Behaviour of the solutions of a traffic flow mathematical model

Pogrebnyak M. A.

P.G. Demidov Yaroslavl State University, Mathematics Faculty, Yaroslavl, Russian Federation

pogrebnyakmaksim@mail.ru

To simulate movement of the traffic flow, a mathematical model was built. The new model presents a system of differential equations with a delay, which describes the movement of $N \in \mathbb{N}$ vehicles that follow each other. All vehicles are considered as material points, thus their internal structure and external dimensions are not taken into account.

$$\ddot{x}_n(t) = R_n\left[a_n\left(\frac{v_{max,n} - V_n}{1 + e^{k_n(-\Delta x_n(t,\tau)+s_n)}} + V_n - \dot{x}_n(t)\right)\right] + (1 - R_n)\left[q_n\left(\frac{\dot{x}_n(t)\,[V_n - \dot{x}_n(t)]}{\Delta x_n(t,\tau) - l_{n,\varepsilon}}\right)\right], \qquad (1)$$

where $\Delta x_n(t,\tau) = x_{n-1}(t-\tau) - x_n(t)$ — distance between adjacent vehicles, $V_n = \min(\dot{x}_{n-1}(t-\tau), v_{max,n})$ — speed function, $R(\Delta x_n(t,\tau))$ — relay function (2), and the rest model parameters are shown in the table 1.

Relay function

The entire vehicle movement is splitted on two phases: acceleration phase and deceleration phase. To describe the "acceleration-deceleration" switch, a relay function is used:

$$R_n = \begin{cases} 1, & \text{if } \Delta x_n(t,\tau) > S + l_n, \\ 0, & \text{if } \Delta x_n(t,\tau) \le S + l_n, \end{cases} \qquad (2)$$

where S — braking distance and the other parameters are given in the table 1.

Braking distance refers to the distance a vehicle will travel from the brake system activation point to when it comes to a complete stop. The following formula is used to calculate this value:

$$S = \frac{\dot{x}_n^2(t)}{2\mu g}.$$

Model parameters

For the model (1), the values and units of parameters measurement are determined based on physical laws, the current decreases of the Russian Federation, logical considerations and observations of real traffic flows:

Table 1: Model parameters (1)

Designation	Description	Value	SI unit
$v_{max,n}$	maximum desired speed	$[0, 16.7]$	m/s
$v_{min,0}$	minimum desired speed	$[0, v_{max}]$	m/s
τ	driver response time	1	s
l_n	safe distance	2	m
k_n	coefficient of deceleration smoothness	≥ 0	1/m
s_n	ceasing distance of the influence	$\ge S$	m
a_n	characteristics of vehicle acceleration	≥ 0	1/s
q_n	characteristics of vehicle deceleration	≥ 0	dimensionless
g	acceleration of gravity	9.8	m/s^2
μ	coefficient of friction	$(0, 1)$	dimensionless

Stability analysis

An analysis of the stability of a uniform driving mode was carried out for the model. This driving mode assumes that all vehicles are moving with the same speed v_{max} and at a distance $\Delta c_n = c_n - c_{n-1}$ from each other, where c_n is a decreasing sequence. For any decreasing sequence c_n a solution of the (1) system exists as follows:

$$x_n(t) = c_n + v_{max} t.$$

The stability of such a solution depends on the signs of expressions:

$$d_n = -\tau v_{max} + c_n - c_{n-1} - l_{n,\varepsilon}.$$

The following theorem is valid.

Theorem 1 *If the inequality $d_n > 0$ holds for $\forall n$, then the uniform mode is stable. If at least for one i inequality $d_i \le 0$ is satisfied, then the uniform mode is unstable.*

It follows from the theorem that if all the vehicles of the flow move at a fairly large distance from each other, then this driving mode is stable. Stability is lost by increasing the speed v_{max}, the driver response time τ, the safe distance between vehicles l_n, or by reducing the distance between two adjacent vehicles Δc_n.

Velocities and coordinates changes graphics

Figure 1 shows the dynamics of the vehicles movement, which can be interpreted as stop in front of a traffic light.

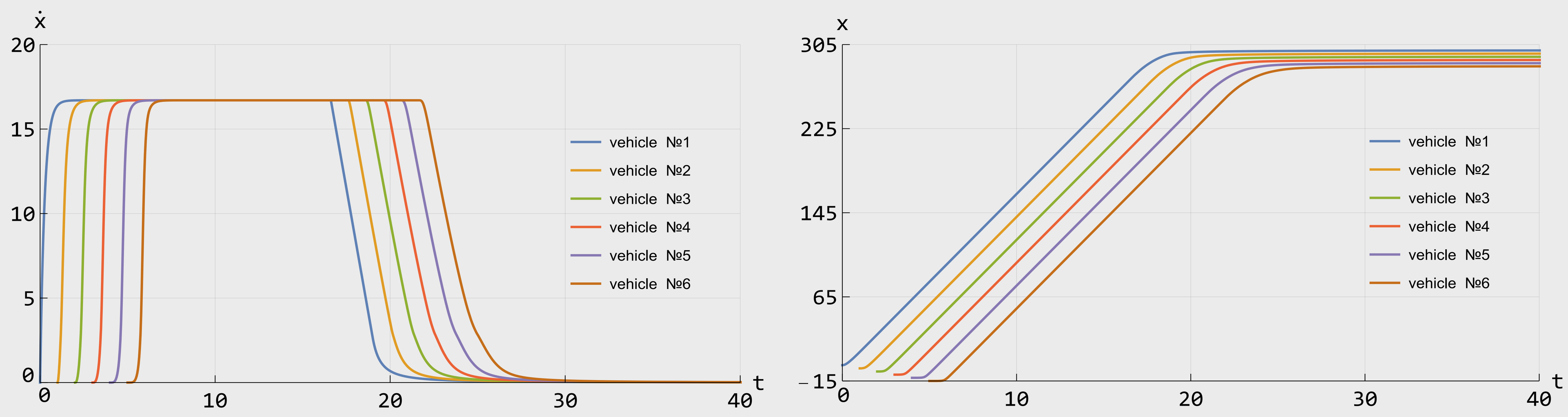

Figure 1: Velocities (left) and coordinates (right) changes graphics of vehicles movement for the model (1) with parameters: $\tau = 1$, $a_n = 4$, $q_n = 3$, $v_{max,n} = 16.7$, $v_{min,0} = 0$, $l_n = 3$, $g = 9.8$, $k = 0.5$, $s = 35$, $\mu = 0.6$.

Figure 2 shows the dynamics of the vehicles movement, which can be interpreted as the speed limit changing. At the first interval v_{max} is equal to 16.7, at the second interval v_{max} is equal to 16.7/2 and at the third interval v_{max} is equal to 16.7.

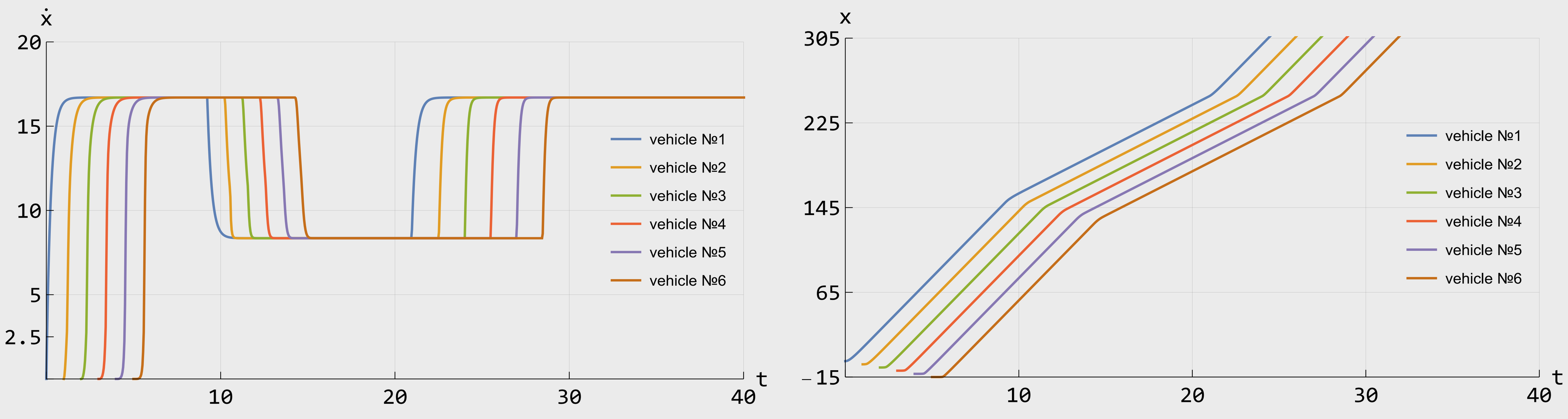

Figure 2: Velocities (left) and coordinates (right) changes graphics of vehicles movement for the model (1) with parameters: $\tau = 1$, $a_n = 4$, $q_n = 3$, $v_{max,n} = 16.7$, $v_{min,0} = 0$, $l_n = 3$, $g = 9.8$, $k = 0.5$, $s = 35$, $\mu = 0.6$.