Impulsive differential equations with nonlinear mixed maxima

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A nonlocal boundary value problem for a first order system of ordinary integro-differential equations with impulsive effects and nonlinear mixed maxima is investigated. The boundary value problem is given by an integral condition. The method of successive approximations in combination with the method of contracting mappings is used. The existence and uniqueness of a solution of the boundary value problem are proved. The continuous dependence of the solutions on the right-hand side of the boundary condition is showed.

One of the interesting fields of the theory of functional differential equations is the differential equations with maxima. The qualitative theory of differential equations with maxima has peculiarities in theoretical investigations [1]. The following type of differential equations

$$x'(t) = f\big(t, x(t), \max\{x(\tau) \mid \tau \in [h_1(t) : | : h_2(t)]\}\big), \qquad t \in [0, T], \tag{1}$$

where $[h_1(t) : | : h_2(t)] = [\min\{h_1(t), h_2(t)\}, \max\{h_1(t), h_2(t)\}]$, is called a differential equation with mixed maxima. We suppose that there exist some points $t_i \in (0, T)$, $i = 1, 2, \ldots, p$, at which $h_1(t) = h_2(t)$. Then on the intervals $\Omega_1^p = [0, t_1] \cup [t_2, t_3] \cup [t_4, t_5] \cup \ldots \cup [t_{p-1}, t_p]$ the differential equation with mixed maxima (1) has the form

$$x'(t) = f\big(t, x(t), \max\{x(\tau) \mid \tau \in [h_1(t), h_2(t)]\}\big). \tag{2}$$

On the complementary intervals $\Omega_2^p = [t_1, t_2] \cup [t_3, t_4] \cup [t_5, t_6] \cup \ldots \cup [t_p, T]$ the differential equation with mixed maxima (1) has the form

$$x'(t) = f\big(t, x(t), \max\{x(\tau) \mid \tau \in [h_2(t), h_1(t)]\}\big). \tag{3}$$

The set of solutions of the differential equation with mixed maxima (1) on the interval $[0, T]$ coincides with the union of the sets of solutions of the two differential equations (2) and (3) on the intervals Ω_1^p and Ω_2^p, respectively. At the points $t_1, t_2, t_3, \ldots, t_{p-1}, t_p$, the solutions of the differential equation (1) with mixed maxima have discontinuities depending on the posed problem for (2) and (3).

Example. On the interval $[0, \infty)$ we consider the following differential equation with mixed maxima:

$$x'(t) = \frac{e^t}{(e^t+1)^2} \frac{e^{(1+(-1)^{[t]})\,t} + 1}{e^{(1+(-1)^{[t]})\,t}} \max\big\{x(\tau) | \tau \in [t{:}|{:}(1+(-1)^{[t]})\,t]\big\}, \quad t \in [0, \infty), \tag{4}$$

where $[t]$ is the integer part of t.

On the intervals $\Omega_1 = [0, 1] \cup [2, 3] \cup [4, 5] \cup \ldots$ the differential equation (4) with mixed maxima has the form

$$x'(t) = \frac{2e^t}{(e^t+1)^2} \max\{x(\tau) \mid \tau \in [0, t]\}. \tag{5}$$

On the complementary intervals $\Omega_2 = [1,2] \cup [3,4] \cup [5,6] \cup \dots$ the differential equation (4) with mixed maxima has the form

$$x'(t) = \frac{e^{2t}+1}{e^t(e^t+1)^2} \max\{x(\tau) \mid \tau \in [t, 2t]\}. \tag{6}$$

The general form of increasing and decreasing solutions of the differential equation (5) with maxima on the intervals $\Omega_1 = [0,1] \cup [2,3] \cup [4,5] \cup \dots$ is

$$x(t) = \begin{cases} \nearrow x(t) = C_i e^{-2(e^t+1)^{-1}}, & t \in \Omega_1, \quad C_i > 0, \\ \searrow x(t) = C_j \dfrac{e^t}{e^t+1}, & t \in \Omega_1, \quad C_j < 0. \end{cases}$$

The general form of increasing and decreasing solutions of the differential equation (6) with maxima on the intervals $\Omega_2 = [1,2] \cup [3,4] \cup [5,6] \cup \dots$ is

$$x(t) = \begin{cases} \nearrow x(t) = D_i \dfrac{e^t}{e^t+1}, & t \in \Omega_2, \quad D_i > 0, \\ \searrow x(t) = D_j (1+e^{-t}) e^{-2e^{-t}(e^t+1)^{-1}}, & t \in \Omega_2, \quad D_j < 0. \end{cases}$$

Therefore, the general form of increasing and decreasing solutions of the differential equation (4) with mixed maxima on the interval $[0, \infty)$ is

$$x(t) = \begin{cases} \begin{cases} \nearrow x(t) = C_i e^{-2(e^t+1)^{-1}}, & t \in \Omega_1, \quad C_i > 0, \\ \searrow x(t) = C_j \dfrac{e^t}{e^t+1}, & t \in \Omega_1, \quad C_j < 0; \end{cases} \\ \begin{cases} \nearrow x(t) = D_i \dfrac{e^t}{e^t+1}, & t \in \Omega_2, \quad D_i > 0, \\ \searrow x(t) = D_j (1+e^{-t}) e^{-2e^{-t}(e^t+1)^{-1}}, & t \in \Omega_2, \quad D_j < 0. \end{cases} \end{cases}$$

It is required to set conditions at each of the points $t_k = t_1, t_2, t_3, \dots, t_n, \dots$. If we do not specify continuous gluing conditions at these points, the solution of the differential equation (4) with mixed maxima suffers a discontinuity of the first kind at these points.

In this paper we consider the questions of existence and uniqueness of a solution to the nonlocal boundary value problem for an impulsive system of differential equations with nonlinear mixed maxima. This paper is the further development of the works [2, 3].

On the segment $[0, T]$ for $t \neq t_i$, $i = 1, 2, \dots, p$, we consider the following first order system of nonlinear differential equations:

$$x'(t) = f\big(t, x(t), \max\{x(\tau) \mid \tau \in [\lambda_1(t, x(t)) : | : \lambda_2(t, x(t))]\}\big) \tag{7}$$

with nonlocal boundary condition

$$Ax(0) + \int_0^T K(t,s)\, x(s)\, ds = B(t) \tag{8}$$

and nonlinear impulsive effect

$$x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \qquad i = 1, 2, \dots, p, \tag{9}$$

where $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$, $A \in \mathbb{R}^{n\times n}$ is a given matrix, $K(t,s)$ is a given $(n\times n)$-dimensional matrix function, and $\det Q(t) \neq 0$, $Q(t) = A + \int_0^T K(t,s)\,ds$, $f\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $I_i \colon \mathbb{R}^n \to \mathbb{R}^n$ are given functions; $0 \leq \lambda_1(t), \lambda_2(t) \leq T$, $x(t_i^+) = \lim_{h\to 0^+} x(t_i + h)$, $x(t_i^-) = \lim_{h\to 0^-} x(t_i - h)$ are the right and left limits of the function $x(t)$ at the point $t = t_i$, respectively.

By $\mathrm{PC}([0,T],\mathbb{R}^n)$ we denote the linear vector space

$$\mathrm{PC}([0,T],\mathbb{R}^n) = \big\{x\colon\ [0,T] \to \mathbb{R}^n;\ x(t) \in C((t_i,t_{i+1}],\mathbb{R}^n),\ i = 1,\ldots,p\big\},$$

where $x(t_i^+)$ and $x(t_i^-)$ $(i = 0,1,\ldots,p)$ exist and are bounded; $x(t_i^-) = x(t_i)$. The linear vector space $\mathrm{PC}([0,T],\mathbb{R}^n)$ is a Banach space with the norm

$$\|x\|_{\mathrm{PC}} = \max\big\{\|x\|_{C((t_i,t_{i+1}])},\ i = 1,2,\ldots,p\big\}.$$

Problem. Find a function $x(t) \in \mathrm{PC}([0,T],\mathbb{R}^n)$ such that for all $t \in [0,T]$, $t \neq t_i$, $i = 1,2,\ldots,p$, it satisfies the differential equation (7), the nonlocal integral condition (8) and for $t = t_i$, $i = 1,2,\ldots,p$, $0 < t_1 < t_2 < \ldots < t_p < T$, it satisfies the nonlinear limit condition (9).

References

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