Linear differential games with Langenhop type constraints on controls

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We study the pursuit–evasion problems in a linear differential game with two players [1, 2], which are called the pursuer and evader, in the space $\mathbb{R}^n$. The controls of the players are subject to Langenhop type constraints [9]. We construct in parallel a pursuit strategy (Π-strategy) [3, 5–10] for the pursuer and an evasion strategy for the evader. We have obtained sufficient solvability conditions of the pursuit and evasion problems. To solve the evasion problem, we have found lower bounds of the convergence depending on the given parameters.

Suppose that in $\mathbb{R}^n$ a controlled object P, called the pursuer, follows another object E, called the evader. Denote by x the position of the pursuer and by y that of the evader in $\mathbb{R}^n$. Let the motions of the objects be represented by the following differential equations, respectively:

$$\dot{x} + kx = u, \qquad x(0) = x_0, \tag{1}$$

$$\dot{y} + ky = v, \qquad y(0) = y_0. \tag{2}$$

Here $x, y, u, v \in \mathbb{R}^n$, $n \geq 2$; k is a positive number; x_0 and y_0 are the initial positions of the objects and it is assumed that $x_0 \neq y_0$; u and v are the velocity vectors functioning as control parameters of the objects, respectively. They should be picked as measurable functions $u(\cdot)\colon [0, +\infty) \to \mathbb{R}^n$ and $v(\cdot)\colon [0, +\infty) \to \mathbb{R}^n$, respectively.

We consider the following classes of admissible controls for the pursuer:

(P_1) the Langenhop type constraint (briefly, La-constraint) [9]

$$|u(t)|^2 \leq \rho^2 - 2k\int_0^t |u(s)|^2\, ds, \qquad \text{for almost every } \ 0 \leq t < \bar{t}, \tag{3}$$

and we denote this class by U_{La}, where $\bar{t} = \sup\{t\colon \rho^2 - 2k\int_0^t |u(s)|^2\, ds \geq 0\}$;

(P_2) the geometric constraint (briefly, G-constraint) in the form

$$|u(t)| \leq \rho e^{-kt} \qquad \text{for almost every } \ t \geq 0, \tag{4}$$

and we denote this class by U_G;

(P_3) the integral constraint (briefly, I-constraint) in the form

$$\int_0^t |u(s)|^2\, ds \leq \frac{\rho^2}{2k}(1 - e^{-2kt}), \qquad t \geq 0, \tag{5}$$

and we denote this class of admissible controls by U_I; in (3)–(5), ρ and k are positive numbers.

Similarly, for the evader we introduce the following classes of admissible controls:

(E_1) the Langenhop constraint (briefly, La-constraint)

$$|v(t)|^2 \le \sigma^2 - 2k \int_0^t |v(s)|^2\, ds \qquad \text{for almost every } \ 0 \le t \le \tilde{t}, \tag{6}$$

and we denote the class of all admissible controls of the form (6) by V_{La}, where $\tilde{t} = \sup\{t\colon \sigma^2 - 2k\int_0^t |v(s)|^2\, ds \ge 0\}$;

(E_2) the geometric constraint (briefly, G-constraint)

$$|v(t)| \le \sigma e^{-kt} \qquad \text{for almost every } \ t \ge 0, \tag{7}$$

and we denote the class of admissible controls satisfying the G-constraint (7) by V_G;

(E_3) the integral constraint (briefly, I-constraint)

$$\int_0^t |v(s)|^2\, ds \le \frac{\sigma^2}{2k}(1 - e^{-2kt}) \qquad \text{for almost every } \ t \ge 0, \tag{8}$$

and we denote the class of admissible controls satisfying the I-constraint (8) by V_I; in (6)–(8), σ and k are positive numbers too.

If U (respectively, V) is one of the introduced classes U_G, U_I or U_{La} (respectively, V_G, V_I or V_{La}), then the pairs $(x_0, u(\cdot))$ and $(y_0, v(\cdot))$ generate the following trajectories:

$$x(t) = e^{-kt}\left(x_0 + \int_0^t u(s)e^{ks}\, ds\right), \qquad y(t) = e^{-kt}\left(y_0 + \int_0^t v(s)e^{ks}\, ds\right)$$

of the pursuer and evader, respectively.

Lemma 1. *$U_G \subset U_{La} \subset U_I$ and $V_G \subset V_{La} \subset V_I$ for almost every $t \ge 0$.*

Let us introduce the notation $z(t) = x(t) - y(t)$, $z(0) = z_0$, $z_0 = x_0 - y_0$.

Definition 1. If $\rho \ge \sigma$, then the function

$$\mathbf{u}_{La}(t, v) = v - \lambda_{La}(t, v)\xi_0, \qquad \lambda_{La}(t, v) = \langle v, \xi_0\rangle + \sqrt{\langle v, \xi_0\rangle^2 + \delta e^{-2kt}},$$

is called the Π_{La}-strategy of the pursuer in the La-game, where $\delta = \rho^2 - \sigma^2$ and $\xi_0 = z_0/|z_0|$. Note that $|\mathbf{u}_{La}(t, v)|^2 = |v|^2 + \delta e^{-2kt}$, $t \ge 0$.

Definition 2. Let there exist a positive root of the equation

$$\sqrt{\Phi^2(t) + \Psi(t)} - \Phi(t) = |z_0| \tag{9}$$

with respect to t, where

$$\Phi(t) = \frac{\sigma}{2ke^{kt}}(e^{2kt} - 1), \qquad \Psi(t) = (\rho^2 - \sigma^2)t^2.$$

Then we call the smallest positive root of equation (9) a guaranteed pursuit time and denote it by T_{La}.

Theorem 1. *If $\rho > \sigma$ and there exists a smallest positive root of equation* (9), *then the Π_{La}-strategy guarantees the completion of pursuit in the La-game on the time interval* $[0, T_{La}]$.

Definition 3. Let $\sigma \geq \rho$ in the La-game. Then by the E_{La}-strategy of the evader we mean the function

$$\mathbf{v}_{La}(t, u_\varepsilon(t)) = \begin{cases} 0 & \text{if } \ 0 \leq t < \varepsilon, \\ -\sqrt{|u(t-\varepsilon)|^2 + \theta e^{-2k(t-\varepsilon)}}\xi_0 & \text{if } \ t \geq \varepsilon, \end{cases} \tag{10}$$

where

$$u_\varepsilon(t) = \begin{cases} 0 & \text{if } \ 0 \leq t < \varepsilon, \\ u(t-\varepsilon) & \text{if } \ t \geq \varepsilon, \end{cases} \qquad u_\varepsilon(t) \in \mathbb{R}^n, \quad \theta = \sigma^2 - \rho^2, \quad \xi_0 = \frac{z_0}{|z_0|}.$$

Theorem 2. *If in the La-game* $\rho \leq \sigma$ *and* $0 < \varepsilon \leq k^{-1} \ln(2k|z_0|/\rho)$, *then the* E_{La}*-strategy* (10) *is winning for* E *and the following estimates for the distance between the players hold for all* $t \geq 0$:

$$|z(t)| > \begin{cases} 0 & \textit{if } \ 0 \leq t < \varepsilon, \\ \sqrt{\Phi_E^2(t-\varepsilon) + \Psi_E(t-\varepsilon)} - \Phi_E(t-\varepsilon) & \textit{if } \ t \geq \varepsilon, \end{cases}$$

where

$$\Phi_E(t) = \frac{\rho}{2ke^{kt}}(e^{2kt} - 1), \qquad \Psi_E(t) = (\sigma^2 - \rho^2)t^2.$$

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