Galois cohomology of real algebraic groups

(joint with Mikhail Borovoi)

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Algebraic groups over $\mathbb{R}$

Let G be a linear algebraic group defined over $\mathbb{R}$.

Naively:

$G \subset GL_n$ is defined by polynomial equations in g_{ij} with coefficients $\in \mathbb{R}$.

$G(\mathbb{C}) = \{\text{solutions with } g_{ij} \in \mathbb{C}\}$, complex Lie group;
$G(\mathbb{R}) = \{\text{solutions with } g_{ij} \in \mathbb{R}\}$, real Lie group.

Complex conjugation: $G(\mathbb{C}) \ni g = (g_{ij}) \mapsto \bar{g} = (\bar{g}_{ij}) \in G(\mathbb{C})$;

$$g \in G(\mathbb{R}) \iff g = \bar{g}.$$

More rigorously:

$G = (G(\mathbb{C}), \sigma)$, where $\sigma : G(\mathbb{C}) \to G(\mathbb{C})$, $g \mapsto \bar{g}$, is a *real structure*
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i.e., $\sigma^* \mathcal{O}_G = \overline{\mathcal{O}}_G$).

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1-cocycles: $\mathrm{Z}^1(\mathbb{R}, G) = \{c \in G(\mathbb{C}) \mid c\bar{c} = 1\}$

Twisted conjugation action $G(\mathbb{C}) \curvearrowright \mathrm{Z}^1(\mathbb{R}, G)$: $c \overset{g}{\longmapsto} gc\bar{g}^{-1}$

1-cohomology: $\mathrm{H}^1(\mathbb{R}, G) = \mathrm{Z}^1(\mathbb{R}, G)/G(\mathbb{C})$ (pointed set, not group!)

Principle

Let X be an object defined over $\mathbb{R}$ (quadratic form, tensor, algebra, algebraic variety/group). Then:

$$\{\mathbb{R}\text{-forms of } X\} \quad \longleftrightarrow \quad \mathrm{H}^1(\mathbb{R}, \operatorname{Aut} X)$$

Exact cohomology sequence:

Let $G \curvearrowright Q = Gq_0$ be a homogeneous variety, $q_0 \in Q(\mathbb{R})$, $H = G_{q_0}$.

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Galois cohomology: computation problem

Problem 1

Compute the Galois cohomology $\mathrm{H}^1(\mathbb{R}, G)$.

Fact

G *unipotent* $\implies \mathrm{H}^1(\mathbb{R}, G) = 1$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{\mathrm{uni}} & \longrightarrow & G & \longrightarrow & G/G_{\mathrm{uni}} & \longrightarrow & 1 \\ & & & & \bigcup & & \| & & \\ & & & & G_{\mathrm{red}} & \xrightarrow{\sim} & G/G_{\mathrm{uni}} & & \end{array}$$

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Conclusion: It suffices to solve Problem 1 for reductive G.

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Galois cohomology of tori

Real structures on tori:

Let (T, σ) be an algebraic torus defined over $\mathbb{R}$; $T(\mathbb{C}) = \underbrace{\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times}_{n}$.

Basic $\mathbb{R}$-structures:

- split: $\sigma(t_1, \ldots, t_n) = (\bar{t}_1, \ldots, \bar{t}_n)$; $T(\mathbb{R}) = (\mathbb{R}^\times)^n$
- anisotropic: $\sigma(t_1, \ldots, t_n) = (\bar{t}_1^{-1}, \ldots, \bar{t}_n^{-1})$; $T(\mathbb{R}) = (S^1)^n$

Proposition

$\exists$ unique decomposition $T = T_0 \cdot T_1$ such that $|T_0 \cap T_1| < \infty$ (almost direct product), T_0 anisotropic, T_1 split.

Notation: $T_0^{(2)} = \{t \in T_0 \mid t^2 = 1\}$.

Proposition

$\mathrm{H}^1(\mathbb{R}, T) \simeq T_0^{(2)}/(T_0 \cap T_1)$

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Galois cohomology: reduction to tori

Notation: $G = G^{\mathrm{ss}} \cdot S$ *connected reductive*, G^{ss} semisimple, $S = Z(G)^{\circ}$.

Choose a maximal anisotropic torus $T_0 \subset G$
$\implies T = Z_G(T_0) = T_0 \cdot T_1$ is a maximal torus in G, T_1 is a split torus.

Twisted normalizer: $N_0 = \{n \in N_G(T_0) \mid n\bar{n}^{-1} \in T_0^{(2)}\}$

Twisted conjugation $N_0 \curvearrowright T_0$, $t \overset{n}{\longmapsto} nt\bar{n}^{-1}$, preserves $T_0(\mathbb{R})$ and $T_0^{(2)}$.
$N_0 \cap T$ acts as translations by $T_0 \cap T_1$.

Theorem (M. Borovoi, 1988)

$\mathrm{H}^1(\mathbb{R}, G) \simeq T_0^{(2)}/N_0$

How to compute?

Effective action: $N_0 \twoheadrightarrow \widehat{W}_0 \curvearrowright T_0$

$$1 \longrightarrow T_0 \cap T_1 \longrightarrow \widehat{W}_0 \longrightarrow W_0 := N_G(T_0)/T \longrightarrow 1$$

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Theorem (M. Borovoi, 1988)

$\mathrm{H}^1(\mathbb{R}, G) \simeq T_0^{(2)}/N_0$

How to compute?

Effective action: $N_0 \twoheadrightarrow \widehat{W_0} \curvearrowright T_0$

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Real structure:

$\sigma = \sigma_c \circ \tau \circ \mathrm{inn}(t_\sigma)$, product of commuting involutions. **Here:**

σ_c is an *anisotropic real structure* on G, i.e., $G(\mathbb{R}, \sigma_c)$ is compact;
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$t_\sigma \in T_0$, $t_\sigma^2 \in Z(G^{\mathrm{ss}})$.

Shift and logarithm:

$$\begin{array}{ccccc} T_0 \cap T_1 & \subset & \widehat{W}_0 & \curvearrowright & T_0(\mathbb{R}) \\ \uparrow & & \uparrow & & \uparrow \mathcal{E} \\ \boldsymbol{i} X_0^\vee & \subset & \widetilde{W}_0 & \curvearrowright & \mathfrak{t}_0(\mathbb{R}) \end{array}$$

$X_0^\vee =$ image of $X^\vee(T)$ under projection $\mathfrak{t} = \mathrm{Lie}\, T \to \mathfrak{t}_0 = \mathrm{Lie}\, T_0$;
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$\widetilde{W}_0 \supset \widetilde{W}_r = \boldsymbol{i}Q_0^\vee \rtimes W_0$, reflection group (*affine Weyl group* of (G,τ)).

Fundamental domain for $\widetilde{W}_r \curvearrowright \mathfrak{t}_0(\mathbb{R})$ is $\Delta_1 \times \cdots \times \Delta_m \times \mathfrak{s}_0(\mathbb{R})$.

Here: $\mathfrak{s}_0 = \mathfrak{s} \cap \mathfrak{t}_0$ and Δ_i are simplices defined by (twisted) affine Dynkin diagrams Dyn_i corresponding to $\mathbb{R}$-simple factors of G.

$x_i \in \Delta_i$ is determined by *barycentric coordinates* $p_{ij} \geq 0$ ($j = 0, \ldots, l_i$) such that $\sum_j n_{ij} p_{ij} = 2$ ($n_{ij} \in \mathbb{Z}_{\geq 0}$).

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Determine $x = (x_1, \ldots, x_m, x_0) \in \Delta_1 \times \cdots \times \Delta_m \times \mathfrak{s}_0(\mathbb{R})$ *such that* $\mathcal{E}(x) = \exp 2\pi(x - x_\sigma) \in T_0^{(2)}$.

Answer

$\mathcal{E}(x) \in T_0^{(2)}$ *if and only if:*

1. $p_{ij} \in \mathbb{Z}_{\geq 0}$ $(\forall i, j)$;
2. $x_0 = \boldsymbol{i}\mu/2, \ \ \mu \in M_0^\vee := X^\vee(S_0/S_0 \cap G^{ss})$;
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Galois cohomology: main theorem

Definition

A *Kac labeling* is $p = (p_{ij})_{i,j}$ with $p_{ij} \in \mathbb{Z}_{\geq 0}$, $\sum_j n_{ij} p_{ij} = 2$.

A *reductive Kac labeling* is $(p, [\mu])$, where p is a Kac labeling and $[\mu] \in M_0^\vee / 2\Lambda_0^\vee$.

$\mathcal{K}(G) = \{$reductive Kac labelings satisfying congruences (3)$\}$.

$F_0 = X_0^\vee / (Q_0^\vee \oplus \Lambda_0^\vee)$ acts on $\mathcal{K}(G)$ via automorphisms of $\mathrm{Dyn}_1, \ldots, \mathrm{Dyn}_m$ and translations on $M_0^\vee / 2\Lambda_0^\vee$.

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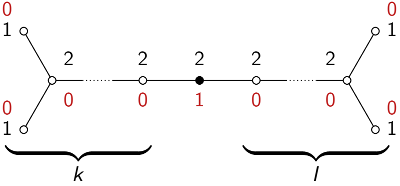
Example

$G = SO_{2k,2l}$, $n = k + l$, $k, l \geq 2$. Here: $\tau = \mathrm{id}$, $T_0 = T$



n_j black
q_j red

$X(T) = \langle \omega_1, \alpha_1, \ldots, \alpha_n \rangle$, $\omega_1 = \sum_{j>0} \lambda_j \alpha_j$, $\lambda_j : {}_1 1 \cdots 1^{1/2}_{1/2}$

Congruence (3): $p_{n-1} \equiv p_n \pmod 2$

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$\mathcal{K}(G)$: ${}^1_1 0\cdots0^0_0$ ${}^2_0 0\cdots0^0_0$ ${}^0_0 2\cdots0^0_0$ $\cdots$ ${}^0_0 0\cdots2^0_0$ ${}^0_0 0\cdots0^2_0$ ${}^0_0 0\cdots0^1_1$
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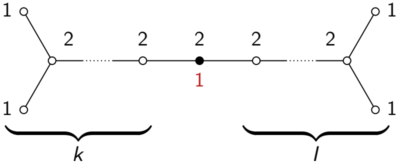

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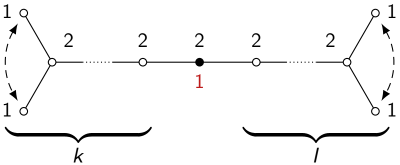

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Component group

Suppose G is a connected (in Zariski topology) linear algebraic group

$\implies G(\mathbb{C})$ is connected (in Hausdorff topology)

But $G(\mathbb{R})$ may be disconnected (in Hausdorff topology)

$G(\mathbb{R})^\circ :=$ identity component of $G(\mathbb{R})$

Problem 2

Compute the component group $\pi_0 G(\mathbb{R}) = G(\mathbb{R})/G(\mathbb{R})^\circ$.

Examples

- G is a split torus

$$\implies G(\mathbb{C}) = \underbrace{\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times}_{n}, \quad G(\mathbb{R}) = \mathbb{R}^\times \times \cdots \times \mathbb{R}^\times$$

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Compute the component group $\pi_0 G(\mathbb{R}) = G(\mathbb{R})/G(\mathbb{R})^\circ$.

Examples

- G is a split torus
  $\implies G(\mathbb{C}) = \underbrace{\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times}_{n}, \quad G(\mathbb{R}) = \mathbb{R}^\times \times \cdots \times \mathbb{R}^\times$
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Component group: reduction to reductive case

Facts

- *G unipotent $\implies G(\mathbb{R})$ connected*
- *$G = G_{\text{uni}} \rtimes G_{\text{red}}$ (Levi decomposition)*
  $\implies G(\mathbb{R}) = G_{\text{uni}}(\mathbb{R}) \rtimes G_{\text{red}}(\mathbb{R}) \implies \pi_0 G(\mathbb{R}) = \pi_0 G_{\text{red}}(\mathbb{R})$

Conclusion: It suffices to solve Problem 2 for connected reductive G.

Known results

(É. Cartan) *G semisimple simply connected $\implies G(\mathbb{R})$ connected*

(Matsumoto, 1964) $\pi_0 G(\mathbb{R}) \simeq \{\pm 1\}^n$

Reason: $T_s \subset G$ is a maximal split torus $\implies G(\mathbb{R}) = T_s(\mathbb{R}) \cdot G(\mathbb{R})^\circ$

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Recall: $G = G^{\mathrm{ss}} \cdot S$, G^{ss} semisimple, $S = Z(G)^{\circ}$.

More notation: G^{sc} = universal cover of G^{ss}, $\mathfrak{s} = \operatorname{Lie} S$.

$$1 \longrightarrow \pi_1 G \xrightarrow{\ i\ } \underset{\text{universal cover}}{\widetilde{G} = G^{\mathrm{sc}} \times \mathfrak{s}} \xrightarrow{\ j\ } G = G^{\mathrm{ss}} \cdot S \longrightarrow 1$$

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Component group: formula

Choose a maximal torus $T \supset T_s$ defined over $\mathbb{R}$.

$T = T_s \cdot T_c$ with T_c anisotropic.

$X^\vee = X^\vee(T)$, cocharacter lattice;

$Q^\vee = Q^\vee(G, T)$, coroot lattice.

$\widetilde{X}_s^\vee =$ image of $X^\vee$ under projection $\mathfrak{t} = \mathrm{Lie}\, T \to \mathfrak{t}_s = \mathrm{Lie}\, T_s$;

$X_s^\vee = X^\vee \cap \mathfrak{t}_s$; $\quad Q_s^\vee = Q^\vee \cap \mathfrak{t}_s$.

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$\widetilde{Z} \simeq \boldsymbol{i}X^\vee / \boldsymbol{i}Q^\vee$

Theorem

$\pi_0 G(\mathbb{R}) \simeq X_s^\vee / (2\widetilde{X}_s^\vee + Q_s^\vee)$

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Example (continued)

$G = PSO_{k,l}, \quad n = k + l = 2m.$

$$X^\vee = \{n_1\varepsilon_1^\vee + \cdots + n_m\varepsilon_m^\vee \mid n_i \in \tfrac{1}{2}\mathbb{Z},\ n_i - n_j \in \mathbb{Z},\ \forall i,j = 1,\dots,m\}$$
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$$Q_s^\vee = \{n_1\varepsilon_1^\vee + \cdots + n_k\varepsilon_k^\vee \mid n_i \in \mathbb{Z},\ n_1 + \cdots + n_k \text{ even}\}$$

$$\pi_0 G(\mathbb{R}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & k \text{ even;} \\ 0, & k \text{ odd.} \end{cases}$$

Split case $k = l = m$: $\quad X_s^\vee = \widetilde{X}_s^\vee = X^\vee, \quad Q_s^\vee = Q^\vee.$

$$\pi_0 G(\mathbb{R}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & m \text{ even;} \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, & m \text{ odd.} \end{cases}$$

Example (continued)

$$G = PSO_{k,l}, \quad n = k + l = 2m.$$

$$X^\vee = \{n_1\varepsilon_1^\vee + \cdots + n_m\varepsilon_m^\vee \mid n_i \in \tfrac{1}{2}\mathbb{Z},\ n_i - n_j \in \mathbb{Z},\ \forall i,j = 1,\dots,m\}$$
$$Q^\vee = \{n_1\varepsilon_1^\vee + \cdots + n_m\varepsilon_m^\vee \mid n_i \in \mathbb{Z},\ n_1 + \cdots + n_m \text{ even}\}$$

Non-split case $k < l$:

$$X_s^\vee = \{n_1\varepsilon_1^\vee + \cdots + n_k\varepsilon_k^\vee \mid n_i \in \mathbb{Z},\ \forall i = 1,\dots,k\}$$
$$\widetilde{X}_s^\vee = \{n_1\varepsilon_1^\vee + \cdots + n_k\varepsilon_k^\vee \mid n_i \in \tfrac{1}{2}\mathbb{Z},\ n_i - n_j \in \mathbb{Z},\ \forall i,j = 1,\dots,k\}$$
$$Q_s^\vee = \{n_1\varepsilon_1^\vee + \cdots + n_k\varepsilon_k^\vee \mid n_i \in \mathbb{Z},\ n_1 + \cdots + n_k \text{ even}\}$$

$$\pi_0 G(\mathbb{R}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & k \text{ even};\\ 0, & k \text{ odd}. \end{cases}$$

Split case $k = l = m$: $\quad X_s^\vee = \widetilde{X}_s^\vee = X^\vee, \quad Q_s^\vee = Q^\vee.$

$$\pi_0 G(\mathbb{R}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & m \text{ even};\\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, & m \text{ odd}. \end{cases}$$

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