Singular Del Pezzo varieties

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Del Pezzo surfaces

Definition

A del Pezzo surface is a smooth projective surface X with ample anticanonical class $-K_X$.

Projective models

Let X be a del Pezzo surface. Then one following holds:

- $K_X^2 = 1$ and $X \simeq X_6 \subset \mathbb{P}(1,1,2,3)$,
- $K_X^2 = 2$ and $X \simeq X_4 \subset \mathbb{P}(1,1,1,2)$,
- $K_X^2 = 3$ and $X \simeq X_3 \subset \mathbb{P}^3$,
- $K_X^2 = 4$ and $X \simeq X_{2,2} \subset \mathbb{P}^4$,
- $K_X^2 = 5$ and $X \simeq \mathrm{Gr}(2,5) \cap \mathbb{P}^5 \subset \mathbb{P}^9$,
- $K_X^2 = 6$ and $X \simeq (1,1,1) \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$,
- $K_X^2 = 7$ and $X \simeq \mathrm{Bl}_{P_1,P_2}(\mathbb{P}^2)$.
- $K_X^2 = 8$ and $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$.
- $K_X^2 = 9$ and $X \simeq \mathbb{P}^2$.

Del Pezzo surfaces

Blowup models

A del Pezzo surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowup of $9 - K_X^2$ points in general position on $\mathbb{P}^2$.

Lattice

The Picard group $\mathrm{Pic}(X)$ of a del Pezzo surface X is a lattice of rank $10 - K_X^2$ equipped with a symmetric bilinear form $\langle -, - \rangle$ of signature $(1, 9 - K_X^2)$ and distinguished element $-K_X$.
The set

$$\Delta := \{\alpha \in \mathrm{Pic}(X) \mid \langle \alpha, K_X \rangle = 0, \quad \langle \alpha, \alpha \rangle = -2\}$$

forms a root system of type

d	1	2	3	4	5	6	8
Δ	E_8	E_7	E_6	D_5	A_4	$\mathrm{A}_2 \times \mathrm{A}_1$	A_1

Del Pezzo varieties

Definition (T. Fujita)

A del Pezzo variety X is a Fano variety with at worst terminal Gorenstein singularities such that

$$-K_X = (n-1)A,$$

where A is an ample line bundle and $n = \dim X$.

Invariants

- The degree of X: $\mathrm{d}(X) := A^n$.
- The lattice $\mathrm{Cl}(X)$ (Weil divisor class group) equipped with a natural symmetric bilinear form

  $$\langle D_1,\, D_1\rangle := A^{n-2} \cdot D_1 \cdot D_2.$$

  and distinguished element $A \in \mathrm{Cl}(X)$ with $\langle A,\, A\rangle = \mathrm{d}(X)$.
- The rank of X: $\mathrm{r}(X) := \mathrm{rk}\,\mathrm{Cl}(X)$.

Warning

In general we have $\mathrm{Cl}(X) \supset \mathrm{Pic}(X)$ but $\mathrm{Cl}(X) \neq \mathrm{Pic}(X)$.

Projective models

We assume that X is not a cone over a lower dimensional variety.

Theorem (V. Iskovskikh, T. Fujita, ...)

Let (X, A) be a del Pezzo variety of dimension $n \geq 3$.
Then $1 \leq \mathrm{d}(X) \leq 8$ and the following holds:

- *If $\mathrm{d}(X) = 1$, then $X \simeq X_6 \subset \mathbb{P}(1^n, 2, 3)$.*
- *If $\mathrm{d}(X) = 2$, then $X \simeq X_4 \subset \mathbb{P}(1^{n+1}, 2)$.*
- *If $\mathrm{d}(X) = 3$, then $X \simeq X_3 \subset \mathbb{P}^{n+1}$.*
- *If $\mathrm{d}(X) = 4$, then $X \simeq X_{2,2} \subset \mathbb{P}^{n+2}$.*
- *If $\mathrm{d}(X) = 5$, then $X \simeq \mathrm{Gr}(2,5) \cap \mathbb{P}^{n+3} \subset \mathbb{P}^9$.*
- *If $\mathrm{d}(X) = 6$, then $X \simeq (\mathbb{P}^2 \times \mathbb{P}^2) \cap \mathbb{P}^{n+4} \subset \mathbb{P}^8$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*
- *If $\mathrm{d}(X) = 7$, then $X \simeq \mathrm{Bl}_P(\mathbb{P}^3)$.*
- *If $\mathrm{d}(X) \geq 8$, then $X \simeq \mathbb{P}^3$.*

Birational geometry

Definition

X is an *almost del Pezzo variety* if X is terminal Gorenstein and $-K_X = (n-1)A$, where $n := \dim(X)$ and A is a nef and big line bundle and the morphism $\Phi_{|mA|}$ given by $|mA|$ for $m \gg 0$ does not contract divisors.

A (small) $\mathbb{Q}$-*factorialization* of a variety X is a projective birational morphism $\xi : \hat{X} \to X$ such that $\hat{X}$ is $\mathbb{Q}$-factorial and ξ does not contract divisors. If X has only klt (in particular, terminal) singularities, then a $\mathbb{Q}$-factorialization exists (but it is not unique!)
A *crepant model* of X is an almost del Pezzo variety X' such that

$$X' \overset{\text{pseu}}{\sim} X.$$

Proposition

If X is a del Pezzo variety, then its $\mathbb{Q}$-factorialization $\hat{X}$ is an almost del Pezzo variety.
Conversely, if X is an almost del Pezzo variety, then its anticanonical model $\Phi_{|mA|}(X)$, $m \gg 0$ is a del Pezzo variety.

Birational geometry: MMP on del Pezzos

Proposition

Let (X, A) be a $\mathbb{Q}$-factorial almost del Pezzo variety with $n := \dim X$ and let $f : X \to Z$ be a Mori contraction. Then one of the following holds:

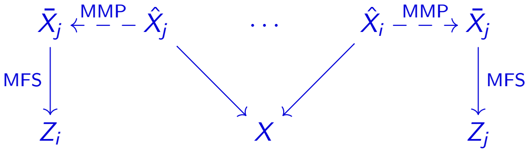
- *Z is a $\mathbb{Q}$-factorial almost del Pezzo variety and f is the blowup of a smooth point $P \in Z$;*
- *Z is a smooth del Pezzo surface and f is a $\mathbb{P}^{n-2}$-bundle;*
- *$Z \simeq \mathbb{P}^1$ and f is a quadric bundle,*
- *Z is a point and $\mathrm{r}(X) = 1$.*

Corollary

In the category $\mathbb{Q}$-factorial almost del Pezzo varieties we can run the D-MMP with respect to any divisor D.
Any step of this MMP is either a flop or inverse to the blowup of a smooth point.

Birational geometry: MMP on del Pezzos

Let X be a del Pezzo variety and $\hat{X}_1, \dots \hat{X}_N \to X$ be its $\mathbb{Q}$-factorializations (finite number).



Definition

An (almost) del Pezzo variety X is called *imprimitive* if it has a $\mathbb{Q}$-factorialization admitting a birational Mori contraction. Otherwise X is called *primitive*.

Example

- Any del Pezzo variety with $\mathrm{r}(X) = 1$ is primitive.
- $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2 \times \mathbb{P}^2$ are primitive del Pezzo.
- Let $\hat{X} \to \mathbb{P}^3$ be the blowup of $n \leq 7$ points in general position. Then the anticanonical model of $\hat{X}$ is an imprimitive del Pezzo with $\mathrm{d}(X) = 9 - n$.

Hyperplane sections of del Pezzo varieties

If X is a del Pezzo variety of dimension $n \geq 4$ and $Y \subset X$ is a general divisor from $|A|$ then Y is a del Pezzo variety and the restriction map

$$\mathrm{Cl}(X) \to \mathrm{Cl}(Y)$$

an isomorphism of lattices (preserving the form $\langle -, - \rangle$ and A).

Definition

We say that X is *maximal* if it cannot be realized as a member of $|A_V|$ of a non-conical del Pezzo variety (V, A_V) with $\dim V = \dim X + 1$.

Example

- $\mathbb{P}^3$, $\mathrm{Gr}(2, 5)$ are maximal del Pezzos;
- $X_3 \subset \mathbb{P}^{n+1}$, $X_{2\cdot 2} \subset \mathbb{P}^{n+2}$ are not maximal.

A del Pezzo variety X is of *unbounded type*, if there is an infinite series of (non-conical) del Pezzo varieties

$$X = X^n \subset X^{n+1} \subset \dots$$

where X^k is a hyperplane section of X^{k+1}. Otherwise we say X is of *bounded type*.

Imprimitive del Pezzo varieties

Theorem

Let X be an imprimitive del Pezzo variety. Then there exists, a $\mathbb{Q}$-factorialization $\hat{X} \to X$, a primitive del Pezzo variety Y, and a collection $P_1, \ldots, P_k \in Y$ of distinct smooth points suits to the following diagram

$$\begin{array}{ccccccc} & & \hat{X} & \xrightarrow[\text{of } P_1, \ldots, P_k]{\text{composition of blowups}} & \bar{X} & & \\ & \swarrow & & & & \searrow & \\ X & & & & & & Y \end{array}$$

where $\bar{X} \to Y$ is the anticanonical model.
Moreover, $\mathrm{d}(X) = \mathrm{d}(Y) - k$ and $\mathrm{r}(X) = \mathrm{r}(Y) + k$.

Lattices

Let (X, A) be a del Pezzo variety.

Definition

We say that an element $x \in \mathrm{Cl}(X)$ is

- *exceptional*, if $\langle A, x\rangle = 1, \ \langle x, x\rangle = -1$;
- *of $\mathbb{P}^1$-type*, if $\langle A, x\rangle = 2, \ \langle x, x\rangle = 0$;
- *of $\mathbb{P}^2$-type*, if $\langle A, x\rangle = 3, \ \langle x, x\rangle = 1$.

Proposition

Let $E \in \mathrm{Cl}(X)$ be an exceptional class. Then there exists a $\mathbb{Q}$-factorialization $\xi : \hat{X} \to X$ and a birational Mori contraction $f : \hat{X} \to \bar{X}$ whose exceptional divisor is the proper transform $\hat{E} \subset \hat{X}$ of E.

Proposition

Let $D \in \mathrm{Cl}(X)$ be a $\mathbb{P}^1$-class (resp. $\mathbb{P}^2$-class). Then there exists a $\mathbb{Q}$-factorialization $\xi : \hat{X} \to X$ such that the proper transform D' is nef and the linear system $|D'|$ defines a contraction $X \to \mathbb{P}^1$ (resp. $X \to \mathbb{P}^2$).

Effective cone

Let X be a weak del Pezzo variety with $\mathrm{r}(X) > 1$.

- Any exceptional class generates an extremal ray of $\mathrm{Eff}(X)$.
- Any $\mathbb{P}^1$- and $\mathbb{P}^2$-class D lies in the boundary of $\mathrm{Eff}(X)$.

Proposition

The effective cone $\mathrm{Eff}(X) \subset \mathrm{Cl}(X)_{\mathbb{R}}$ *is polyhedral and generated by a finite number of extremal rays. Each extremal ray is generated by an exceptional,* $\mathbb{P}^1$*-, or* $\mathbb{P}^2$*-class.*

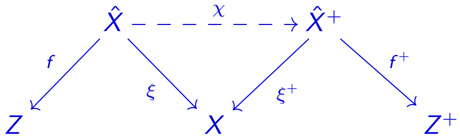
Proposition

The moving cone $\mathrm{Mov}(X) \subset \mathrm{Cl}(X)_{\mathbb{R}}$ *is polyhedral and generated by a finite number of extremal rays. Each extremal ray is generated by a* $\mathbb{P}^1$*-,* $\mathbb{P}^2$*-class, or a big class* D *such that the linear system* $|nD|$, $n \gg 0$ *defines a birational map to a del Pezzo variety* X' *with* $\mathrm{r}(X') = 1$.

Primitive del Pezzos with $\mathrm{r}(X) = 2$

Proposition

Let X be a primitive del Pezzo variety with $n := \dim X$ and $\mathrm{r}(X) = 2$. Then there exists the following diagram:



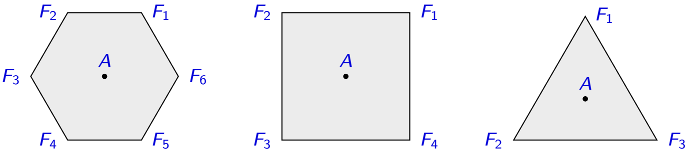
where ξ and ξ^+ are $\mathbb{Q}$-factorializations and f and f^+ are Mori contractions. There are the following possibilities.

f	f^+	$\mathrm{d}(X)$
$\mathbb{P}^{n-2}$*-bundle over* $\mathbb{P}^2$	$\mathbb{P}^{n-2}$*-bundle over* $\mathbb{P}^2$	1,2,3,6
$\mathbb{P}^{n-2}$*-bundle over* $\mathbb{P}^2$	*quadric bundle over* $\mathbb{P}^1$	5
quadric bundle over $\mathbb{P}^1$	*quadric bundle over* $\mathbb{P}^1$	1,2,4

Primitive del Pezzo varieties with $\mathrm{r}(X) = 3$

Proposition

Let X be a primitive del Pezzo variety with $n := \dim X$ and $\mathrm{r}(X) \geq 3$. Then $\mathrm{r}(X) = 3$, any $\mathbb{Q}$-factorialization $\hat{X}$ of X is a $\mathbb{P}^{n-2}$-bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ and $d := \mathrm{d}(X) \in \{2, 4, 6\}$. A transversal section of the cone $\mathrm{Eff}(X)$ of effective divisors has the form



$\mathrm{d}(X) = 2$ $\mathrm{d}(X) = 4$ $\mathrm{d}(X) = 6$

where F_i are $\mathbb{P}^1$-classes.

Corollary

Let X be an (almost) del Pezzo variety. Then $\mathrm{d}(X) + \mathrm{r}(X) \leq 9$.

Primitive del Pezzo varieties

Proposition

Let X be a primitive del Pezzo variety with $\mathrm{r}(X) \geq 2$ and let $\hat{X}$ be its $\mathbb{Q}$-factorialization.

- *If $\hat{X}$ has a structure of $\mathbb{P}^{n-2}$-bundle over $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, then X is of bounded type.*
- *If $\hat{X}$ has a structure of quadric bundle over $\mathbb{P}^1$ and $d \neq 5$, then X is of unbounded type.*
- *If $\hat{X}$ has a structure of quadric bundle over $\mathbb{P}^1$ and $d = 5$, then another $\mathbb{Q}$-factorialization $\hat{X}'$ has a structure of $\mathbb{P}^{n-2}$-bundle and X is of bounded type.*

Del Pezzo varieties of unbounded type satisfy the inequalities

$$\mathrm{d}(X) + \mathrm{r}(X) \leq 6, \quad \mathrm{d}(X) \leq 5.$$

Primitive del Pezzo varieties of the form $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})$

Proposition

Let $\mathscr{E}_{\mathbb{P}^2,k}$ be a maximal primitive del Pezzo bundle on $\mathbb{P}^2$ with $c_2(\mathscr{E}) = k$, $c_1(\mathscr{E}) = K_{\mathbb{P}^2}$ and we write $d = \mathrm{d}(\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}_{\mathbb{P}^2,k})) = 9 - k$. Then

- *if $d = 6$ then $\mathscr{E}_{\mathbb{P}^2,3} \simeq \mathscr{O}(-1)^{\oplus 3}$;*
- *if $d = 5$ then $\mathscr{E}_{\mathbb{P}^2,4} \simeq \mathscr{O}(-1)^{\oplus 2} \oplus \Omega(1)$;*
- *if $d = 3$ then $\mathscr{E}_{\mathbb{P}^2,6} \simeq \Omega(1)^{\oplus 3}$;*
- *if $d = 2$ there is an exact sequence* $0 \to \mathscr{E}_{\mathbb{P}^2,7} \to \mathscr{O}^{\oplus 9} \to \mathscr{O}(1) \oplus \mathscr{O}(2) \to 0$;
- *if $d = 1$ there is an exact sequence* $0 \to \mathscr{E}_{\mathbb{P}^2,8} \to \mathscr{O}^{\oplus 9} \to \mathscr{O}(3) \to \mathscr{O}_p \to 0$, *where $p \in \mathbb{P}^2$ is a point.*

Primitive del Pezzos of the form $\mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathscr{E})$

Proposition

Let $\mathscr{E}_{\mathbb{P}^1\times\mathbb{P}^1,k}$ *be a maximal primitive del Pezzo bundle on* $\mathbb{P}^1\times\mathbb{P}^1$ *with* $c_2(\mathscr{E})=k$, $c_1(\mathscr{E})=K_{\mathbb{P}^1\times\mathbb{P}^1}$ *and we write* $d=\mathrm{d}(\mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathscr{E}_{\mathbb{P}^1\times\mathbb{P}^1,k}))=8-k$. *Then*

- *if* $d=6$ *then* $\mathscr{E}_{\mathbb{P}^1\times\mathbb{P}^1,2}\simeq\mathscr{O}(-1,-1)^{\oplus 2}$;
- *if* $d=4$ *then* $\mathscr{E}_{\mathbb{P}^1\times\mathbb{P}^1,4}\simeq\mathscr{O}(-1,0)^{\oplus 2}\oplus\mathscr{O}(0,-1)^{\oplus 2}$;
- *if* $d=2$ *then* $\mathscr{E}_{\mathbb{P}^1\times\mathbb{P}^1,6}\simeq\Omega_{\mathbb{P}^3}(1)|^{\oplus 2}_{\mathbb{P}^1\times\mathbb{P}^1}$;

Primitive del Pezzos: quadric bundles

Proposition

Let X be a primitive $\mathbb{Q}$-factorial almost del Pezzo variety with $\mathrm{d}(X) = d$ and $\mathrm{r}(X) = 2$ having quadric bundle structure $X \to \mathbb{P}^1$. Then $X \subset \mathbb{P}_{\mathbb{P}^1}(\oplus \mathscr{O}(-a_i))$ is a divisor of type $2A - kF$, where

- *if $d = 5$ then $\dim(X) \leq 5$, $a = (0, \dots, 0, 1, 1, 1)$, $k = 1$;*
- *if $d = 4$ then $a = (0, 0, \dots, 0, 1, 1)$, $k = 0$, and X is a complete intersection in $\mathbb{P}^1 \times \mathbb{P}^{n+2}$ of three divisors of types $(1, 1)$, $(1, 1)$, and $(0, 2)$.*
- *if $d = 2$ then $a = (0, 0, \dots, 0, 0, 0)$, $k = -2$, and X is a divisor in $\mathbb{P}^1 \times \mathbb{P}^n$ of type $(2, 2)$;*
- *if $d = 1$ then $a = (0, 0, \dots, 0, 0, -1)$, $k = -3$, and X is a complete intersection in $\mathbb{P}^1 \times \mathbb{P}^{2n}$ of n divisors of type $(1, 1)$ and one divisor of type $(1, 2)$.*

Case $\mathrm{d}(X) + \mathrm{r}(X) = 9$

The varieties obtained by blowups of $\mathbb{P}^3$ are quite classical. Remind that distinct points $P_1, \ldots, P_k$ are in general position if

- no plane that contains 4 points,
- no quadric is singular at one point and contains 6 others.

The curves contracted by the anticanonical morphism are

- proper transforms of lines through pairs of points;
- proper transforms of twisted cubic curves through sextuples of points.

For$k \geq 2$ the corresponding del Pezzo varieties are

- $d = 6$: 1-nodal hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$;
- $d = 5$: 3-nodal linear section of $\mathrm{Gr}(2, 5)$ of codimension 3;
- $d = 4$: 6-nodal intersection of two quadrics $\{x_1x_2 = x_3x_4 = x_5x_6\} \subset \mathbb{P}^5$;
- $d = 3$: 10-nodal Segre cubic $\{x_1 + \cdots + x_6 = x_1^3 + \cdots + x_6^3 = 0\} \subset \mathbb{P}^5$;
- $d = 2$: 16-nodal quartic double solids ramified over Kummer quartic surfaces;
- $d = 1$: 28-nodal double Veronese cones.

Lattices

An element $\boldsymbol{\alpha} \in \mathrm{Cl}(X)$ is a *root* if

$$\langle A, \boldsymbol{\alpha} \rangle = 0, \qquad \langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle = -2.$$

The set

$$\Delta := \{\boldsymbol{\alpha} \in A^{\perp} \mid \langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle = -2\}$$

of roots in $A^{\perp}$ forms a root system.

Example

Case $\mathrm{d}(X) + \mathrm{r}(X) = 9$.

d	1	2	3	4	5	6	$6'$
Δ	E_7	D_6	A_5	$\mathrm{A}_1 \times \mathrm{A}_3$	A_2	A_1	A_2

Case $\mathrm{d}(X) + \mathrm{r}(X) = 8$.

d	1	2	3	4	5	6
Δ	E_6	A_5	$2\mathrm{A}_2$	$2\mathrm{A}_1$	A_1	A_1

Geography

Theorem

Let X be a del Pezzo variety of dimension $n \geq 3$. Let $i\colon S \hookrightarrow X$ be a general surface linear section of X. The restriction map

$$\mathrm{Cl}(X) \xrightarrow{i^*} \mathrm{Cl}(S)$$

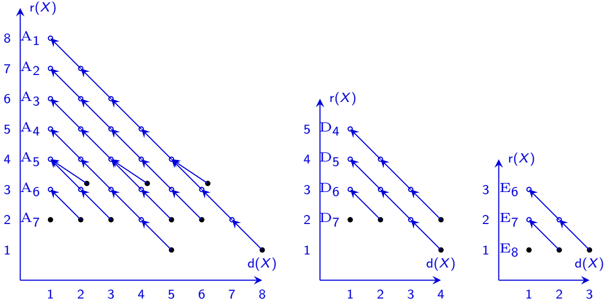
is a linear isomorphism $\mathrm{Cl}(X) \cong \Xi(X)^{\perp}$ onto the orthogonal complement of a negative definite sublattice $\Xi(X) \subset K_S^{\perp} \subset \mathrm{Cl}(S)$ of rank $m = 10 - \mathrm{d}(X) - \mathrm{r}(X)$ which has Dynkin type

- A_m, $1 \leq m \leq 7$, *or*
- D_m, $4 \leq m \leq 7$, *or*
- E_m, $6 \leq m \leq 8$.

In the case A_m we have $\dim X \leq 12 - \mathrm{d}(X) - \mathrm{r}(X)$, types D_m and E_m are unbounded.

Geography

The three “maps” of this “atlas” correspond to varieties of types A, D, and E, respectively. Black dots stand for primitive varieties and white dots for imprimitive varieties, and the arrows correspond to the operation of blowup of a general point (which decreases $\mathsf{d}(X)$ by 1 and increases $\mathsf{r}(X)$ by 1, keeping their sum constant), followed by passing to the anticanonical model.



Classification

Theorem

Let X be a del Pezzo variety of dimension $n \geq 3$.

- $\Xi(X)$ *has type* $\mathrm{A}_m \iff X \cong \mathbb{P}^3$, *or* $X \cong \mathrm{Gr}(2,5) \cap \mathbb{P}^{n+3}$, *or*

$$X \overset{\text{pseudo}}{\leftsquigarrow\rightsquigarrow} \mathbb{P}_Z(\mathscr{E}),$$

- $\Xi(X)$ *has type* $\mathrm{D}_m \iff X = X_{2\cdot 2} \subset \mathbb{P}^{n+2}$ *with* $\mathrm{r}(X) = 1$, *or*

$$X \overset{\text{pseudo}}{\leftsquigarrow\rightsquigarrow} \mathrm{Bl}_{P_1,\dots,P_{r-2}}(X_0) \; .$$

where X_0 *is an almost del Pezzo variety with* $\mathrm{r}(X_0) = 2$ *having quadric bundle structure.*

- $\Xi(X)$ *has type* $\mathrm{E}_m \iff$

$$X \overset{\text{pseudo}}{\leftsquigarrow\rightsquigarrow} \mathrm{Bl}_{P_1,\dots,P_{r-1}}(X_0)$$

where X_0 *is a del Pezzo variety with* $\mathrm{r}(X_0) = 1$ *and* $\mathrm{d}(X_0) \leq 3$*.*

Double Veronese cones

Theorem

There is a bijection between the sets of isomorphism classes of

- *maximal del Pezzo varieties* X *of dimension* $n \neq 9$ *with* $\mathrm{d}(X) = 1$, *and*
- *smooth del Pezzo surfaces* S *of degree* $n - 1$,

defined by the rules

$$X \longmapsto \Delta(X)^\vee \qquad \text{and} \qquad S \longmapsto X_{S,L}.$$

In particular, $X_{S,L} \simeq X_{S,L'}$ *for any pair of lines* $L, L' \subset S$. *Moreover, there is a pseudoisomorphism* $\mathrm{Bl}_P(X) \overset{\text{pseu}}{\sim} \mathrm{H}(S)$ *over* $\mathbb{P}(W^\vee)$.