Recent progress on rationality

Alena Pirutka

Courant Institute, New York University

June 22, 2022

ADVANCES IN ALGEBRA AND APPLICATIONS

MINSK, JUNE 22-26

Goal

Goal: overview of a recent (2013-2022) progress on rationality questions, after

> *C. Voisin, J.-L. Colliot-Thélène, A. Pirutka, B. Totaro, A. Beauville, B. Hassett, A. Kresch, Y. Tschinkel, A. Chatzistamatiou, M. Levine, A. Auel, C. Böhning, H.-C. Graf von Bothmer, S. Schreieder, H. Ahmadinezhad, T. Okada, J. Nicaise, E. Shinder, M. Kontsevich, I.Krylov, J.-C. Ottem et al.*

Plan

Plan

1. Reminders on «rationality», classical examples.

Plan

1. Reminders on «rationality», classical examples.
2. Recent progress, specialization method.

Plan

1. Reminders on «rationality», classical examples.
2. Recent progress, specialization method.
3. Examples of computation of some birational invariants in the tradition of Belarusian school: Brauer group, conic and quadric bundles (V.I. Yanchevskiĭ, V.P. Platonov, S.V. Tikhonov, D.F. Bazyleu, and others).

Reminders on «rationality»

Objects/questions of interest

- X a (projective) algebraic variety over a field k;
- For this talk: X_d **a hypersurface of degree** d: $f(x_0, \ldots, x_n) = 0, f$ homogeneous of degree d;;

Objects/questions of interest

- X a (projective) algebraic variety over a field k;
- For this talk: X_d **a hypersurface of degree** d: $f(x_0, \ldots, x_n) = 0, f$ homogeneous of degree d;;
- Question: find or parametrize all the «soltions».

I.I. Voronovich example:

Look at $x, y, z \in \mathbb{Z}$ integers with $x^2 + y^2 - z^2 = 0$
(a **Pythagorean triple**).

I.I. Voronovich example:

Look at $x, y, z \in \mathbb{Z}$ integers with $x^2 + y^2 - z^2 = 0$
(**a Pythagorean triple**).
Trivial solutions: $x = y = z = 0$

I.I. Voronovich example:

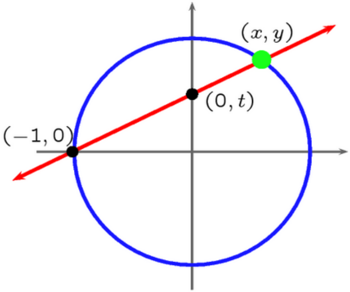
Look at $x, y, z \in \mathbb{Z}$ integers with $x^2 + y^2 - z^2 = 0$
(**a Pythagorean triple**).
Trivial solutions: $x = y = z = 0$
Nontrivial solutions: $x^2 + y^2 - z^2 = 0 \Leftrightarrow (\frac{x}{z})^2 + (\frac{y}{z})^2 = 1$

I.I. Voronovich example:

Look at $x, y, z \in \mathbb{Z}$ integers with $x^2 + y^2 - z^2 = 0$
(**a Pythagorean triple**).
Trivial solutions: $x = y = z = 0$
Nontrivial solutions: $x^2 + y^2 - z^2 = 0 \Leftrightarrow (\frac{x}{z})^2 + (\frac{y}{z})^2 = 1$
$\Leftrightarrow x^2 + y^2 = 1$ with $(x, y) = (\frac{x}{z}, \frac{y}{z})$ rational points on a circle

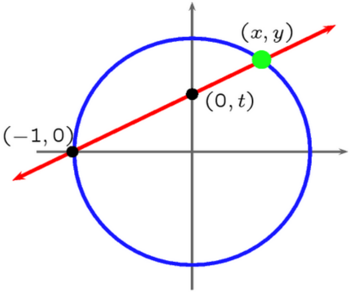
I.I. Voronovich example:

Look at $x, y, z \in \mathbb{Z}$ integers with $x^2 + y^2 - z^2 = 0$
(**a Pythagorean triple**).
Trivial solutions: $x = y = z = 0$
Nontrivial solutions: $x^2 + y^2 - z^2 = 0 \Leftrightarrow (\frac{x}{z})^2 + (\frac{y}{z})^2 = 1$
$\Leftrightarrow x^2 + y^2 = 1$ with $(x, y) = (\frac{x}{z}, \frac{y}{z})$ rational points on a circle



I.I. Voronovich example:

Look at $x, y, z \in \mathbb{Z}$ integers with $x^2 + y^2 - z^2 = 0$
(**a Pythagorean triple**).
Trivial solutions: $x = y = z = 0$
Nontrivial solutions: $x^2 + y^2 - z^2 = 0 \Leftrightarrow (\frac{x}{z})^2 + (\frac{y}{z})^2 = 1$
$\Leftrightarrow x^2 + y^2 = 1$ with $(x, y) = (\frac{x}{z}, \frac{y}{z})$ rational points on a circle



$\Leftrightarrow$ lines through $(-1, 0)$ and (x, y)
with rational slope t.

Conclusion: rational points on a circle

$$\{x^2 + y^2 = 1, x, y \in \mathbb{Q}\} \leftarrow \mathbb{Q} \overset{def}{=} \mathbb{A}^1_{\mathbb{Q}}$$

$$(x, y) = (\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2}) \leftarrow t$$

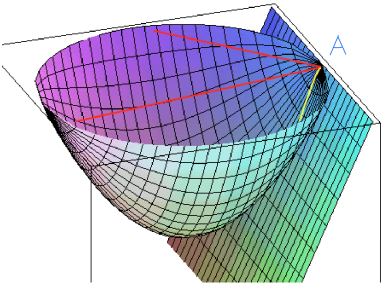
is a **rational** parametrization.

Quadric hypersurfaces

Q is defined by $q(x_0, \ldots, x_n) = 0$, homogeneous of degree 2.

Quadric hypersurfaces

Q is defined by $q(x_0, \ldots, x_n) = 0$, homogeneous of degree 2.
Assume: there is a point $A \in Q$.



Rational parametrization:
nontangent lines through $A \to$ second intersection point with Q

Rational, unirational, and stably rational varieties

- X is **rational** if there is an almost everywhere defined isomorphism:

$$k^n = \mathbb{A}^n_k \overset{\sim}{\dashrightarrow} X,$$

i.e. X is, intrinsically, a linear space.

Rational, unirational, and stably rational varieties

- X is **rational** if there is an almost everywhere defined isomorphism:

$$k^n = \mathbb{A}^n_k \overset{\sim}{\dashrightarrow} X,$$

i.e. X is, intrinsically, a linear space.

Examples:

- A smooth quadric with a point is rational (*geometric construction*).

Rational, unirational, and stably rational varieties

- X is **rational** if there is an almost everywhere defined isomorphism:

$$k^n = \mathbb{A}_k^n \overset{\sim}{\dashrightarrow} X,$$

i.e. X is, intrinsically, a linear space.

Examples:

- A smooth quadric with a point is rational (*geometric construction*).
- A smooth cubic (of dimension at least 2) with a point is unirational (Kollár):

- X is **unirational** if there is an almost everywhere defined *dominant* map:

$$\mathbb{A}_k^n \dashrightarrow X.$$

Rational, unirational, and stably rational varieties

- X is **rational** if there is an almost everywhere defined isomorphism:

  $$k^n = \mathbb{A}^n_k \overset{\sim}{\dashrightarrow} X,$$

  i.e. X is, intrinsically, a linear space.

  Examples:
  - A smooth quadric with a point is rational (*geometric construction*).
  - A smooth cubic (of dimension at least 2) with a point is unirational (Kollár):

- X is **unirational** if there is an almost everywhere defined *dominant* map:

  $$\mathbb{A}^n_k \dashrightarrow X.$$

- X is **stably rational** if $X \times \mathbb{P}^n_k$ is rational, for some $n \geq 1$.

Rationality and unirationality

A classical problem: find unirational nonrational varieties defined by polynomials with complex coefficients.

Rationality and unirationality

A classical problem: find unirational nonrational varieties defined by polynomials with complex coefficients.

- (Lüroth, 1875) A curve X is unirational $\Leftrightarrow$ X is rational;

Rationality and unirationality

A classical problem: find unirational nonrational varieties defined by polynomials with complex coefficients.

- (Lüroth, 1875) A curve X is unirational $\Leftrightarrow$ X is rational;
- (Castelnuovo-Enriques, 1890-1900) A surface X is unirational $\Leftrightarrow$ X is rational;

Rationality and unirationality

A classical problem: find unirational nonrational varieties defined by polynomials with complex coefficients.

- (Lüroth, 1875) A curve X is unirational $\Leftrightarrow$ X is rational;
- (Castelnuovo-Enriques, 1890-1900) A surface X is unirational $\Leftrightarrow$ X is rational;
- (Fano, 1908, 1915, 1947) There are unirational nonrational threefolds and fourfolds:

Rationality and unirationality

A classical problem: find unirational nonrational varieties defined by polynomials with complex coefficients.

- (Lüroth, 1875) A curve X is unirational $\Leftrightarrow$ X is rational;
- (Castelnuovo-Enriques, 1890-1900) A surface X is unirational $\Leftrightarrow$ X is rational;
- (Fano, 1908, 1915, 1947) There are unirational nonrational threefolds and fourfolds: but all proofs had gaps!

Rationality and unirationality of complex varieties

- 1970th: examples of unirational nonrational threefolds:

Rationality and unirationality of complex varieties

- 1970th: examples of unirational nonrational threefolds:
  - (Clemens-Griffiths): cubics;

Rationality and unirationality of complex varieties

- 1970th: examples of unirational nonrational threefolds:
    - (Clemens-Griffiths): cubics;
    - (Iskovskikh-Manin): quartics;

Rationality and unirationality of complex varieties

- 1970th: examples of unirational nonrational threefolds:
  - (Clemens-Griffiths): cubics;
  - (Iskovskikh-Manin): quartics;
  - (Artin-Mumford): conic bundles;

Rationality and unirationality of complex varieties

- 1970th: examples of unirational nonrational threefolds:
  - (Clemens-Griffiths): cubics;
  - (Iskovskikh-Manin): quartics;
  - (Artin-Mumford): conic bundles;
- (Kollár, 1990th): a very general hypersurface X_d of dimension n is not rational if $d \geq 2\lceil \frac{n+3}{3} \rceil$.

Note: «very general» is outside of a countable union of proper closed conditions.

Irrationality proofs

- How to establish the failure of rationality of a variety X?

Irrationality proofs

- How to establish the failure of rationality of a variety X?
- Classical methods (1970th):
  - compute some invariant $i(X)$;
  - $i(X) \neq 0 \Rightarrow X$ is irrational.

Irrationality proofs

- How to establish the failure of rationality of a variety X?
- Classical methods (1970th):
  - compute some invariant $i(X)$;
  - $i(X) \neq 0 \Rightarrow X$ is irrational.
- Specialization method (2014, Voisin, Colliot-Thélène–P.):
  - consider a family of varieties, specializing X to (a possibly singular) X_0;
  - compute some invariant $i(X_0)$;
  - $i(X_0) \neq 0 \Rightarrow X$ is irrational

Irrationality proofs

- How to establish the failure of rationality of a variety X?
- Classical methods (1970th):
  - compute some invariant $i(X)$;
  - $i(X) \neq 0 \Rightarrow X$ is irrational.
- Specialization method (2014, Voisin, Colliot-Thélène–P.):
  - consider a family of varieties, specializing X to (a possibly singular) X_0;
  - compute some invariant $i(X_0)$;
  - $i(X_0) \neq 0 \Rightarrow X$ is irrational, in some cases, all previously computable $i(X)$ vanish.

Implementation of the specialization method

- Key invariant:

  $CH_0(X)$, the Chow group of zero-cycles on X;

- Key property: if X is rational, then $CH_0(X_L) = \mathbb{Z}$, «*universally*» for any field L.

Implementation of the specialization method

(Voisin, Colliot-Thélène-P. 2014)

$X \to B$ a family of complex projective varieties, $0 \in B$.

Implementation of the specialization method

(Voisin, Colliot-Thélène-P. 2014)

$X \to B$ a family of complex projective varieties, $0 \in B$.

- Assume that:
  - X_0 has mild singularities;
  - $i(X_0) \neq 0$ (examples later).

Implementation of the specialization method

(Voisin, Colliot-Thélène-P. 2014)

$X \to B$ a family of complex projective varieties, $0 \in B$.

- Assume that:
  - X_0 has mild singularities;
  - $i(X_0) \neq 0$ (examples later).
- Then:
  - $CH_0(X_t) \neq \mathbb{Z}$ (universally), for t very general;
  - hence X_t is not rational.

RECENT PROGRESS

Applications of the specialization method

For this talk: case of smooth hypersurfaces X_d of dimension n.

Applications of the specialization method

For this talk: case of smooth hypersurfaces X_d of dimension n.
A very general X_d is not rational:

	n=3	n=4	n=5
$d = 3$	Clemens-Griffiths	open	open
$d = 4$	Colliot-Thélène-Pirutka	Totaro	Nicaise-Ottem
$d = 5$		Kollár	Schreieder

Applications of the specialization method

For this talk: case of smooth hypersurfaces X_d of dimension n.
A very general X_d is not rational:

	n=3	n=4	n=5
$d = 3$	Clemens-Griffiths	open	open
$d = 4$	Colliot-Thélène-Pirutka	Totaro	Nicaise-Ottem
$d = 5$		Kollár	Schreieder

In general: a very general X_d is not (stably) rational if

1. (Totaro, 2016) $d \geq 2\lceil \frac{n+2}{3} \rceil$;
2. (Schreieder, 2019) $d \geq log_2 n + 2$.

Global properties of rationality

- (Hassett-Tschinkel-P., 2016)
  Rationality is not a deformation invariant in smooth families of complex projective varieties (specialization method).

Global properties of rationality

- (Hassett-Tschinkel-P., 2016)
  Rationality is not a deformation invariant in smooth families of complex projective varieties (specialization method).
- (Kontsevich-Tschinkel, Nicaise-Shinder, 2017)
  Rationality is a specialization invariant! (invariants of motivic nature)

COMPUTATION

Reminders on $H^i(K, \mathbb{Z}/2)$ and residues

- $H^1(K, \mathbb{Z}/2) \simeq K^*/K^{*2}$ (Kummer),
  for $a \in K^*$, we wil still denote by a its class in $H^1(K, \mathbb{Z}/2)$.
- $Br(K)[2] = H^2(K, \mathbb{Z}/2)$ (Kummer);
  symbols: $(a, b) := a \cup b \in H^2(K, \mathbb{Z}/2), a, b \in K^*$.

Reminders on $H^i(K,\mathbb{Z}/2)$ and residues

- $H^1(K,\mathbb{Z}/2) \simeq K^*/K^{*2}$ (Kummer),
  for $a \in K^*$, we wil still denote by a its class in $H^1(K,\mathbb{Z}/2)$.
- $Br(K)[2] = H^2(K,\mathbb{Z}/2)$ (Kummer);
  symbols: $(a,b) := a \cup b \in H^2(K,\mathbb{Z}/2), a, b \in K^*$.
- Similarly: $a_1 \cup \cdots \cup a_i \in H^i(K,\mu_n^{\otimes i})$, where
  $a_1,\ldots a_i \in H^1(K,\mu_n) \simeq K^*/K^{*n}$.
- Any element is a sum of symbols (Bloch-Kato conjecture: Merkurjev, Merkurjev-Suslin, Voevodsky).

Residues

- $v : K \to \mathbb{Z} \cup \infty$ a discrete valuation of rank 1:
  Recall: $v(x) = \infty \Leftrightarrow x = 0$
  $v(xy) = v(x) + v(y)$
  $v(x + y) \geq min(v(x), v(y))$
- A be the valuation ring: $A = \{x, v(x) \geq 0\}$,
- $\kappa(v)$ the residue field: $\kappa(v) = A/m$,
  $m = \{x, v(x) > 0\} = (\pi_A)$, π_A is a uniformizer

Residues

- $v : K \to \mathbb{Z} \cup \infty$ a discrete valuation of rank 1:
  Recall: $v(x) = \infty \Leftrightarrow x = 0$
  $v(xy) = v(x) + v(y)$
  $v(x + y) \geq min(v(x), v(y))$
- A be the valuation ring: $A = \{x, v(x) \geq 0\}$,
- $\kappa(v)$ the residue field: $\kappa(v) = A/m$,
  $m = \{x, v(x) > 0\} = (\pi_A)$, π_A is a uniformizer
- this gives $\partial_v^i : H^i(K, \mathbb{Z}/2) \to H^{i-1}(\kappa(v), \mathbb{Z}/2)$,
- $\partial_v^1(a) = v(a) \bmod 2 \in H^0(\kappa(v), \mathbb{Z}/2) \simeq \mathbb{Z}/2$,

$$\partial_v^2(a, b) = (-1)^{v(a)v(b)} \overline{\frac{a^{v(b)}}{b^{v(a)}}}$$

where $\overline{\frac{a^{v(b)}}{b^{v(a)}}}$ is the image of the unit $\frac{a^{v(b)}}{b^{v(a)}}$ in $\kappa(v)^*/\kappa(v)^{*2}$.

Residues

- $v : K \to \mathbb{Z} \cup \infty$ a discrete valuation of rank 1:
  Recall: $v(x) = \infty \Leftrightarrow x = 0$
  $v(xy) = v(x) + v(y)$
  $v(x + y) \geq min(v(x), v(y))$
- A be the valuation ring: $A = \{x, v(x) \geq 0\}$,
- $\kappa(v)$ the residue field: $\kappa(v) = A/m$,
  $m = \{x, v(x) > 0\} = (\pi_A)$, π_A is a uniformizer
- this gives $\partial_v^i : H^i(K, \mathbb{Z}/2) \to H^{i-1}(\kappa(v), \mathbb{Z}/2)$,
- $\partial_v^1(a) = v(a) \bmod 2 \in H^0(\kappa(v), \mathbb{Z}/2) \simeq \mathbb{Z}/2$,

$$\partial_v^2(a, b) = (-1)^{v(a)v(b)} \overline{\frac{a^{v(b)}}{b^{v(a)}}}$$

  where $\overline{\frac{a^{v(b)}}{b^{v(a)}}}$ is the image of the unit $\frac{a^{v(b)}}{b^{v(a)}}$ in $\kappa(v)^*/\kappa(v)^{*2}$.
- In particular, $\partial_v^2(a, b) = 0$ is a, b are units in A (i.e. $v(a) = v(b) = 0$).

Properties of residues: compatibility

- Setting:
  - $A \subset B$ be discrete valuation rings
  - $K \subset L$ fields of fractions
  - e is the valuation of the uniformizer π_A in B.

Properties of residues: compatibility

- Setting:
  - $A \subset B$ be discrete valuation rings
  - $K \subset L$ fields of fractions
  - e is the valuation of the uniformizer π_A in B.
- Compatibility:

$$\begin{array}{ccc} H^i(L, \mathbb{Z}/2) & \xrightarrow{\partial^i_{v_B}} & H^{i-1}(\kappa(B), \mathbb{Z}/2) \\ \uparrow & & \uparrow e \\ H^i(K, \mathbb{Z}/2) & \xrightarrow{\partial^i_{v_A}} & H^{i-1}(\kappa(A), \mathbb{Z}/2) \end{array}$$

the vertical arrows are the restriction maps in Galois cohomology.

H^i_{nr}: definition

Definition

X/k an integral variety, then

$$H^2_{nr}(X) = H^2_{nr}(k(X)/k) = \cap_v \mathrm{Ker}\partial^2_v$$

where the intersection is over all discrete valuations v on $k(X)$ (of rank one), trivial on the field k.

H^i_{nr}: definition

Definition

X/k an integral variety, then

$$H^2_{nr}(X) = H^2_{nr}(k(X)/k) = \cap_v \mathrm{Ker}\partial^2_v$$

where the intersection is over all discrete valuations v on $k(X)$ (of rank one), trivial on the field k.

Birational invariant by definition (Saltman, Bogomolov, Colliot-Thélène-Ojanguren).
Advantage: No need to compute a smooth model of X!

H^i_{nr}: definition

Definition

X/k an integral variety, then

$$H^2_{nr}(X) = H^2_{nr}(k(X)/k) = \cap_v \mathrm{Ker}\partial^2_v$$

where the intersection is over all discrete valuations v on $k(X)$ (of rank one), trivial on the field k.

Birational invariant by definition (Saltman, Bogomolov, Colliot-Thélène-Ojanguren).
Advantage: No need to compute a smooth model of X!
Fact: if X is smooth and projective, $H^2_{nr}(X) \simeq Br(X)[2]$.

Similarly: $H^i_{nr}(k(X)/k) = \cap_v \mathrm{Ker}\partial^i_v$.

$H^i_{nr}(k(X)/k) = \cap_v \mathrm{Ker}\partial^i_v$: applications

1. X/k is a stably rational ($X \times \mathbb{P}^n_k$ is rational for some n): $H^i(k) \simeq H^i_{nr}(k(X)/k)$, ex: $H^i_{nr}(k(X)/k) = 0$, if $k = \mathbb{C}$.

$H^i_{nr}(k(X)/k) = \cap_v \mathrm{Ker}\partial^i_v$: applications

1. X/k is a stably rational ($X \times \mathbb{P}^n_k$ is rational for some n): $H^i(k) \simeq H^i_{nr}(k(X)/k)$, ex: $H^i_{nr}(k(X)/k) = 0$, if $k = \mathbb{C}$.
2. Failure of stable rationality (specialization method): $X \to B/\mathbb{C}$ projective, generically smooth, $0 \in B$. If
   - $H^i_{nr}(X_0) \neq 0$;
   - $+$ properties of X_0 or $\alpha \in H^2_{nr}(X_0)$: mild singularities, or (easier) restrictions of α (Schreieder)

   Then X_b is not stably rational for a very general $b \in B(\mathbb{C})$.

$H^i_{nr}(k(X)/k) = \cap_v \mathrm{Ker}\partial^i_v$: applications

1. X/k is a stably rational ($X \times \mathbb{P}^n_k$ is rational for some n): $H^i(k) \simeq H^i_{nr}(k(X)/k)$, ex: $H^i_{nr}(k(X)/k) = 0$, if $k = \mathbb{C}$.
2. Failure of stable rationality (specialization method): $X \to B/\mathbb{C}$ projective, generically smooth, $0 \in B$. If
   - $H^i_{nr}(X_0) \neq 0$;
   - $+$ properties of X_0 or $\alpha \in H^2_{nr}(X_0)$: mild singularities, or (easier) restrictions of α (Schreieder)

   Then X_b is not stably rational for a very general $b \in B(\mathbb{C})$.
3. H^3_{nr} is related to properties of CH^2 (Colliot-Thélène, Voisin, Kahn).

H^1_{nr}: easy example

- Recall: $H^1_{nr}(k(X)/k) = \cap_v \mathrm{Ker} \partial^1_v$,
- $\partial^1_v = v$ mod 2.

H^1_{nr}: easy example

- Recall: $H^1_{nr}(k(X)/k) = \cap_v \mathrm{Ker} \partial^1_v$,
- $\partial^1_v = v \bmod 2$.
- X is an elliptic curve/$\mathbb{C}$, example: $y^2 = x(x-1)(x+1)$.

H^1_{nr}: easy example

- Recall: $H^1_{nr}(k(X)/k) = \cap_v \mathrm{Ker}\partial^1_v$,
- $\partial^1_v = v \bmod 2$.
- X is an elliptic curve/$\mathbb{C}$, example: $y^2 = x(x-1)(x+1)$.
- Then $x \in H^1_{nr}(\mathbb{C}(X)/\mathbb{C})$:
  - $v(x) \neq 0 \Rightarrow v(x-1) = v(x+1) = 0$,
  - so $v(x) = v(y^2) = 2v(y)$.

H^i_{nr} for quadrics

- Recall: $H^i_{nr}(k(X)/k) = \cap_v \mathrm{Ker}\partial^i_v$, v are trivial on k.
- We have $\tau_i : H^i(k, \mathbb{Z}/2) \to H^i_{nr}(k(X)/k) \subset H^i(k(X), \mathbb{Z}/2)$.
- τ_i is an isomorphism if X is rational.

H^i_{nr} for quadrics

- Recall: $H^i_{nr}(k(X)/k) = \cap_v \mathrm{Ker}\partial^i_v$, v are trivial on k.
- We have $\tau_i : H^i(k, \mathbb{Z}/2) \to H^i_{nr}(k(X)/k) \subset H^i(k(X), \mathbb{Z}/2)$.
- τ_i is an isomorphism if X is rational.
- τ_2 is an isomorphism if X is a quadric surface with nonzero discriminant (Arason):
  - $Q \subset \mathbb{P}^{n-1}_k$ is a quadric $q = a_1x_1^2 + \ldots + a_nx_n^2 = 0$, write $q \simeq \langle a_1, \ldots, a_n \rangle$ is a nondegenerate quadratic form, $n = dim\, q$.
  - cases of interest: $dim\, q = 3$ (a conic) or $dim\, q = 4$ (a quadric surface).
  - the **discriminant** of q is the class $disc(q) = (-1)^{n(n-1)/2} a_1 \ldots a_n$ in $H^1(k, \mathbb{Z}/2) = k^*/k^{*2}$.

More formulas for $\tau_i : H^i(K, \mathbb{Z}/2) \to H^i_{nr}(K(Q)/K, \mathbb{Z}/2)$

- $ker\, \tau_1 = (a),\ q \simeq \langle 1, -a \rangle$ generated by a
- $ker\, \tau_1 = 0,\ dim(q) > 2$
- $ker\, \tau_2 = (a, b),\ q \simeq < 1, -a, -b >$
- $ker\, \tau_2 = (a, b),\ q \simeq < 1, -a, -b, ab >$ a Pfister form
- τ_i for $i \geq 3$: Kahn, Rost, Sujatha.

Next

Cohomological properties of function fields of quadrics make computation of unramified cohomology of a total space of a fibration in quadrics more tractable.

Fibrations in quadrics: strategy (Colliot-Thélène-Ojanguren)

- Example: $X \to \mathbb{P}^2_{\mathbb{C}}$ with generic fiber $X_\eta \simeq Q = \{q = 0\}$ a quadric over $K = \mathbb{C}(\mathbb{P}^2)$;
- Note: $\mathbb{C}(X) = K(Q)$
- Note: $H^2_{nr}(K(Q)/K) \supset H^2_{nr}(\mathbb{C}(X)/\mathbb{C})$
  $(\Leftrightarrow \cap_{v|_K=0} Ker\partial^2_v \supset \cap_v Ker\partial^2_v \ )$

Fibrations in quadrics: strategy (Colliot-Thélène-Ojanguren)

- Example: $X \to \mathbb{P}^2_{\mathbb{C}}$ with generic fiber $X_\eta \simeq Q = \{q = 0\}$ a quadric over $K = \mathbb{C}(\mathbb{P}^2)$;
- Note: $\mathbb{C}(X) = K(Q)$
- Note: $H^2_{nr}(K(Q)/K) \supset H^2_{nr}(\mathbb{C}(X)/\mathbb{C})$ $(\Leftrightarrow \cap_{v|_K=0} Ker\partial^2_v \supset \cap_v Ker\partial^2_v\ )$
- Case of interest for the talk: Q is a quadric surface with nonzero discriminant.

Fibrations in quadrics: strategy (Colliot-Thélène-Ojanguren)

- Example: $X \to \mathbb{P}^2_{\mathbb{C}}$ with generic fiber $X_\eta \simeq Q = \{q = 0\}$ a quadric over $K = \mathbb{C}(\mathbb{P}^2)$;
- Note: $\mathbb{C}(X) = K(Q)$
- Note: $H^2_{nr}(K(Q)/K) \supset H^2_{nr}(\mathbb{C}(X)/\mathbb{C})$ $(\Leftrightarrow \cap_{v|_K=0} Ker\partial^2_v \supset \cap_v Ker\partial^2_v\ )$
- Case of interest for the talk: Q is a quadric surface with nonzero discriminant.
- We know:

$$\tau_2 : H^2(K, \mathbb{Z}/2) \simeq H^2_{nr}(K(Q)/K, \mathbb{Z}/2)$$

- Unramified over $\mathbb{C}$? What is $Im\, \tau_2 \cap H^2_{nr}(\mathbb{C}(X)/\mathbb{C})$?

Types of valuations

1. Data:
   - $X \to \mathbb{P}^2_{\mathbb{C}}$ with $X_\eta = Q/K$, $K = \mathbb{C}(\mathbb{P}^2)$ a quadric
   - $\alpha \in Im\tau_2$
     $\tau_2 : H^2(K, \mathbb{Z}/2) \to H^2_{nr}(K(Q)/K) \subset H^2(K(Q), \mathbb{Z}/2) = H^2(\mathbb{C}(X), \mathbb{Z}/2)$.

Types of valuations

1. Data:
   - $X \to \mathbb{P}^2_{\mathbb{C}}$ with $X_\eta = Q/K$, $K = \mathbb{C}(\mathbb{P}^2)$ a quadric
   - $\alpha \in Im\tau_2$
     $\tau_2 : H^2(K, \mathbb{Z}/2) \to H^2_{nr}(K(Q)/K) \subset H^2(K(Q), \mathbb{Z}/2) = H^2(\mathbb{C}(X), \mathbb{Z}/2)$.
2. Valuation data:
   - v on $\mathbb{C}(X)$, induces v_K on K via $K \subset K(Q)$;
   - A the valuation ring of v.
   - The rational map $\operatorname{Spec} A \dashrightarrow \operatorname{Spec} K(Q) \to \operatorname{Spec} K$ extends to $\operatorname{Spec} A \to \mathbb{P}^2$

Types of valuations

1. Data:
   - $X \to \mathbb{P}^2_{\mathbb{C}}$ with $X_\eta = Q/K$, $K = \mathbb{C}(\mathbb{P}^2)$ a quadric
   - $\alpha \in Im\tau_2$
     $\tau_2 : H^2(K, \mathbb{Z}/2) \to H^2_{nr}(K(Q)/K) \subset H^2(K(Q), \mathbb{Z}/2) = H^2(\mathbb{C}(X), \mathbb{Z}/2)$.
2. Valuation data:
   - v on $\mathbb{C}(X)$, induces v_K on K via $K \subset K(Q)$;
   - A the valuation ring of v.
   - The rational map $\operatorname{Spec} A \dashrightarrow \operatorname{Spec} K(Q) \to \operatorname{Spec} K$ extends to $\operatorname{Spec} A \to \mathbb{P}^2$
   - the **center** of v in $\mathbb{P}^2$:
     $x_v \in \mathbb{P}^2$ the image of the closed $\kappa(v)$-point of $\operatorname{Spec} A$
   - Cases to consider:
     1. x_v is the generic point of $\mathbb{P}^2_{\mathbb{C}}$, then v is trivial on K: $\partial^2_v(\alpha) = 0$ $(\partial^2_v(a, b) = (-1)^{v(a)v(b)} \overline{\frac{a^{v(b)}}{b^{v(a)}}})$.

Types of valuations

1. Data:
   - $X \to \mathbb{P}^2_{\mathbb{C}}$ with $X_\eta = Q/K$, $K = \mathbb{C}(\mathbb{P}^2)$ a quadric
   - $\alpha \in Im\tau_2$
     $\tau_2 : H^2(K, \mathbb{Z}/2) \to H^2_{nr}(K(Q)/K) \subset H^2(K(Q), \mathbb{Z}/2) = H^2(\mathbb{C}(X), \mathbb{Z}/2)$.
2. Valuation data:
   - v on $\mathbb{C}(X)$, induces v_K on K via $K \subset K(Q)$;
   - A the valuation ring of v.
   - The rational map $\operatorname{Spec} A \dashrightarrow \operatorname{Spec} K(Q) \to \operatorname{Spec} K$ extends to $\operatorname{Spec} A \to \mathbb{P}^2$
   - the **center** of v in $\mathbb{P}^2$:
     $x_v \in \mathbb{P}^2$ the image of the closed $\kappa(v)$-point of $\operatorname{Spec} A$
   - Cases to consider:
     1. x_v is the generic point of $\mathbb{P}^2_{\mathbb{C}}$, then v is trivial on K: $\partial^2_v(\alpha) = 0$ $(\partial^2_v(a, b) = (-1)^{v(a)v(b)} \overline{\frac{a^{v(b)}}{b^{v(a)}}})$.
     2. x_v is the generic point of a curve $C_v \subset \mathbb{P}^2$
     3. x_v is a closed point $P_v \in \mathbb{P}^2$.

Properties of residues: local rings and completion

1. Gersten conjecture for A a dvr:

$$0 \to H^i_{ét}(A, \mathbb{Z}/2) \to H^i(K, \mathbb{Z}/2) \xrightarrow{\partial^i_v} H^{i-1}(\kappa(v), \mathbb{Z}/2) \to 0$$

Properties of residues: local rings and completion

1. Gersten conjecture for A a dvr:

$$0 \to H^i_{\acute{e}t}(A, \mathbb{Z}/2) \to H^i(K, \mathbb{Z}/2) \xrightarrow{\partial^i_v} H^{i-1}(\kappa(v), \mathbb{Z}/2) \to 0$$

2. Completion: ∂^i_v factorizes through the completion K_v as

$$\partial^i_v : H^i(K, \mathbb{Z}/2) \to H^i(K_v, \mathbb{Z}/2) \to H^{i-1}(\kappa(v), \mathbb{Z}/2).$$

Properties of residues: local rings and completion

1. Gersten conjecture for A a dvr:

$$0 \to H^i_{ét}(A, \mathbb{Z}/2) \to H^i(K, \mathbb{Z}/2) \xrightarrow{\partial^i_v} H^{i-1}(\kappa(v), \mathbb{Z}/2) \to 0$$

2. Completion: ∂^i_v factorizes through the completion K_v as

$$\partial^i_v : H^i(K, \mathbb{Z}/2) \to H^i(K_v, \mathbb{Z}/2) \to H^{i-1}(\kappa(v), \mathbb{Z}/2).$$

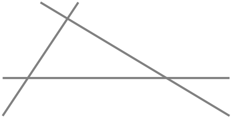
Application: If a is a square in K_v, then $(a, b) = 0$ in K_v $\Rightarrow \partial^2_v(a, b) = 0$.

Example

- $X \to \mathbb{P}^2_{\mathbb{C}}$ defined by $yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$.
- $X_\eta \simeq Q = \{q = 0\}$ a quadric over $K = \mathbb{C}(\mathbb{P}^2)$;
- $\alpha = (x, y) \in Im\tau_2$, where
  $\tau_2 : H^2(K, \mathbb{Z}/2) \to H^2(K(Q), \mathbb{Z}/2) = H^2(\mathbb{C}(X), \mathbb{Z}/2)$.

Example

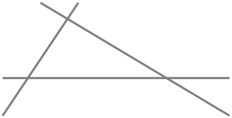
- $X \to \mathbb{P}^2_{\mathbb{C}}$ defined by $yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$.
- $X_\eta \simeq Q = \{q = 0\}$ a quadric over $K = \mathbb{C}(\mathbb{P}^2)$;
- $\alpha = (x, y) \in Im\tau_2$, where
  $\tau_2 : H^2(K, \mathbb{Z}/2) \to H^2(K(Q), \mathbb{Z}/2) = H^2(\mathbb{C}(X), \mathbb{Z}/2)$.
- α has nonzero residues on $\mathbb{P}^2_{\mathbb{C}}$ at $x = 0, y = 0, z = 0$:



Recall:
$\partial_v^2(a, b) = (-1)^{v(a)v(b)} \overline{\frac{a^{v(b)}}{b^{v(a)}}}$
In particular, $\partial_v^2(a, b) = 0$ is a, b are units in A (i.e. $v(a) = v(b) = 0$).

Example

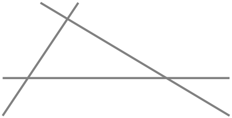
- $X \to \mathbb{P}^2_{\mathbb{C}}$ defined by $yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$.
- $X_\eta \simeq Q = \{q = 0\}$ a quadric over $K = \mathbb{C}(\mathbb{P}^2)$;
- $\alpha = (x, y) \in Im\tau_2$, where
  $\tau_2 : H^2(K, \mathbb{Z}/2) \to H^2(K(Q), \mathbb{Z}/2) = H^2(\mathbb{C}(X), \mathbb{Z}/2)$.
- α has nonzero residues on $\mathbb{P}^2_{\mathbb{C}}$ at $x = 0, y = 0, z = 0$:



- Recall: τ_2 is injective if $F(x, y, z)$ is not a square.

Example

- $X \to \mathbb{P}^2_{\mathbb{C}}$ defined by $yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$.
- $X_\eta \simeq Q = \{q = 0\}$ a quadric over $K = \mathbb{C}(\mathbb{P}^2)$;
- $\alpha = (x, y) \in Im\tau_2$, where
  $\tau_2 : H^2(K, \mathbb{Z}/2) \to H^2(K(Q), \mathbb{Z}/2) = H^2(\mathbb{C}(X), \mathbb{Z}/2)$.
- α has nonzero residues on $\mathbb{P}^2_{\mathbb{C}}$ at $x = 0, y = 0, z = 0$:



- Recall: τ_2 is injective if $F(x, y, z)$ is not a square.
- Let v on $\mathbb{C}(X)$ a valuation, A the valuation ring, $x_v \in \mathbb{P}^2_{\mathbb{C}}$ its center:
  - x_v is the generic point of a curve C_v
  - $x_v = P_v$ a point.

Example: $x_v = C_{v,\eta}$

$X \to \mathbb{P}^2_{\mathbb{C}} : yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$, $\alpha = (x,y)$,
$X_\eta \simeq Q = \{q = 0\}$ a quadric over $K = \mathbb{C}(\mathbb{P}^2)$
A the valuation ring, $i : \mathcal{O}_{\mathbb{P}^2, x_v} \to A$.

Example: $x_v = C_{v,\eta}$

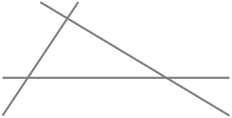
$X \to \mathbb{P}^2_{\mathbb{C}} : yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$, $\alpha = (x,y)$,
$X_\eta \simeq Q = \{q = 0\}$ a quadric over $K = \mathbb{C}(\mathbb{P}^2)$
A the valuation ring, $i : \mathcal{O}_{\mathbb{P}^2, x_v} \to A$.

- C_v is not the line $x = 0$, or $y = 0$ or $z = 0$: x, y are units in $\mathcal{O}_{\mathbb{P}^2, x_v}$, hence in A (via i), hence $\partial_v^2(\alpha) = 0$.

Example: $x_v = C_{v,\eta}$

$X \to \mathbb{P}^2_{\mathbb{C}} : \ yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$, $\alpha = (x,y)$,
$X_\eta \simeq Q = \{q = 0\}$ a quadric over $K = \mathbb{C}(\mathbb{P}^2)$
A the valuation ring, $i : \mathcal{O}_{\mathbb{P}^2, x_v} \to A$.

- C_v is not the line $x = 0$, or $y = 0$ or $z = 0$: x, y are units in $\mathcal{O}_{\mathbb{P}^2, x_v}$, hence in A (via i), hence $\partial_v^2(\alpha) = 0$.
- C_v is the line $x = 0$ (similarly $y = 0$ or $z = 0$): use the completion argument.



Example: $x_v = C_{v,\eta}$

$X \to \mathbb{P}^2_{\mathbb{C}} : yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$, $\alpha = (x,y)$,
$X_\eta \simeq Q$ a quadric over $K = \mathbb{C}(\mathbb{P}^2)$
A the valuation ring, $i : \mathcal{O}_{\mathbb{P}^2, x_v} \to A$.

- C_v is not the line $x = 0$, or $y = 0$ or $z = 0$: x, y are units in $\mathcal{O}_{\mathbb{P}^2, x_v}$, hence in A (via i), hence $\partial_v^2(\alpha) = 0$.

Example: $x_v = C_{v,\eta}$

$X \to \mathbb{P}^2_{\mathbb{C}} \,:\, yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$, $\alpha = (x,y)$,
$X_\eta \simeq Q$ a quadric over $K = \mathbb{C}(\mathbb{P}^2)$
A the valuation ring, $i : \mathcal{O}_{\mathbb{P}^2, x_v} \to A$.

- C_v is not the line $x = 0$, or $y = 0$ or $z = 0$: x, y are units in $\mathcal{O}_{\mathbb{P}^2, x_v}$, hence in A (via i), hence $\partial_v^2(\alpha) = 0$.
- C_v is the line $x = 0$ (similarly $y = 0$ or $z = 0$): use the completion argument.
  - $\hat{K}$ the completion at x_v, $K(Q)_v$ completion at v
  - compute $\partial_v^2(x,y)$ as
    $$H^2(K, \mathbb{Z}/2) \to H^2(\hat{K}, \mathbb{Z}/2) \xrightarrow{\tau_2} H^2(\hat{K}(Q), \mathbb{Z}/2) \to H^2(K(Q)_v, \mathbb{Z}/2) \xrightarrow{\partial} H^1(\kappa(v), \mathbb{Z}/2).$$

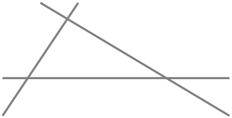
Example: $x_v = C_{v,\eta}$

$X \to \mathbb{P}^2_{\mathbb{C}}$: $yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$, $\alpha = (x,y)$,
$X_\eta \simeq Q$ a quadric over $K = \mathbb{C}(\mathbb{P}^2)$
A the valuation ring, $i : \mathcal{O}_{\mathbb{P}^2, x_v} \to A$.

- C_v is not the line $x = 0$, or $y = 0$ or $z = 0$: x, y are units in $\mathcal{O}_{\mathbb{P}^2, x_v}$, hence in A (via i), hence $\partial_v^2(\alpha) = 0$.
- C_v is the line $x = 0$ (similarly $y = 0$ or $z = 0$): use the completion argument.
  - $\hat{K}$ the completion at x_v, $K(Q)_v$ completion at v
  - compute $\partial_v^2(x,y)$ as
    $H^2(K, \mathbb{Z}/2) \to H^2(\hat{K}, \mathbb{Z}/2) \xrightarrow{\tau_2} H^2(\hat{K}(Q), \mathbb{Z}/2) \to H^2(K(Q)_v, \mathbb{Z}/2) \xrightarrow{\partial} H^1(\kappa(v), \mathbb{Z}/2)$.
  - $\alpha \in ker[H^2(\hat{K}, \mathbb{Z}/2) \to H^2(\hat{K}(Q), \mathbb{Z}/2)]$ if the discriminant is a square in $\hat{K}$:
    - then $Q_{\hat{K}}$ is given by $\langle 1, x, y, xy \rangle$.
    - Hence for $Q_{\hat{K}}$ we know $ker\, \tau_2 = (x,y)$.
  - for $\partial_v^2(\alpha) = 0$ it is enough to have $F(0,y,z)$ is a square.

Example: $x_v = P_v$

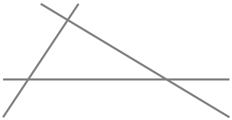
$X \to \mathbb{P}^2_{\mathbb{C}} : yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$, $\alpha = (x,y)$, A the valuation ring, $i_v : \widehat{\mathcal{O}_{\mathbb{P}^2, x_v}} \to A_v$ the completions.



- P_v is not on the lines $x = 0$, or $y = 0$, or $z = 0$, then x, y are units in A_v (via i_v).

Example: $x_v = P_v$

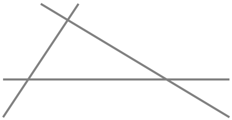
$X \to \mathbb{P}^2_{\mathbb{C}} : \; yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$, $\alpha = (x, y)$,
A the valuation ring, $i_v : \widehat{\mathcal{O}_{\mathbb{P}^2, x_v}} \to A_v$ the completions.



- P_v is not on the lines $x = 0$, or $y = 0$, or $z = 0$, then x, y are units in A_v (via i_v).
- P_v is on one line, for example $x = 0$, then $y \in \widehat{\mathcal{O}_{\mathbb{P}^2, x_v}}$ is a square (nonzero complex number in the residue field), hence $\alpha = 0$ over $\widehat{\mathcal{O}_{\mathbb{P}^2_{x_v}}}$, hence in A_v.

Example: $x_v = P_v$

$X \to \mathbb{P}^2_{\mathbb{C}}$: $yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$, $\alpha = (x, y)$,
A the valuation ring, $i_v : \widehat{\mathcal{O}_{\mathbb{P}^2, x_v}} \to A_v$ the completions.



- P_v is not on the lines $x = 0$, or $y = 0$, or $z = 0$, then x, y are units in A_v (via i_v).
- P_v is on one line, for example $x = 0$, then $y \in \widehat{\mathcal{O}_{\mathbb{P}^2, x_v}}$ is a square (nonzero complex number in the residue field), hence $\alpha = 0$ over $\widehat{\mathcal{O}_{\mathbb{P}^2_{x_v}}}$, hence in A_v.
- P_v is on two curves: $x = 0$ and $y = 0$.
  Enough: $F(0,0,1)$ is a nonzero complex number, hence a square. (Similarly to C_v case).

Conclusion

$X \to \mathbb{P}^2_{\mathbb{C}} : \; yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0,$
$\alpha = (x,y) \in H^2_{nr}(\mathbb{C}(X)/\mathbb{C})$ if:

1. $F(x,y,z)$ is not a square,
2. $F(0,y,z), F(x,0,z), F(x,y,0)$ are squares,
3. $F(0,0,1), F(0,1,0), F(1,0,0)$ are nonzero.

Conclusion

$X \to \mathbb{P}^2_{\mathbb{C}}$: $yzX_0^2 + xzX_1^2 + xyX_2^2 + F(x,y,z)X_3^2 = 0$,
$\alpha = (x,y) \in H^2_{nr}(\mathbb{C}(X)/\mathbb{C})$ if:

1. $F(x,y,z)$ is not a square,
2. $F(0,y,z), F(x,0,z), F(x,y,0)$ are squares,
3. $F(0,0,1), F(0,1,0), F(1,0,0)$ are nonzero.

For example, $F(x,y,z) = x^2 + y^2 + z^2 - 2(yz + xz + xy)$ works.

General formulas:

1. For conic bundles (Colliot-Thélène)
   - $X \to \mathbb{P}^2_{\mathbb{C}}$, generic fiber is a conic $Q = \langle 1, a, b \rangle$.
   - Assume the ramification divisor of (a, b) is $C = \cup_I C_i$ is snc.
   - $\gamma_i = \partial^2_{C_i}(a, b)$.
   - $H = \{(n_i)_{i \in I} \in \{0, 1\} \mid$ such that :
     $\Leftrightarrow n_i = n_j$ if $\exists\, P \in C_i \cap C_j, \partial_P(\gamma_i) = \partial_P(\gamma_j) \neq 0\}$.
   - Then $H^2_{nr}(\mathbb{C}(X)/\mathbb{C})$ is the quotient of H by the diagonal $(1, \ldots, 1)\mathbb{Z}/2$.
2. For quadric surface bundles (P.)
   - $X \to \mathbb{P}^2_{\mathbb{C}}$, generic fiber is a quadric surface Q with nonzero discriminant d and Clifford invariant α.
   - Similar formula in terms of d and the ramification divisor of α.

General formula for quadric surface bundles

- $X \to \mathbb{P}^2_{\mathbb{C}}$, generic fiber is a quadric surface Q with nonzero discriminant d and Clifford invariant α.
- Assume the ramification divisor of α is $C = \cup C_i$ is snc, $\mathcal{P}$ is the set of singular points of C.
- $T = \{x \in \mathbb{P}^{2,(1)}_{\mathbb{C}}\} \subset C$ such that:
  - $\partial_x(\alpha) \neq 0$,
  - d, up to a multiplication by a square, is a unit in $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}},x}$,
  - the image of d in $\kappa(x)$ is a square.
- $T = \cup_{i=1}^n T_i$ with T_i irreducible, $\gamma_i := \partial_{T_i}(\alpha)$.
- $H^2_{nr}(\mathbb{C}(X)/\mathbb{C}) = ker[(\mathbb{Z}/2)^n \to \oplus_{P\in\mathcal{P}} H^0(\kappa(P), \mathbb{Z}/2)]$, $(n_i)_{i=1}^n \mapsto (\oplus n_i\, \partial^1_P(\gamma_i))$.
- (n_i) corresponds to a unique class $\beta \in H^2(K, \mathbb{Z}/2)$ with

$$\partial^2(\beta) = (n_i\gamma_i)_i \in \oplus_{i=1}^n H^1(\kappa(T_i), \mathbb{Z}/2).$$

THANK YOU MESSAGE:

- to the audience;

THANK YOU MESSAGE:

- to the audience;
- to my parents;
- to the Belarusian math olympiads school (S.A. Mazanik, I.I. Voronovich, V.I. Kaskevich, and others)
- Minsk shool 51 (now 29 Gymnaisum), BSU Lyceum, and Belarusian State University, Faculty of Applied Mathematics and Computer Science.