The l-catch and evasion differential games for inertial motions

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In this work, we study the l-catch and evasion problems in a differential game with two inertial players, a pursuer and an evader. On the controls of both players, exponential damping geometric constraints are imposed, which ensure the values of the acceleration vectors are monotone decreasing in time. In the l-catch problem, we construct an approach strategy for the pursuer, which provides an approach to the evader at a distance l, $l > 0$, and a sufficient condition of l-catch is defined. In the evasion problem, a specific admissible strategy is proposed for the evader and a sufficient evasion condition is found.

Assume that a player X (pursuer) chases another player Y (evader) in the space $\mathbb{R}^n$. Represent a state of the pursuer in $\mathbb{R}^n$ by x and that of the evader by y. Let us consider the l-catch and evasion problem when the motions of the players are described by the differential equations with initial conditions

$$\ddot{x} = u, \qquad x(0) = x_0, \quad \dot{x}(0) = x_1,$$

$$\ddot{y} = v, \qquad y(0) = y_0, \quad \dot{y}(0) = y_1,$$

respectively. Here $x, y, u, v \in \mathbb{R}^n$, $n \geq 2$; x_0 and y_0 are the initial states of the players, respectively, and it is presumed that $|x_0 - y_0| > l$, $l > 0$; x_1 and y_1 are the initial velocity vectors of the players, respectively, and we suppose $x_1 = y_1$; the acceleration vectors u and v act as control parameters of the players, and they depend on time t, $t \geq 0$.

The temporal variations of u and v must be Lebesgue measurable functions $u(\cdot)\colon [0, \infty) \to \mathbb{R}^n$ and $v(\cdot)\colon [0, \infty) \to \mathbb{R}^n$. We denote by $\mathbb{U}$ the set of all measurable functions $u(\cdot)$ satisfying the geometric constraint (briefly, G-constraint)

$$|u(t)| \leq \alpha e^{-kt} \qquad \text{for almost every} \;\; t \geq 0,$$

where $\alpha > 0$, $k > 0$.

Similarly, we denote by $\mathbb{V}$ the set of all measurable functions $v(\cdot)$ satisfying the G-constraint

$$|v(t)| \leq \beta e^{-kt} \qquad \text{for almost every} \;\; t \geq 0,$$

where $\beta > 0$, $k > 0$.

Definition 1. The measurable functions $u(\cdot) \in \mathbb{U}$ and $v(\cdot) \in \mathbb{V}$ are called controls of the pursuer and evader, respectively.

If $u(\cdot) \in \mathbb{U}$ and $v(\cdot) \in \mathbb{V}$, then the triplets $(x_0, x_1, u(\cdot))$ and $(y_0, y_1, v(\cdot))$ yield the trajectories of motion

$$x(t) = x_0 + x_1 t + \int_0^t (t-s)u(s)\,ds, \qquad y(t) = y_0 + y_1 t + \int_0^t (t-s)v(s)\,ds$$

of the players X and Y, respectively.

The main target of the player X is to approach the player Y at the distance $l > 0$ (l-catch problem [1, 4, 7–11]), that is, to obtain the inequality

$$|x(\eta) - y(\eta)| \le l$$

for some $\eta > 0$. But the basic target of the player Y is to avoid the above inequality, i.e., to keep the relation (evasion problem)

$$|x(t) - y(t)| > l$$

for all t, $t \ge 0$, and if it is impossible, then the player Y struggles to postpone an instant of l-catch.

Let us introduce the notation $z(t) = x(t) - y(t)$, $z(0) = z_0$. Then we have $z_0 = x_0 - y_0$ and $\dot{z}(0) = 0$, and therefore, we come to the unique Cauchy problem

$$\ddot{z} = u - v, \qquad z(0) = z_0, \quad \dot{z}(0) = 0.$$

Now we will define an approach strategy for the pursuer on the basis of the works [2, 3, 5, 6].

Definition 2. For $\alpha \ge \beta$, we term the function

$$\mathbf{u}(t, v) = v - \lambda(t, v) \frac{\alpha e^{-kt} z_0 + vl}{\alpha e^{-kt} + \lambda(t, v) l}$$

an approach strategy or Π_l-strategy of the pursuer in the l-catch problem, where

$$\lambda(t, v) = \frac{1}{h^2} \left[\langle v, z_0 \rangle + \alpha l e^{-kt} + \sqrt{(\langle v, z_0 \rangle + \alpha l e^{-kt})^2 + h^2(\alpha^2 e^{-2kt} - |v|^2)} \right],$$

$h^2 = |z_0|^2 - l^2$, and $\langle v, z_0 \rangle$ is the scalar product of the vectors v and z_0 in $\mathbb{R}^n$.

Note that the function $\lambda(t, v)$ is generally termed the resolving function.

Proposition 1. *If $\alpha \ge \beta$, then $\lambda(t, v)$ is bounded as*

$$\frac{\alpha - \beta}{|z_0| - l} e^{-kt} \le \lambda(t, v) \le \frac{\alpha + \beta}{|z_0| - l} e^{-kt}.$$

Proposition 2. *If $\alpha > \beta$, then the equation*

$$e^{-kt} = -kt + b, \qquad b = 1 + \frac{k^2(|z_0| - l)}{\alpha - \beta},$$

has a positive root with respect to t, and we denote it by T_l.

Definition 3. In the l-catch problem, we say that the Π_l-strategy is winning on some time interval $[0, T_l]$ if, for any $v(\cdot) \in \mathbb{V}$,

(a) there exists a τ, $\tau \in [0, T_l]$, that generates $|z(\tau)| \le l$;

(b) the inclusion $\mathbf{u}(t, v(\cdot)) \in \mathbb{U}$ is satisfied on $[0, \tau]$.

Here the number T_l is termed a guaranteed time of l-catch.

Theorem 1. *Let $\alpha > \beta$ be valid in the l-catch problem. Then the Π_l-strategy is winning on the interval $[0, T_l]$.*

Definition 4. In the evasion problem, we call the control function

$$\mathbf{v}^*(t) = -\beta e^{-kt} \hat{z}_0$$

a strategy of the player Y, where $\hat{z}_0 = z_0 / |z_0|$.

Definition 5. We say that the strategy $\mathbf{v}^*(t)$ is winning if, for any control $u(\cdot) \in \mathbb{U}$, the solution $z(t)$ of

$$\ddot{z} = u(t) - \mathbf{v}^*(t), \qquad z(0) = z_0, \quad \dot{z}(0) = 0,$$

satisfies the inequality $|z(t)| > l$ for all t, $t \geq 0$.

Theorem 2. *Let $\alpha \leq \beta$ be fulfilled. Then in the evasion game, the strategy $\mathbf{v}^*(t)$ is winning on the time interval $[0, +\infty)$.*

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