimension of the Hamiltonian system is 16, and the results obtained in [5] cannot be applied. But using the same approach, we show that in this problem there are spiral-like extremals that attain the singular point (on the singular surface of the second order) in finite time, wherein the corresponding controls perform an infinite number of rotations along the circle S^1.

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