Properties of ongoing critical branching processes with countable particle types

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Abstract: Let $\mathbf{X}^{(l)}(n) = \big(X_j^{(l)}(n)\big)_{j\in\mathcal{Z}}$, $n \in \mathcal{N}_0 := \mathcal{N} \cup 0$, be the Galton-Watson branching process starting from one particle of type $l \in \mathcal{Z}$, where $Z_j^{(l)}(n)$ is the number of particles of type j at time n. Let's put $\mathbf{X}(n) := \mathbf{X}^{(0)}(n) =: \big(X_j(n)\big)_{j\in\mathcal{Z}}$. Define the generating function $f(\mathbf{s})$ for the random vector $\xi := (\xi_i)_{i\in\mathcal{Z}} \in \mathcal{N}_0^{\mathcal{Z}}$, where ξ_i is the type i offspring number for a particle of type 0, with its own distribution $p_{\mathbf{j}} := \mathcal{P}\big(\xi = \mathbf{j}\big)$

$$f(\mathbf{s}) := \mathbf{E}\mathbf{s}^{\xi} = \sum\nolimits_{\mathbf{j}\in\mathcal{N}_0^{\mathcal{Z}}} p_{\mathbf{j}}\mathbf{s}^{\mathbf{j}}, \quad \mathbf{j} = \big(j_i\big)_{i\in\mathcal{N}} \in \mathcal{N}_0^{\mathcal{Z}},$$

where $\mathbf{s} = \big(s_i\big)_{i\in\mathcal{Z}} \in [0,1]^{\mathcal{Z}}$, and $\mathbf{s}^{\mathbf{j}} := \prod_{i\in\mathcal{Z}} s_i^{j_i}$. An analogous generating function for a particle of type $m \in \mathcal{Z}$ has the form $f_m(\mathbf{s}) = f\big(\mathbf{s}^{(m)}\big)$, where $\mathbf{s}^{(m)} := \big(s_{i+m}\big)_{i\in\mathcal{Z}} \in [0,1]^{\mathcal{Z}}$, $\mathbf{s} = \mathbf{s}^{(0)}$.

Process $X(n) := \big|\mathbf{X}(n)\big| = \sum_{j=-\infty}^{\infty} X_j(n)$ with generating function $p(s) := \mathbf{E}s^{|\xi|} = f(\mathbf{1}s) =: \sum_{i=0}^{\infty} p_i s^i$ will be called the accompanying one. In terms of $X(n)$, we study only the critical case with a finite variance for the number of offspring.

We fix n and from the processes $X(k)$ and $\mathbf{X}(k)$, $k = 0,1,\cdots,n$, we exclude all particles that have no offspring at time n. The resulting processes are called reduced and are denoted by $X(k,n)$ and $\mathbf{X}(k,n) = \big(X_j(k,n)\big)_{j\in\mathcal{Z}}$, $k = 0,1,\cdots,n$. Set $\eta := \sum_{j=-\infty}^{\infty} jX_j(1)$ and $V(k,n) := \sum_{j=-\infty}^{\infty} jX_j(k,n)$. It is obvious that $X(n,n) = \{X(n)|X(n) > 0\}$ and $\mathbf{X}(n,n) = \{\mathbf{X}(n)|X(n) > 0\}$.

It is well known (see [1], Ch. I, §10) that in the critical case for $\mathbf{D}\xi = \sigma^2$ and finite third moment $p'''(1) < \infty$ for $Q_n := \mathbf{P}(X(n) > 0)$

$$Q_n^{-1} = 0.5\sigma^2 n + O\big(\ln n\big),$$

while Yaglom's theorem asserts the convergence $\lim_{n\to\infty} \mathbf{P}\big\{Q_n X(n) > x \big| X(n) > 0\big\} \to e^{-x}$.

In [2] a generalization of Yaglom's theorem for processes with a countable number of particle types is proved. The history of the problem is also described in some detail there. The essence of this generalization was that if in the limit, particles with small numbers of types are mainly preserved.

Suppose that $\mathbf{E}\xi = 1$, $\mathbf{D}\xi = \sigma^2$, $\mathbf{E}\eta = a_1 \neq 0$ and among the $p_{\mathbf{j}}$, only a finite number are nonzero. Than

$$\mathbf{E}X(k,n) = \frac{Q_{n-k}}{Q_n};\ \mathbf{D}X(k,n) = \frac{kQ^2_{n-k}}{nQ^2_n}\big(1 + o_n\big(1\big)\big);\ \mathbf{E}V(k,n) = ka_1\frac{Q_{n-k}}{Q_n};$$

$$\mathbf{D}V(k,n) = \big(a_1 k + a_1^2(k-1)\big)\big(1 + o_n\big(1\big)\big), \quad k = O(1);$$

$$\mathbf{D}V(k,n) = (a_1 + a_1^2)k(1 - 0.5kn^{-1})\frac{Q^2_{n-k}}{Q^2_n}\big(1 + o_n\big(1\big)\big), \quad k \to \infty.$$

$$\lim_{M\to+\infty}\lim_{n\to\infty} \mathbf{P}\left(\frac{\big|Q^{-1}_{n-k}V(k,n) - 0.5a_1\sigma^{-2}n^2\big|}{n\sqrt{n}} > M\right) = 0, \text{ for } n - k = o(n).$$

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References

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