On sizes of trees in a Galton-Watson forest with power-law distribution

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Abstract: Let G_N be a critical Galton – Watson branching process with N initial particles and let the number of offspring of each particle be a random variable ξ following the distribution

$$p_k = \mathbf{P}\left\{\xi = k\right\} = \frac{1}{(k+1)^\tau} - \frac{1}{(k+2)^\tau},\ k = 0, 1, 2, \dots \qquad (4)$$

The process G_N induces a conditional probability distribution on the subset $F_{N,n}$ of its trajectories with $N+n$ vertices provided that the number of vertices is equal to $N+n$. We denote by $\mathcal{F}_{N,n}$ the thus constructed Galton – Watson forest with N trees and n non-rooted vertices. It is easy to show that $\mathbf{E}\xi = \zeta(\tau, 2)$, where $\zeta(s, v) = \sum\limits_{k=0}^{\infty}(k+v)^{-s}$ is the generalized zeta-function. Since the branching process G_N is critical, the equality $\zeta(\tau, 2) = 1$ holds and therefore $\tau \approx 1.728$. For such a parameter value only the first moment of the distribution (4) is finite.

Let $\eta(\mathcal{F})$ be a random variable equal to the maximum tree size and $\mu_r(\mathcal{F})$ be a random variable equal to the number of trees of size r in the forest $\mathcal{F}_{N,n}$. Limit distributions of $\eta(\mathcal{F})$ and $\mu_r(\mathcal{F})$ are obtained as $N, n \to \infty$, $n/N^\tau \geq C > 0$.

We denote by $g(x)$ a stable distribution density with a parameter τ and a characteristic function

$$f(t) = \exp\left\{-\Gamma(1-\tau)|t|^\tau e^{-i\pi\tau t/2|t|}\right\},$$

and let $p(x)$ be a stable distribution density with a parameter $1/\tau$ and a characteristic function

$$h(t) = \exp\left\{-\left(-\Gamma(1-\tau)\right)^{-1/\tau}|t|^{1/\tau}e^{-i\pi t/2\tau|t|}\right\}.$$

In particular the following statements hold.

Theorem 1. *Let $N, n \to \infty$ in such a way that $n/N^\tau \to \gamma$, where γ is a positive constant. Then for any positive z*

$$\mathbf{P}\left\{\frac{\eta(\mathcal{F})}{n} \le z\right\} \to \frac{1}{2\pi p\,(\gamma)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} I_k(\gamma z, \gamma),$$

where

$$I_0(u,v) = p(v),\ \ I_k(u,v) = \int\limits_{x_k(u,v)} \frac{p\,(v - x_1 - \ldots - x_k)\,dx_1 \ldots dx_k}{(2\pi C(\tau))^k\,(x_1 \ldots x_k)^{(\tau+1)/\tau}},$$

$$x_k(u,v) = \{x_i \ge u,\ \ i = 1, \ldots, k,\ \ x_1 + \ldots + x_k \le v\}\,,\quad k = 1, 2, \ldots,$$

$$C(\tau) = 1/\tau\Gamma(1 - 1/\tau)\,(-\Gamma(1-\tau))^{1/\tau}.$$

Theorem 2. *Let $N, n \to \infty$ in such a way that $n/N^\tau \to \infty$. Then for any fixed positive z*

$$\mathbf{P}\left\{\frac{n - \eta(\mathcal{F})}{N^\tau} < z^{-\tau}\right\} \to \tau \int\limits_{-\infty}^{-z} g(y)dy.$$