Khovanskii-Pukhlikov rings and divided difference operators

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Khovanskii-Pukhlikov rings

$P \subset \mathbb{R}^d$ - convex polytope

goal

$R_P = \bigoplus_{i=0}^{d} R_P^i$ - graded commutative ring

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Motivation: P – simple lattice $\Rightarrow R_P = CH^*(X_P)$

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smooth projective toric variety

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$P \rightsquigarrow$ semigroup S_P of (lattice) polytopes analogous to P

Khovanskii-Pukhlikov rings

with respect to Minkowski sum

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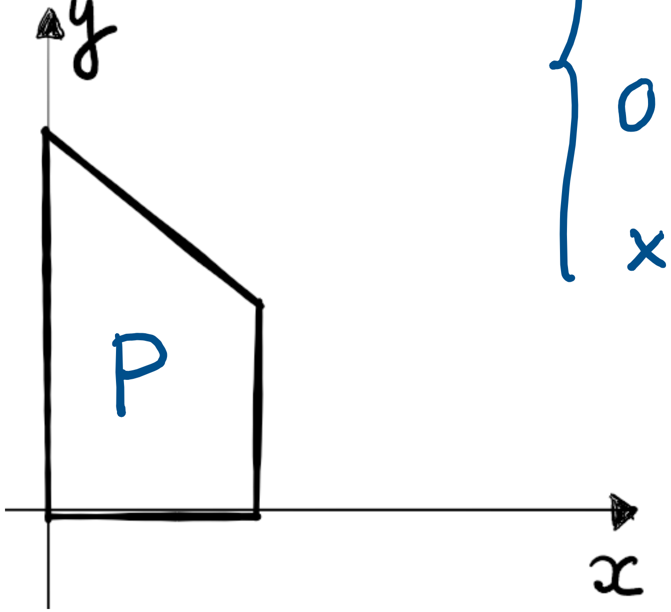
Example:

$$\begin{cases} 0 \leq x \leq a \\ 0 \leq y \\ x + y \leq a + b \end{cases}$$



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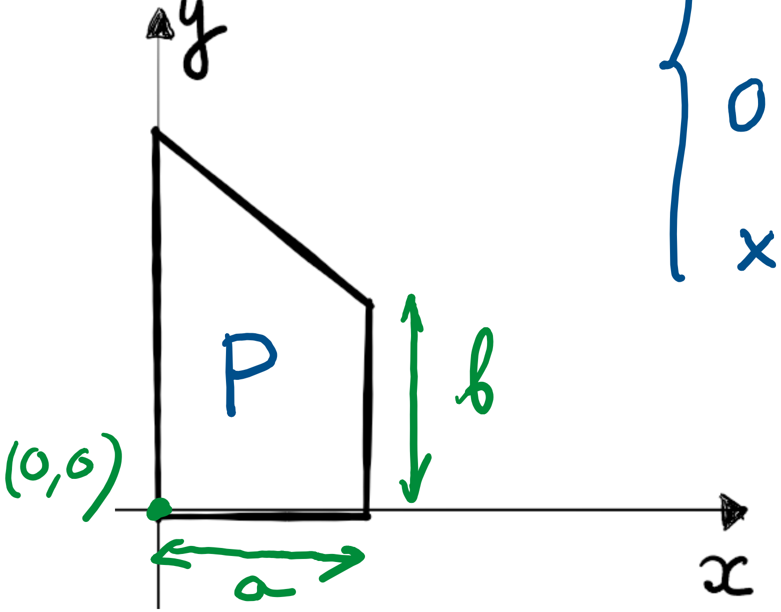

$$\begin{cases} 0 \le x \le a \\ 0 \le y \\ x + y \le a + b \end{cases} \rightsquigarrow \begin{cases} 0 \le (x - \underline{\underline{x_0}}) \le \underline{\underline{a}} \\ 0 \le (y - \underline{\underline{y_0}}) \\ (x - x_0) + (y - y_0) \le a + \underline{\underline{b}} \end{cases}$$

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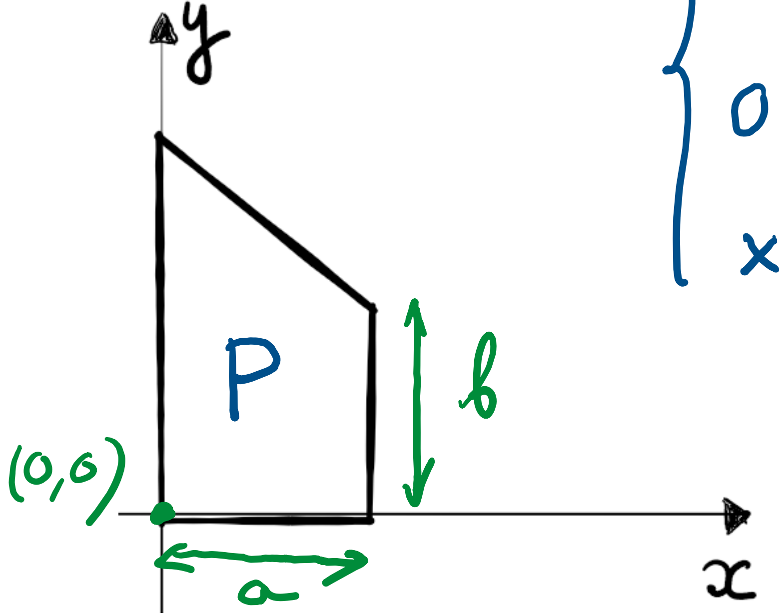

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P

polygon from S_P

Khovanskii-Pukhlikov rings

$\mathcal{S}_P \rightsquigarrow \mathcal{L}_P$ - Grothendieck group of virtual (lattice) polytopes analogous to P

Khovanskii-Pukhlikov rings

$S_P \rightsquigarrow L_P$ - Grothendieck group of virtual (lattice) polytopes analogous to P

Example (continued):

$$\begin{cases} 0 \leq (x-x_0) \leq a \\ 0 \leq (y-y_0) \\ (x-x_0)+(y-y_0) \leq a+b \end{cases} \rightsquigarrow (x_0, y_0, a, b) \in \mathbb{Z}^4$$

polygon from S_P

virtual polygon from L_P

Khovanskii-Pukhlikov rings

$P \rightsquigarrow \mathcal{S}_P \rightsquigarrow \mathcal{L}_P \subseteq \mathbb{R}^\ell$ * vector space of virtual polytopes analogous to P

$(x_1, \dots, x_\ell)$ - coordinates

<u>Definition:</u> $R_P = \mathbb{Z}[\partial_1, \dots, \partial_\ell] \big/ \mathrm{Ann}(\mathrm{vol}_P)$

Khovanskii-Pukhlikov rings

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$\partial_i := \dfrac{\partial}{\partial x_i}$

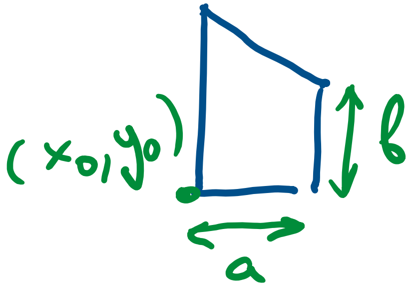
volume polynomial on $\mathcal{L}_P$

Пухликов и Хованский

Khovanskii-Pukhlikov rings

Example (continued) (x_0, y_0, a, b)

$\partial_1, \partial_2, \partial_3, \partial_4$



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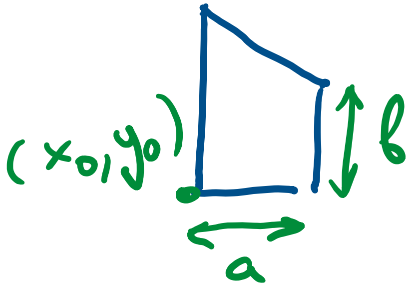

$$vol_P = ab + \frac{a^2}{2}$$

does not depend on x_0, y_0

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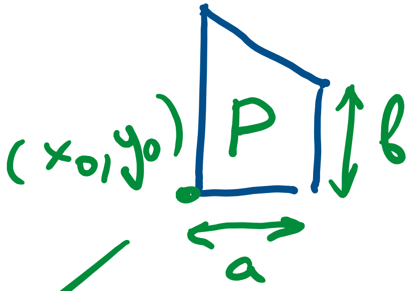

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Definition: $R_P = \mathbb{Z}[\partial_a, \partial_b] \,/\, \mathrm{Ann}(\mathrm{vol}_P)$

Remark: $R_P \simeq CH^*$ (blow-up of $\mathbb{P}^2$ at one point)

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Good news: relations in R_P can be found using geometry!

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Good news: relations in R_P can be found using geometry!

$$\partial_1, \dots, \partial_\ell \longleftrightarrow \text{linear combinations of facets of } P$$

$$\text{product} \longleftrightarrow \text{intersection}$$

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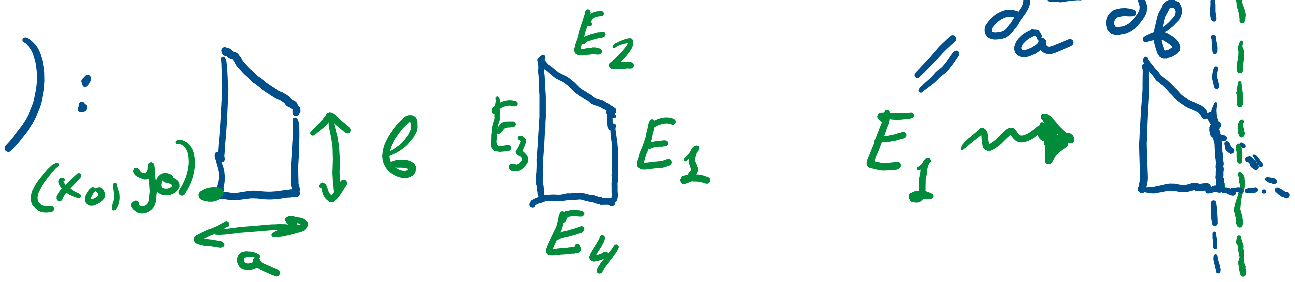

Khovanskii-Pukhlikov rings

$[E_1] = \partial_a - \partial_b$

$[E_2] = \partial_b = [E_4]$

$[E_3] = \partial_a$

$\Rightarrow$

$\partial_a(\partial_a - \partial_b) = 0$

$\partial_b^2 = 0$

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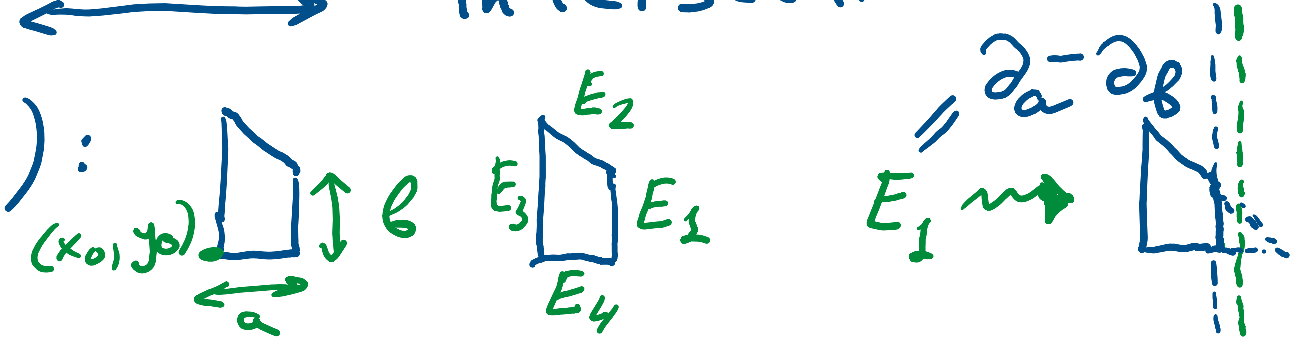
<u>Example</u> (continued):



Applications

- Chow rings of smooth projective toric varieties (Khovanskii-Pukhlikov, 1992)

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based on representation theory

Schubert calculus (brief reminder)

Type A   flag $V^1 \subset V^2 \subset \ldots \subset V^n \subset \mathbb{C}^{n+1}$

$\rightsquigarrow$ complete flag variety $F(1,2,\ldots,n)$

Schubert calculus (brief reminder)

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Goal: enumerate flags in geometric problems.

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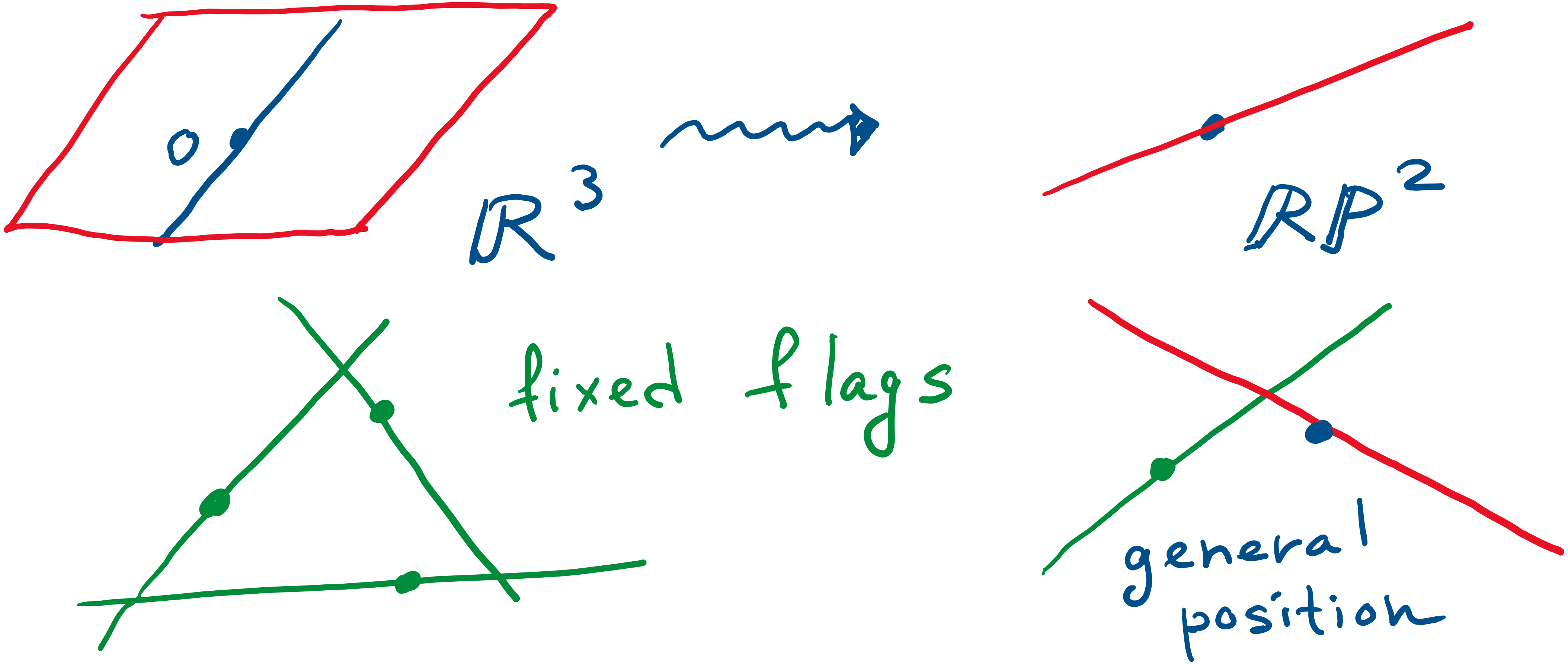
Example: (high school geometry)

How many flags in 3-space are NOT in general position with 3 given flags?

Schubert calculus (brief reminder)



Schubert calculus (brief reminder)

$$CH^*(F(1,2,\dots,n)) \simeq \mathbb{Z}[\partial_1,\dots,\partial_{n+1}] \big/ I$$

$$I = \mathrm{Ann}\Big(\prod_{1\le i<j\le n+1} (x_i - x_j) \Big)$$

(Borel description in Kostant form)

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Question: how to represent Schubert classes by positive linear combinations of faces of Δ?

Divided difference operators

X – smooth variety

$E \to X$ – vector bundle of rank 2

$Y = \mathbb{P}(E)$ – variety of lines in E

$$p : Y \to X$$

$$p_* : CH^*(Y) \to CH^*(X)$$

Divided difference operators

$$R_\Delta$$
$$\|$$
$$p^* p_* : CH^*(Y) \longrightarrow CH^*(Y)$$

$\Big\Downarrow$ goal

Operation on faces of Δ that yields $p^* p_* : R_\Delta \longrightarrow R_\Delta$

Divided difference operators

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goal

simple geometric mitosis

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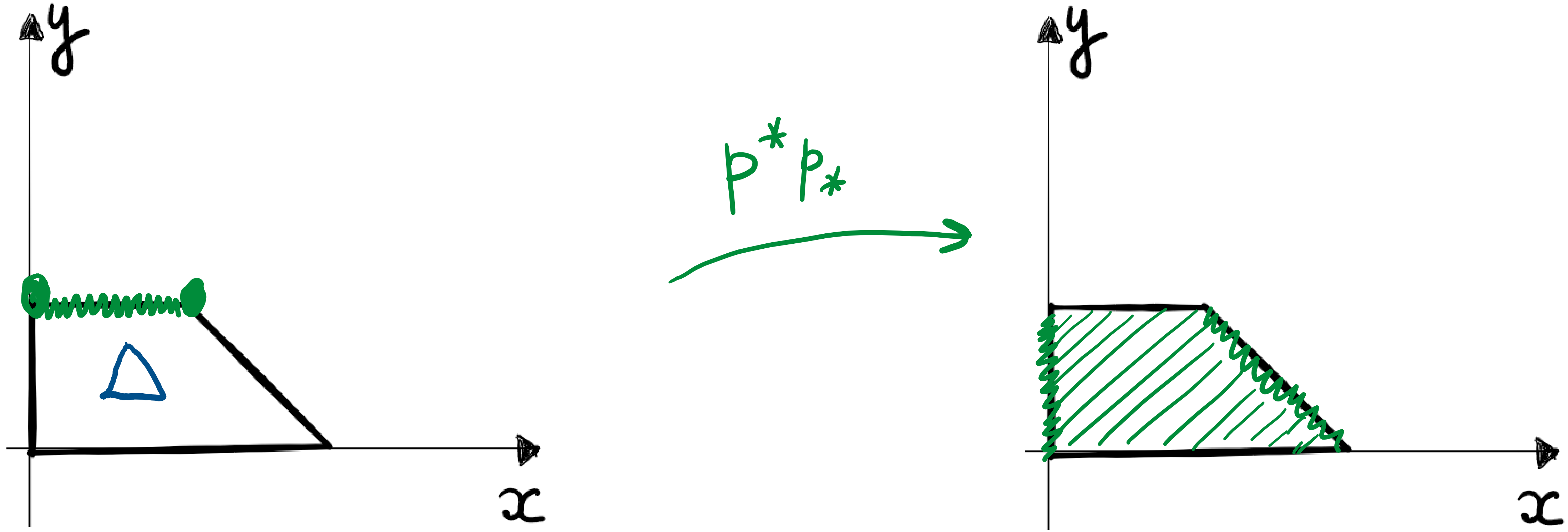
Divided difference operators

Examples: 1 $X = \mathbb{P}^1$, $Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$



Divided difference operators

blow-up of $\mathbb{P}^2$ at one point

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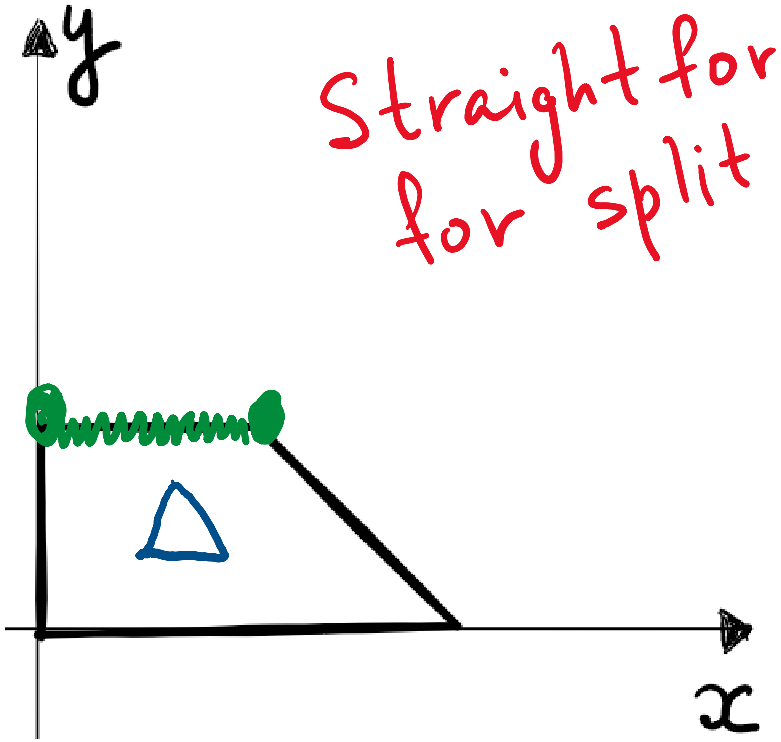
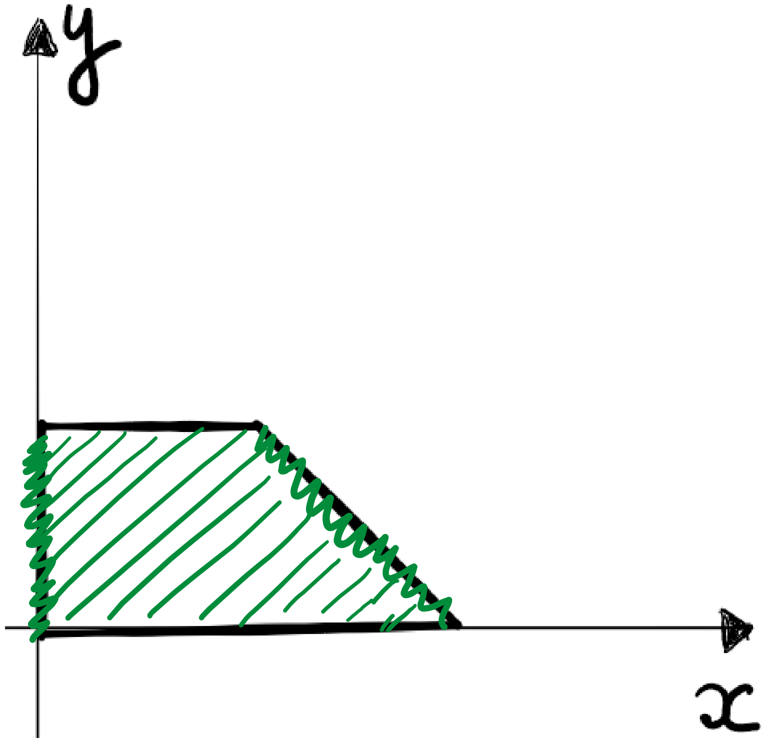


Straightforward for split E

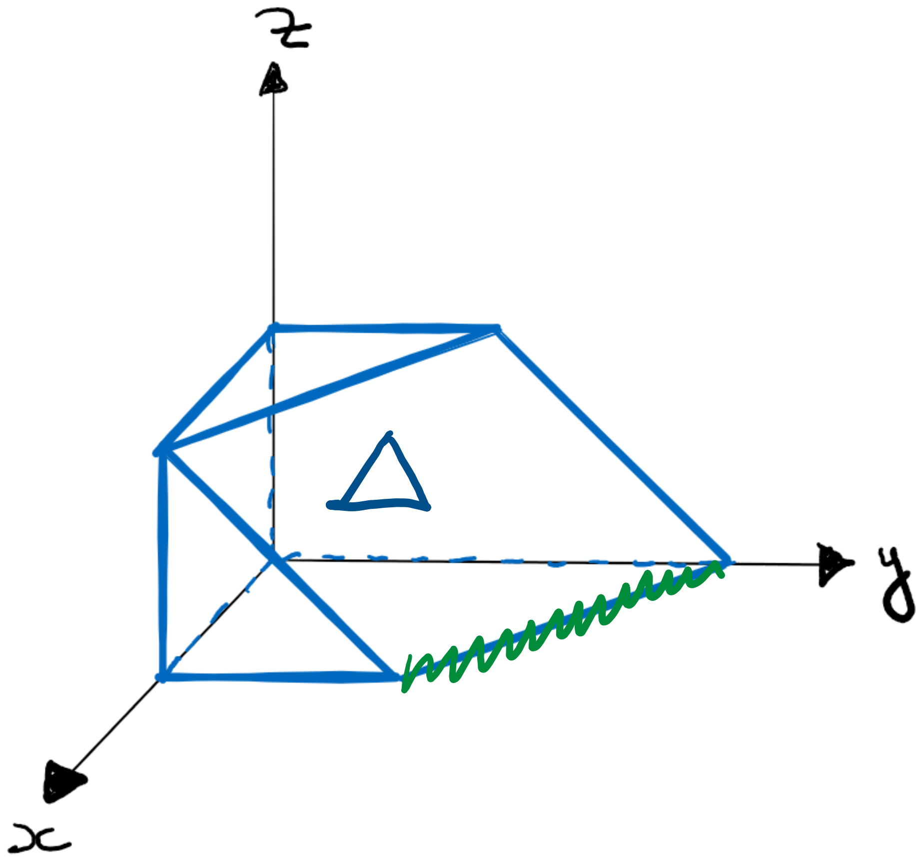
$p^* p_*$



Divided difference operators

flag variety

Examples: 2 $X = \mathbb{P}^2$, $Y = \mathbb{P}(T\mathbb{P}^2)$



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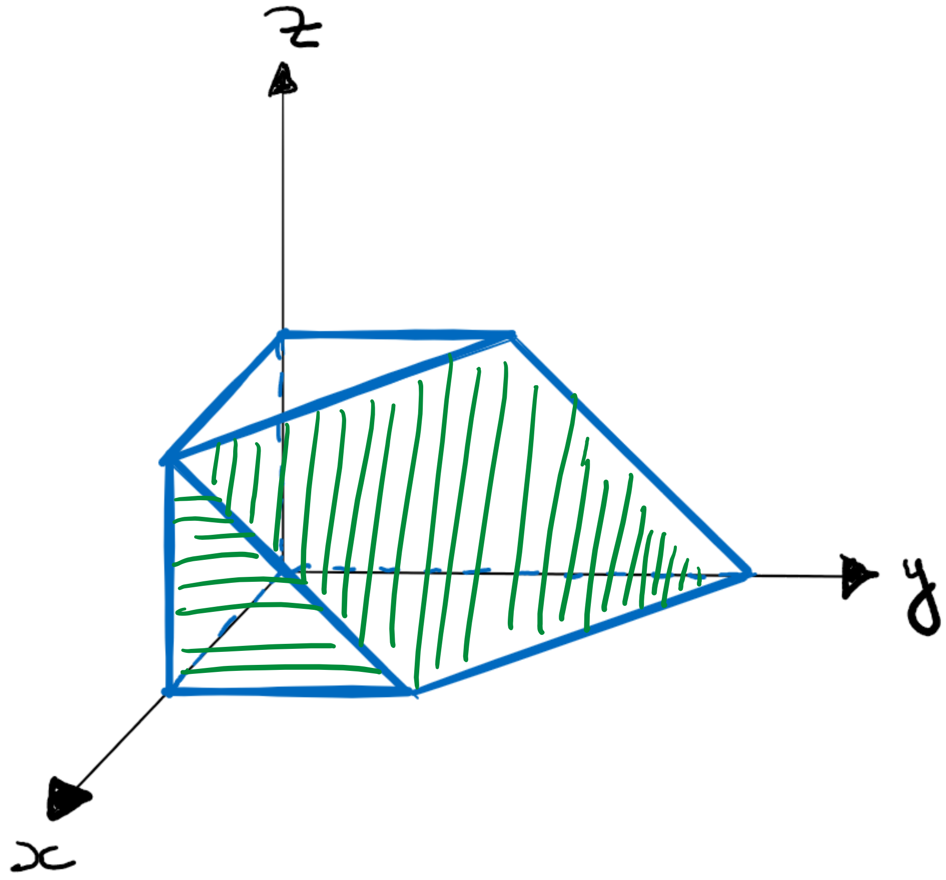

Divided difference operators

<u>Examples:</u> 3) $X = F(1,2,\dots,i-1,i+1,\dots,n)$

$Y = F(1,2,\dots,n)$

<u>Remark</u>: There are n divided difference operators on $CH^*(F(1,2,\dots,n))$

Divided difference operators

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General case: $Y = G/B$, $X_i = G/P_i$

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General case: $Y = G/B$, $X_i = G/P_i$

connected reductive group

Borel subgroup

minimal parabolic subgroup

Divided difference operators

<u>Examples:</u> ③ $X = F(1,2,\dots,i-1,i+1,\dots,n)$

Type A $\quad Y = F(1,2,\dots,n)$

$G = GL_{n+1}(\mathbb{C})$, $B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$, $P_i = \begin{pmatrix} * & \cdots & & * \\ & \boxed{\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}} & & \\ 0 & & \ddots & \\ & & & * \end{pmatrix}$.

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<u>Remark:</u> There are n divided difference operators on $CH^*(F(1,2,\dots,n))$

<u>General case:</u> $Y = G/B, \quad X_i = G/P_i$

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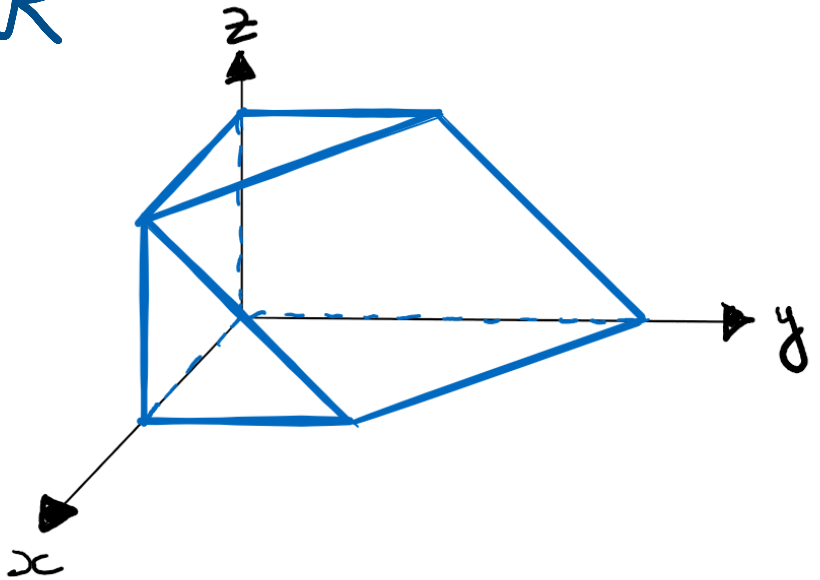
Simple geometric mitosis

$\Delta \subset \mathbb{R}^{d+1}$ – convex polytope such that

$$\Delta = P * Q$$

for convex polytopes $P, Q \subset \mathbb{R}^d$



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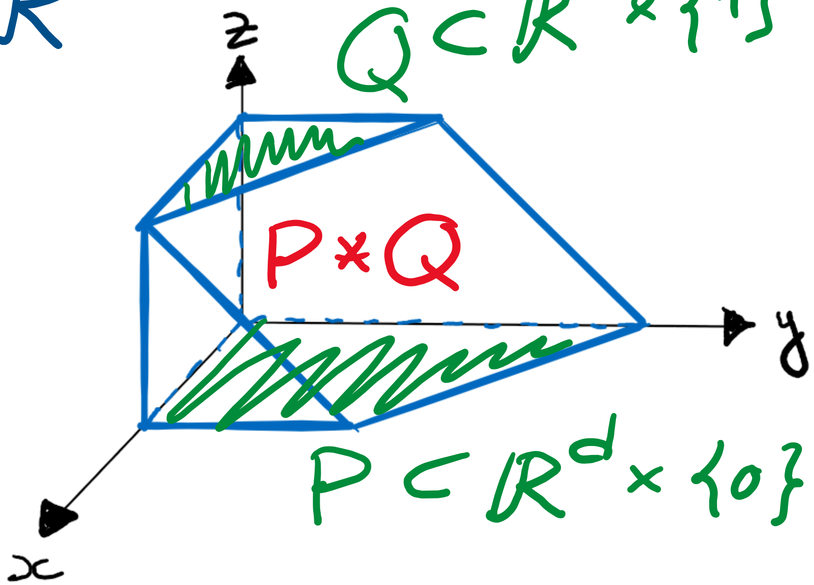

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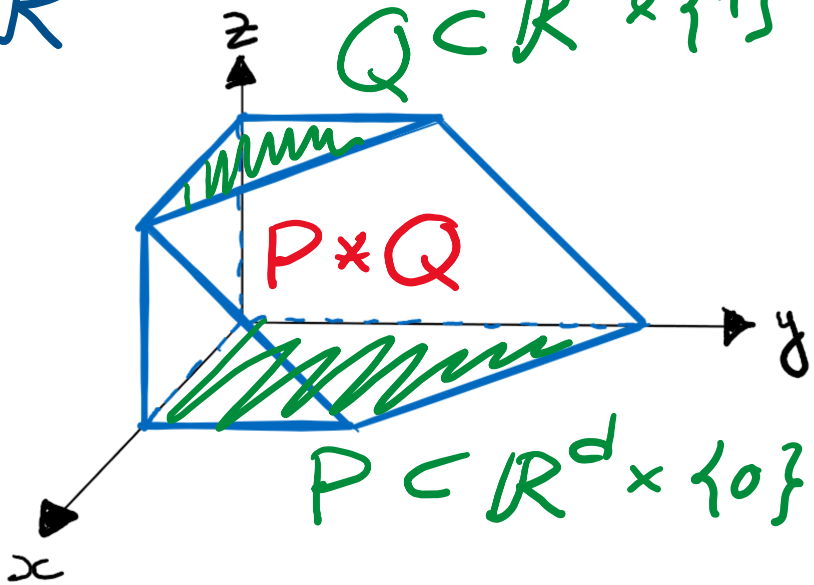

Goal: $F \subset P$ – face

$$\mathrm{mitosis}^{v}(F) = \{E_1, \dots, E_k\}$$

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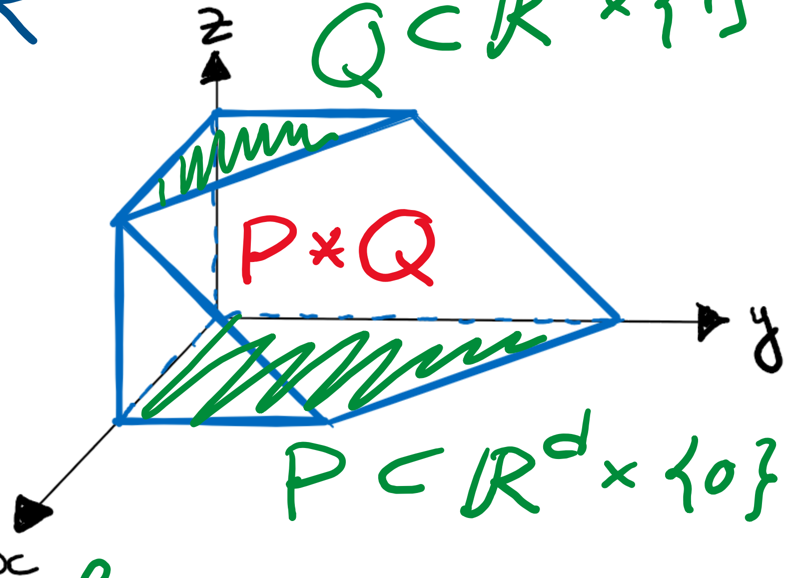


<u>Goal:</u> $F \subset P$ – face
$\dim F = \ell$

$$\Big\Downarrow$$

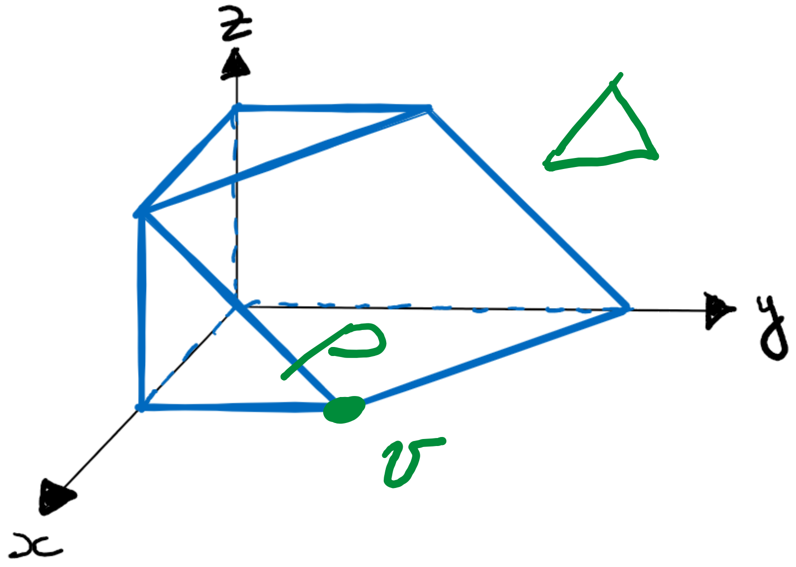
$$\mathrm{mitosis}^v(F) = \{E_1, \dots, E_k\}$$

$E_i \subset \Delta$ – face, $\dim E_i = \ell + 1$

Simple geometric mitosis

Definition: mitosis$^{v}(F)$

a vertex $v \in P$



Simple geometric mitosis

Definition: $\mathrm{mitosis}^{v}(F)$ — a vertex $v \in P$

$F, E_i \ni v$

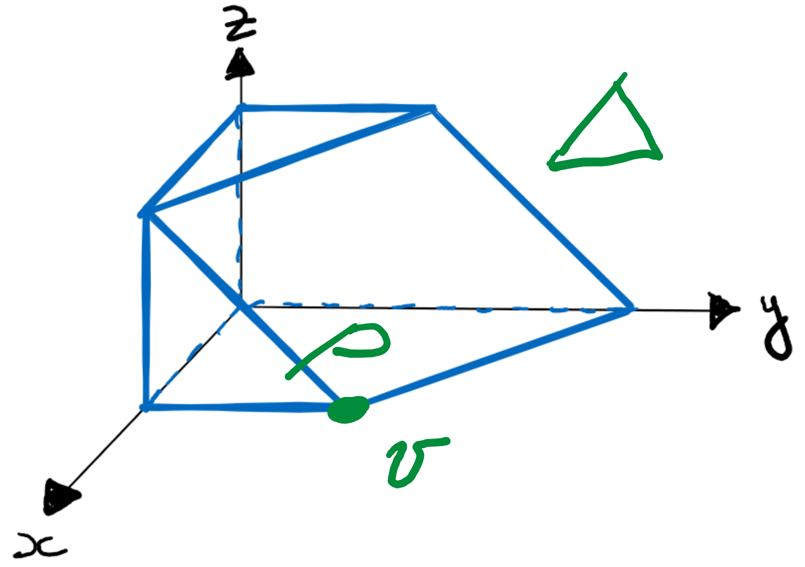
$E_i \in \mathrm{mitosis}^{v}(F)$

$\Updownarrow$

(1)

(2)

(3)



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(1) E_i & P, Q

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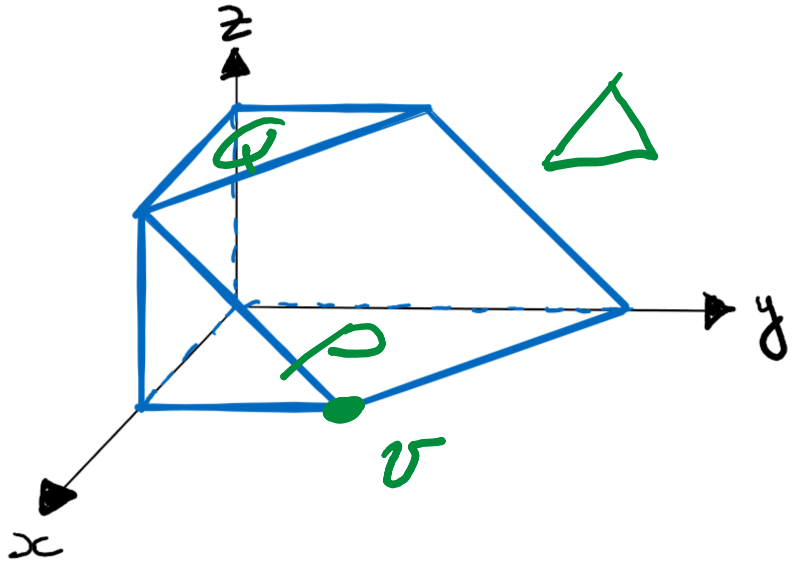

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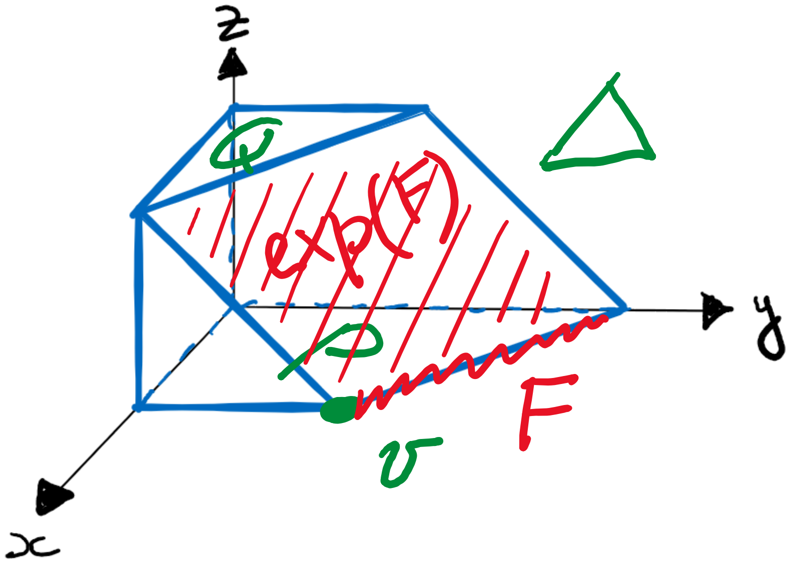
$\exp(F) \cap P = F$

(1) E_i & P, Q

(2) $E_i \cap Q \subset \exp(F) \cap Q$

(3)



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Definition: $\text{mitosis}^{v}(F)$ ← a simple vertex $v \in P$

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$\Updownarrow$

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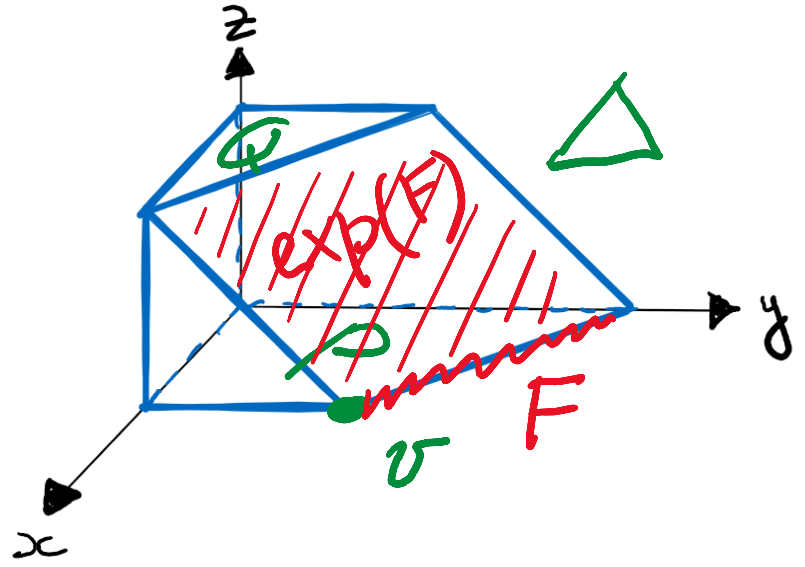


Simple geometric mitosis

simple

a vertex $v \in P$

$F, E_i \ni v$

Definition: $\text{mitosis}^{v}(F)$

$E_i \in \text{mitosis}^{v}(F) \ni \exp(F)$

$\Updownarrow$

$\exp(F) \cap P = F$

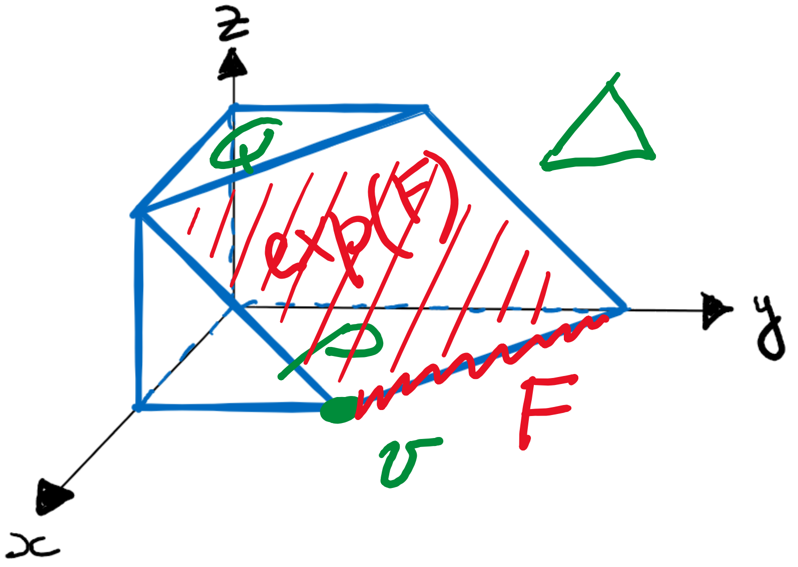
(1) $E_i \pitchfork P, Q$

(2) $E_i \cap Q \subset \exp(F) \cap Q$

(3) $\Gamma \supset E_i$ and $\Gamma \not\supset F \Rightarrow$

facet

$v \in \Gamma$



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$E_i \in \text{mitosis}^{v}(F) \ni \exp(F)$

$\Updownarrow$

$\exp(F) \cap P = F$

(1) $E_i \not\subset P, Q$

(2) $E_i \cap Q \subset \exp(F) \cap Q$

(3) $\Gamma \supset E_i$ and $\Gamma \not\supset F \Rightarrow$

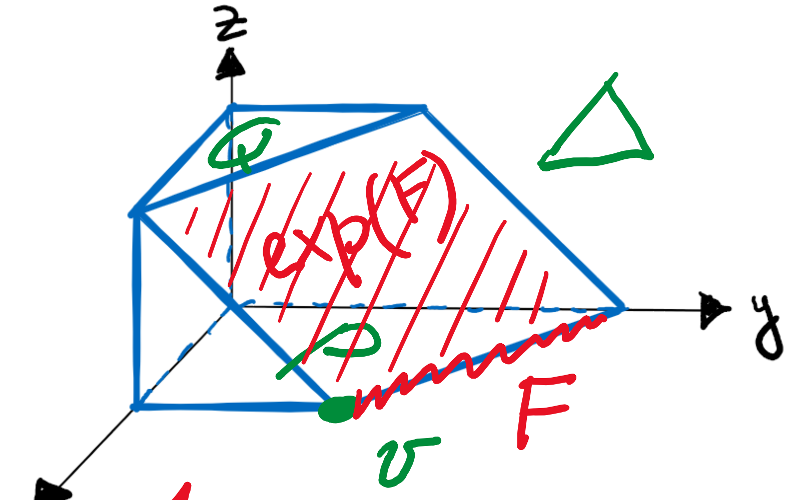
facet $v \in \Gamma$

$\exists\, H \subset \Delta$ such that:

face $v \in H$

$\dim(H \cap Q \cap \Gamma) < \dim(H \cap Q) - 1$

$\neq \emptyset$



Simple geometric mitosis

simple

a vertex $v \in P$

$F, E_i \ni v$

Definition: mitosis$^{v}(F)$

$E_i \in \mathrm{mitosis}^{v}(F) \ni \exp(F)$

$\Updownarrow$

$\exp(F) \cap P = F$

(1) $E_i \not\subset P, Q$

(2) $E_i \cap Q \subset \exp(F) \cap Q$

(3) $\Gamma \supset E_i$ and $\Gamma \not\supset F \Rightarrow$

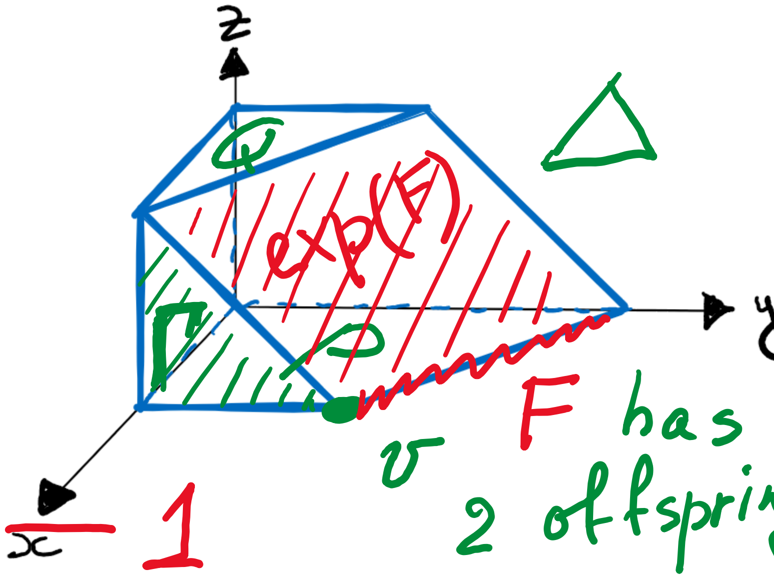
$\exists\, H \subset \Delta$ such that:

$\dim(H \cap Q \cap \Gamma) < \dim(H \cap Q) - 1$

facet $v \in \Gamma$

face $v \in H$

Δ



v

F has 2 offsprings

Applications

- Knutson-Miller mitosis on Gelfand-Zetlin polytopes in type A
- (Simplified) Fujita mitosis on Gelfand-Zetlin polytopes in type C
- Mitosis in type D?

Applications

- Knutson-Miller mitosis on Gelfand-Zetlin polytopes in type A
- (Simplified) Fujita mitosis on Gelfand-Zetlin polytopes in type C
- Mitosis in type D?

References:

- *Push–pull operators on convex polytopes*, IMRN, 2021 https://doi.org/10.1093/imrn/rnab331
- *Simple geometric mitosis,* coming soon

Вялікі дзякуй!