Geometry in ancient Jaina works; a review

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In the context of their pursuit of cosmography the ancient Jaina scholars enunciated various geometric ideas in their compositions. After a lull for a period, in later centuries of the first millennium the earlier geometric understanding was vigorously brought forth and improved upon by various scholars, including *Śrīdhara*, *Vīrasēna*, and *Mahāvīra*, and still later by Ṭhakkura Phērū.

Many formulae from the ancient Jaina compositions have indeed been recalled, in the broader context of study of Jaina works, by various authors, starting with Bibhutibhushan Datta's 1929 paper. The aim of this article is to discuss some of the crucial formulae, and analyse their significance from a mathematical point of view, placing them in the global context. An attempt is also made to place the material in its natural setting, rather than looking at it purely through the prism of present day mathematics.

Keywords: Ancient Jaina Geometry, formulae for arcs and chords, āyatavr̥tta, Śrīdhara, Vīrasēna, Mahāvīra,Ṭhakkura Pherū.

Mathematics Subject Classification 2020: 01-06, 01A32, 01A99.

1 Introduction

Various ancient Jaina canonical works are found to have, embedded within them, many mathematical ideas. Geometry in particular was involved in this since the early times, in the cognition of the cosmographical model they adopted[1]. *Sthānānga sūtra*, a canonical work, estimated to be from 300 BCE or earlier ([5], p. 119) gives a systematic classification of the mathematical topics studied, which includes "*rajju*", a term referring to planar geometry, and "*rāsi*", dealing with some aspects of solid geometry; see [18], pp. 67-70. Other ancient works *Candraprajñapti, Sūryaprajñapti, Jambudvīpaprajñapti*, and others, also devote sections to exposition of geometric principles. The works of *Umāsvāti*, estimated to be from between the 2nd and 4th centuries CE (see [19], p. 20), *Tatvārthādhigamasūtrabhāṣya* and *Jambudvīpasamāsa* stand out as convenient references among

[1] A similar model was involved also in the *purāṇic* tradition, but a mathematical ideas discussed in the sequel seem to be unique to Jaina literature on the topic.

the sources of ancient geometric knowledge in the Jaina tradition.

Our current knowledge concerning the period of many ancient works involves considerable uncertainty, and in many cases even the knowledge of their contents is itself based on the commentaries by later scholars, the originals being no longer extant; see [19] for information on available texts. Given this context, our focus here will be on a critical appreciation of the available material from a mathematical point of view, within the historical framework associated with it, and while we shall endeavor, to the extent possible, to indicate the relevant dating of the individual segments of mathematical development, the exposition will be meant to be tentative in this respect.

After a lull for some centuries, near the end of the first millennium the earlier geometric understanding was built upon and carried forward by various scholars, including Śrīdhara, Vīrasēna, Mahāvīra, Nemicandra, Ṭhakkura Phērū, and others. For this later period we are relatively on firmer grounds with regard to the historical specificity, compared to the older period, though some uncertainties remain. Works in this later period also seem to have played an important role in inducting mathematics into various activities including commerce and artisanship, thereby popularizing it. They also made an impact in terms of pedagogy, introducing mathematical ideas to a wider populace. Our aim in this article will be to bring out the significance of the contents of the works, their overall impact, their priority in the overall historical context and influence on other traditions, etc.

2 Geometry in early Jaina works

Inspiration for study of geometry seems to have been intricately connected with engagements with the model of the cosmos that was envisioned. We briefly recall here the model, to set the context.

Jambudvīpa (= Earth) was visualized as a flat disc of diameter 100000 *yojana*, surrounded alternately by annular rings (*mandalas*), of water and land, of sizes doubling with each ring. The Sun and Moon move in concentric circles in a plane around the earth. The universe was envisioned to consists of three trapezoids piled over one another, with the top and bottom sides of the middle one matching with the top and bottom sides of the ones below and over it respectively.

Engagement with shapes in this way led to introduction of various basic geometric notions. Here are some geometric notions mentioned in *Sūryaprajñapti* (see [28], p. 62, [18], p. 148):

Rectilinear figures: *Trikoṇa* (triangle), *catuṣkoṇa* (quadrilateral), *pañcakoṇa* (pentagon). The reader may note that these terms are also in use in contemporary mathematics in many Indian languages. The term *koṇa*, for angle, is also found in *Sūryaprajñapti*; it may be noted however that numeration of angles, such as in degrees, minutes etc. is not found here. The term *samacatuṣkoṇa* is used for rectangle, and the square was called *samacaturaśra*. Oblique versions of these regular figures were referred to with a prefix *viṣama*, e.g. *viṣamacatuṣkoṇa* stood for parallelogram.

Curved figures: The circle was a frequent occurrence, and was called *cakravāla.* The semicircles were referred to *cakrārdha cakravāla*. There is also mention of *viṣamacakravāla* which has generally been interpreted in current literature as ellipse, but such precise association with the latter figure seems to be of doubtful validity; the term which translates as "oblique circle" is more likely to have stood for a general figure which is an oblique version of the circle in some, rather nonspecific, sense. No evidence is seen in the literature, justifying the interpretation as "ellipse" either in terms of it being realized as a conic section or in terms of its being the locus of a point whose distances from a pair of points (a posteriori the foci) add up to a constant. Thus the allusion to the concept of the ellipse in the Greek tradition, whose origin is traced to Menaechmus, is problematic; I shall also dwell on this point again when discussing the work of Mahāvīra.

Through the rest of this section we discuss various aspects of the geometry of the circle in the ancient Jaina tradition.

2.1 Circumference of a circle

One of the notable features of early Jaina geometry is the adoption of $\sqrt{10}$ as the factor involved in obtaining the circumference of a circle from its diameter, at an early stage.

Reckoning the size of the circumference of a circle in terms of its diameter has engaged human civilizations since the early times of development. In all early traditions, including the Chinese, Egyptian, Biblical, as well as in the early Vedic culture in India, the factor involved was taken to be 3. Thus the circumference was understood to be 3 times the diameter, and the same is seen to have been the case in early Jaina tradition; *Sūryaprajñapti* enumerates the circumferences of the *mandalas* around the *Jambudvīpa* applying the traditional factor of 3, but discards them (see [28], page 62), favoring the factor being $\sqrt{10}$.[2]

The factor $\sqrt{10}$ here corresponds to an approximation to π in the modern context, which is about 3.1623, in place of 3.1416...; thus the error involved is less than 0.7%. The same factor $\sqrt{10}$ was adopted by Chang Heng (also written as Zhang Heng, 78-139 CE), correcting the value 3 that was then prevalent in China. There was considerable contact between India and China during the immediate preceding period, and it may be surmised that adoption of the value in China may have its origin in India.[3]

In India the formula was also used in the Hindu, or *Siddhānta* tradition, by Brahmagupta and others. A more accurate value 3.1416 was described by Āryabhaṭa in *Āryabhaṭīya* (499 CE) but it did not turn up in common usage. In the overall historical context $\sqrt{10}$ is sometimes referred to as the "Jaina value" of π. It continued to be in use as late as the 15th century; see[5], page 131. Along with the Indian arithmetic and astronomy, this value of π also got incorporated in to Arabic mathematics; in particular it is found in Al-Khwarismi's work and is noted to have been from Indian sources; (see [23], page 166).

In various Jaina works (see [5], p. 132 for details) the circumference of *Jambudvīpa* is described, to quite an accuracy, to be 316227 *yojana*, 3 *gavyuti*, 128 *dhanu*, 131 *aṅgula* and a little over. Here *gavyuti*, *dhanu* and *aṅgula* are smaller units of distance prevailing at the time.[4] One *gavyuti* was equivalent to 1 *yojana*, a *dhanu* was a 2000th part of a *gavyuti,* and an *aṅgula* was 96th part of a *dhanu*; thus 1 *yojana* = 4 *gavyuti* = 8000 *dhanu* = 768000 *aṅgula*. It was apparently computed as the square root of $10 \times (100000)^2 = (10)^{11}$ in *yojanas*. The mention of "a little over", following such an accurate value, is notable; while knowing that to be so would of course be a natural outcome of the computational process by which it was arrived at, that it was considered

[2] In the Vedic *śulvasūtras* a revision, to $3\frac{1}{5}$, first appeared in Mānava śulvasūtra (in 7th century BCE, which antedates the Jainas); see [2], [3] for the details. The value is however not found used in the *Śulva* tradition. Other than Mānava's sūtra proposing that value as above, the only other place where the factor is involved in the extant *Sūlvasūtras* is a passing reference in *Baudhayana śulvasūtra*, where it is taken to be 3; see [3].

[3] Bibhutibhushan Datta recalls, in [5], that (Yoshio) Mikami has stated, in (the first edition of) [22] (p. 70), that "the value $\sqrt{10}$ is found recorded in a Chinese work before it appeared in any Indian work." and proceeds to assert it is incorrect.

[4] The terms *krośa* and *daṅḍa* are used in certain works in place of *gavyuti* and *dhanu* respectively.

worth recording shows the importance that they associated with the calculation. In *Tiloyapaṇṇattī* (composed by *Yativṛṣabha*, who is believed to have flourished sometime between 4th and 7th centuries) the circumference is described to an even finer unit of length, called *avasannāsanna skandha*, which is 8×10^{12}th part of an aṅgula!; see [9] for details.

The square root was apparently computed using the approximate formula $\sqrt{a^2 + r} \approx a + \frac{r}{2a}$, where *a* and *r* are positive numbers (see [9] for a detailed discussion on this), which is generally known after Heron of Alexandria (ca. 10 - 70 CE); the formula typically was used to find the square root of an integer, say x, by writing it as $a^2 + r$, where *a* is the largest square integer not exceeding *x*, and using the expression as above; the formula is valid for *r* negative as well, but that seems to have been rarely used in ancient times. The formula was known to the ancient Babylonians as well (see, for instance, [7]), but no square root of such a large number seems to have been determined to comparable accuracy prior to this instance. Analogous computations are also found reported in other ancient Jaina works (see [14] for numerous examples).

2.2 On the choice of the value $\sqrt{10}$

The choice of a factor in the form of a square root is rather unique. Generally in ancient traditions there has been an inclination to choose a fractional number for such factors. Proficiency acquired in computing square roots, may have been a factor involved in adopting a square root for the factor.

We do not know how the factor was arrived at. Various methods have been suggested in literature in this respect. One of these goes back to a commentator Mādhavachandra Traividya (ca. 1000 CE), in his commentary on *Tiloyasāra* of *Nemicandra* (ca. 975 CE). There are also suggestions coming from later authors, in particular by K. Hunrath in the 19th century (see [28], p. 65), G. Chakravarti (1934) and R.C. Gupta (1986) (see [10]). While Hunrath's argument involves comparison with an inscribed 12-sided regular polygon, the later arguments base themselves on consideration of (inscribed) octagons. The arguments however are mostly unsatisfactory. As explained in [10] the method consists of determining the perimeter of a regular inscribed octagon in a circle with unit diameter, but in the course of the computation involving square roots, approximations by rational numbers used, causing substantial change in the value, which leads to an answer that is near to $\sqrt{10}$, but still off from it by quite a margin; and yet this is taken as a justification for the choice of the number as the circumference of the circle; it may also be noted, as done in [10] that the perimeter of the regular octagon is substantially less than the circumference, while $\sqrt{10}$ is significantly greater (by over 1 %). There is also nothing natural about the approximations adopted, that one may grant it as being likely to be picked also by others, in particular the ancient mathematicians. The reasoning of Hunrath described in [28] is also open to similar criticism, to quite an extent, if not fully. In [10] Gupta introduces another possibility, involving a process of averaging. The author mentions that "The process of averaging is known to be a popular and useful ancient technique especially when the exact result of derivation was unknown or difficult". He then proceeds to compute the perimeters of two regular octagons, one inscribed and another circumscribing the circle of unit diameter, and observing that the square of the former is greater than 9 and that of the latter is less than 11, makes a case for $\sqrt{10}$ as the choice for the circumference of the circle. In the present author's view, while the idea of involving the process of averaging may indeed be of significance, too much leeway is taken here in trying to fit the answer; firstly, the actual computations produce values of the perimeters themselves, but averaging is done only of bounds on their squares, without justification for such a step, and secondly the lower and upper bounds 9 and 11 as above are not the precise bounds obtained and thus the "average" is not anything of intrinsic significance. Thus while there is something to the idea of involving the averaging process for an explanation, many questions remain unanswered, in making it plausible explanation.

2.3 Chords and arrows

Consider a circle with diameter *d* and a chord of the circle. The chord divides the circle into two arcs and the chord together with the smaller of the arcs resembles a bow figure. In an obvious extension of the imagery the straight line segment joining the midpoint of the chord to the midpoint of the arc is called the arrow corresponding to the chord. We shall denote by *c* a chord and by *h* the corresponding arrow; (the terms stand for the geometric form as well as the associated numerical size, in some fixed units). Various early works contain the following formula relating these quantities[5] :

$$c = \sqrt{4h(d-h)} \text{ .... } (1)$$

This appears in Verse 180 of *Jyotiṣkaraṇḍaka* which purports to expound the knowledge contained in *Sūryaprajñapti* (see [28], p. 63;[6] it may be pointed out here that in a footnote on this page, it is also noted that the same verse, with a modification in the last line, appears in Bhāskara I's commentary on *Āryabhaṭīya*). The formula is also mentioned in Umāsvāti's works *Tattvārthādhigama sūtrabhāṣya* and *Jambudvīpasamāsa*; see [5], pages 124-125.

The following formulae which are deducible from (1) by simple algebraic manipulations have also been noted in the sources recalled above. [7]

$$h = \frac{1}{2}(d - \sqrt{d^2 - c^2}) \text{ .... } (2) \text{ and } d = (h^2 + \frac{c^2}{4})/h \text{ .... } (3)$$

In turn, either of these formulae readily implies Formula 1.

While there has been a general appreciation of the ancients having discovered the formulae, there does not seem to have been a detailed discussion in literature on specifically how they may have arrived at the formulae. Here we propose two possible routes which may have led them to the (interrelated) set of the three formulae. One way of deducing them would be as follows.

Let AD be a semi-chord corresponding to the given chord, and BC be the diameter of the circle

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[5] It may be emphasized that here while we describe here various formulae here using algebraic symbolism, for convenience of exposition, as has been the practice in the recent literature on the topic, the original descriptions are in terms of names of the entities and verbal presentation of the operations involved.

[6] It appears that [28] cites the former for want of access to the text of the latter - relating to the former, [28] mentions in the Bibliograpy, on page 268, a work "*Jyotiṣkaraṇḍaka* of *Vallabhāchārya* with *Malaygiri*'s commentary, Rishabhdevji Kesarimalji Samstha, Ratlam, 1928"; the book however does not seem to be accessible any longer (private communication from Anupam Jain). With regard to the period of *Jyotiṣkaraṇḍaka* I may recall here a passing mention in [28], page 70: "alleged to have been codified at the Valabhi council of the 4th or the 6th century."

[7] By (1) we have $c^2 = 4(dh - h^2) = 4dh - 4h^2 = d^2 - (2h - d)^2$, and equality of the first term with the third and fourth terms leads to (3) and (2) respectively. Though the algebraic operations and the symbolic notation involved here were unavailable in the ancient times, with some practice the calculations involved here can indeed be performed mentally, or with the aid of simple geometric constructions, which is presumably how it would have been done.

passing through D, perpendicular to the chord (see Figure 1). Then, as the angle subtended by the diameter at a point on the circumference of the circle, the angle BAC is a right angle. As BC and AD are perpendicular to each other it can be deduced from this that the triangles ABD and CAD are similar triangles. Since the corresponding sides of similar triangles are proportional, it follows that the proportion BD:AD is the same as AD:DC, which is the desired result, as AD, DC and AC are $c/2$, h and $d - h$ respectively.

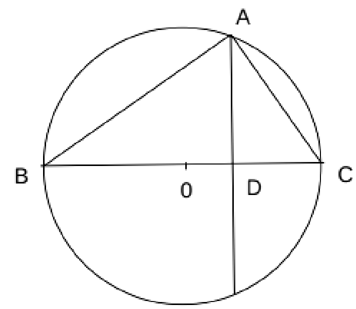

Figure 1: Chord and arc

Before discussing another possibility we observe the following. In the three formulae as above the manifest objective has been to express each of the three quantities, the chord c, the arrow h and the diameter d in terms of the other two, so that a potential user can determine the third one, upon knowing any two of them.[8] Underlying these is the quadratic relation

$$\left(\frac{d}{2}\right)^2 = \left(\frac{c}{2}\right)^2 + \left(\frac{d}{2} - h\right)^2 \quad \text{............}(4)$$

The latter is simply the relation $OA^2 = AD^2 + OD^2$ corresponding exactly to Pythagoras' theorem applied to the right angled triangle OAD; (note that OD is the difference of OC and DC). While the Pythagoras theorem is not found explicitly stated in the extant ancient Jaina works, there are good grounds to believe that they were familiar with it (see below for more on this). This suggests another possibility that Formula (4) could have been noted first, by application of the theorem, and the expressions in the first three formulae, meant for computation of the individual entities in terms of the other two, would have been deduced from it by simple manipulations. Which route they may have followed, it would be difficult to ascertain, but to the present author the second possibility seems more natural and likely in the overall context. If it turns out that it is the first method that was adopted, the discussion above shows how the arguments could have led them to conclude the validity of the Pythagoras theorem in their context (namely obtain a proof for it in the realm of visual geometry, though not in an axiomatic formalism as in Euclidean geometry).

2.4 Chords and arcs

Now let c be a chord in a circle of diameter d, as before, and let a be the smaller arc segment cut out by the chord from the circumference. What is the relation between (the lengths) c and a ? An

[8] The author would have aware of the mutual relationship between the formulae; in the historical literature on the topic the expressions are listed as if they are independent formulae, which is rather misleading.

intriguing formula for this is found along with the formulae as in the last subsection, in the sources mentioned there. Given c and a as above and h the corresponding arrow, the prescription is:

$$a = \sqrt{6h^2 + c^2} \quad \ldots\ldots\ldots\ldots\ldots (6)$$

Variations of this, describing h and c in terms of the remaining two quantities are also given in *Jyotiṣkaraṅḍaka*, as

$$h = \sqrt{(a^2 - c^2)}/6 \text{ and } c = \sqrt{a^2 - 6h^2}$$

(see [28], p. 63).

The reasoning involved in arriving at such a formula seems to be broadly the following: Consider the triangle-like region (see Figure 1) between the semi-chord AD, the arrow DC and the half the arc-segment joining A to C; it resembles a right angled triangle, except for the diagonal segment being curved. The length of the straight segment AC, joining one of the endpoints to the midpoint of the arc, is given by the Pythagoras theorem to be $\sqrt{h^2 + (c/2)^2}$ = ½ $\sqrt{4h^2 + c^2}$. On account of being curved the arc segment between the midpoint and the endpoint has to be greater than that. Taking into account the similarity of the situation with the right angled triangle ADC and the Pythagoras theorem one may seek an expression in the form : ½ $\sqrt{sh^2 + c^2}$, with a multiplier $s > 4$ in place of 4. We see that for the expression to tally in the case when the chord is the diameter (so $c = d$ and $h = \frac{1}{2}d$, in which case $a = \frac{1}{4}\pi d$) we should have - $\sqrt{\frac{1}{4}s + 1} = \frac{1}{2}\pi$; since π was taken to be $\sqrt{10}$ this leads to $s = 6$, which was chosen constant. [9]

The issue of relating the length of an arc to the corresponding chord was also involved in the work of Heron of Alexandria (ca. 10-70 CE). He introduced two formulae:

The first one is $a = \sqrt{4h^2 + c^2} + ¼\, h$ (in the notation as above, making a simple linear Increment in terms of h in the term $\sqrt{4h^2 + c^2}$ corresponding to the length of the line segment path noted above). The second one, which is apparently meant to be a more refined one, is

$$\sqrt{4h^2 + c^2} + \{\sqrt{4h^2 + c^2} - c\}\frac{h}{c} = \sqrt{4h^2 + c^2}(1 + \frac{h}{c}) - h$$

([16], p. 331, and [5], p. 130). It is also mentioned in [5] (p. 130), citing (an earlier edition of) [22], p. 62, that Ch'en Huo (in China, 11th century CE) gives the formula $a = c + 2h^2/d$.

These formulae are significant as the earliest attempts, in the pre-trigonometry era, towards

[9] We may also recall here the following: a formula for the semi-arc in terms of the half-chord (sine) and the arrow (versine) is described by *Nīlakaṇṭha Somayājin* in his commentary on *Āryabhaṭīya*, which in the notation as above corresponds to :

$$\frac{a}{2} = \sqrt{\frac{4}{3}h^2 + \left(\frac{c}{2}\right)^2}, \text{ or } a = \sqrt{\frac{16}{3}h^2 + c^2};$$

the formula is derived using 'infinitesimal' methods and is recommended, by *Nīlakaṇṭha* to be used only for small arcs; (see [28], pp, 179 – 182 for details). It is also mentioned in [28] (page 182), interestingly, that the commentator states that the derivation "is implied" by the second part of *sūtra* 17 of *Gaṇitapāda* of *Āryabhaṭīya* (following the statement of the 'Pythagoras thorem'); linguistically however that part actually corresponds to Formula (1) described in § 1.3 above, relating the diameter and the arrow; see [31], page 59.

understanding the interrelations between the chord, the arrow, and the corresponding arc.[10] In the following table we list the values obtained from the formulae as above, together with the true values, upto 4 decimal digits, for the arclength corresponding to chords in the unit circle subtending an angle 2θ (viewed for convenience as the "vertical" chord subtending the angle from −θ to θ with respect to the usual coordinatization) for various values of θ; we note that in this case $c = 2\sin\theta$ and $h = (1 - \cos\theta) = 2\sin^2(\frac{1}{2}\theta)$, and hence the expressions involved in the Jaina and Heron's formulae are given, respectively, by the following:[11]

$$\sqrt{6h^2 + c^2} = 2\sin(\theta/2)\sqrt{5 - \cos\theta},$$

$$\sqrt{4h^2 + c^2} + \frac{1}{4}h = 4\sin(\theta/2) + \frac{1}{2}\sin^2(\theta/2), \text{ and}$$

$$\sqrt{4h^2 + c^2}(1 + \frac{h}{c}) - h = 4\sin(\theta/2)(1 + \frac{1}{2}\tan(\theta/2)) - (1 - \cos\theta).$$

As $d = 2$ Ch'en Huo's expression corresponds to $2\sin\theta + (1 - \cos\theta)^2$ in this instance.

Angle	arc-length true value	Jaina value	Heron's 1st value	Heron's 2nd value	Ch'en Huo's value
15°	0.5236	0.5243	0.5306	0.5224	0.5188
		(+0.0007)	(+0.0070)	(-0.0012)	
30°	1.0472	1.0525	1.0688	1.04	1.0179
		(+0.0053)	(+0.0216)	(-0.0072)	
45°	1.5708	1.5858	1.6049	1.5549	1.5
		(+0.0150)	(+0.0341)	(-0.0159)	
60°	2.0944	2.1213	2.125	2.0774	1.9821
		(+0.0269)	(+0.0306)	(-0.0170)	
75°	2.618	2.6511	2.6203	2.6281	2.4812
		(+0.0331)	(+0.0023)	(+0.0101)	
90°	3.1416	3.1623	3.0784	3.2426	3

[10] We note also the following: In the *Siddhānta* astronomy there is a well-known approximate formula, with considerable accuracy, for sin θ, known after *Bhāskara* the first (though it is found also in *Brahmagupta*'s work, and may also have been known earlier); see [8] for details. The formula expresses sin θ as a rational function of θ. The function is a ratio of two quadratic polynomials and may be readily be inverted to produce a formula for θ in terms of sin θ, which corresponds to the issue at hand. We shall however not go into the details here.

[11] We have $6h^2 + c^2 = 6(1 - \cos\theta)^2 + 4\sin\theta^2 = 2(5 - 6\cos\theta + \cos^2\theta) = 2(1 - \cos\theta)(5 - \cos\theta) = 4\sin^2(\theta/2)(5 - \cos\theta)$, which gives the first equality as above. Also $4h^2 + c^2 = 4(1 - \cos\theta)^2 + 4\sin^2\theta = 4(2 - 2\cos\theta) = 16\sin^2(\theta/2)$, which readily implies the other two equations.

The table shows that the values from the Jaina formula give good estimates to the corresponding true values. In [5] (p. 130) Datta states "It will be observed that the Hindu value of the arc is older and more accurate than the other two." It may firstly be clarified that by "Hindu" value the author means the values from the Jaina tradition as discussed above.[12] Secondly, while Datta mentions "value" in singular, presumably meaning the statement to be true for any chord and arrow, the table shows that the statement regarding comparisons does not hold consistently for all values involved, except with regard to the value of Ch'en Huo. Little is known about the background of the latter (see [22] for some details confirming this), and the prescription may have been meant for usage in some practical context - it may be noted that the stipulated computation is much simpler compared to the other formulae, and that the formula implicitly takes the value of π to be 3, whereas much better values were known in China since the times of Liu Hui (3rd century) and Zu Chongzhi (5th century). Also, Ch'en Huo seems to be a relatively minor figure in Chinese mathematics, not commonly mentioned by historians of Chinese mathematics. On the other hand the poor choice involved, as late as the 11th century, could also be related to the fact the mathematical, and general scientific, tradition in China has witnessed major ups and downs over the centuries. On the whole a comparison with values of Ch'en Huo does not seem to be of much relevance. With regard to comparisons with Heron's formula the relative level of accuracy of the two formulae is seen to depend on the range of the angle involved; it is remarkable however that the formula is more accurate for the most part than Heron's first formula, and more elegant than his second refined one.

2.5 Area formulae

Along with the various formulae concerning various lengths that we discussed with reference to *Jyotiṣkaraṅḍaka*, representing the knowledge from *Sūryaprajñapti*, as also *Tattvārthādhigamasūtrabhāṣya* of *Umāsvāti*, there is also a formula describing the area A of the circle being given by

$$A = \frac{1}{4} C\, d \quad \ldots\ldots\ldots\ldots\ldots (7)$$

where C is the circumference and d is the diameter of the circle.[13]

This relation may have been recognized by thinking of the partition of the circle into thin isosceles triangles with a common vertex at the centre of the circle and the other two on the circumference. Each of these triangles have height $\frac{1}{2}d$ and base a little segment of the circle. The area would have been realized, heuristically, to be the sum of areas of these triangles, thereby deducing that the total area is $\frac{1}{4}Cd$. There however does not seem to be any specific evidence towards this, and a claim for such reasoning involving "infinitesimals" may not be sustainable.

While in the later Jaina works we find formulae for areas of segments of circles cut off by

[12] The issue did not feature, as far as is known, in the *Siddhānta* tradition which may be suggested by the reference; elsewhere in the article, as also at various other places in literature, the term Hindu has been used in a similar fashion, which includes "Jaina" with "Hindu".

[13] In ancient cultures typically the ratio of circumference to the diameter and that of the area of a circle to its radius appear independently, and the corresponding numerical values are often not equal; the equality of the two ratios, which is an a posteriori fact, was historically inferred at some stage, through reasoning, whether analytical or heuristic. In the Jaina context, once the area was realized by such a formula, in terms of the circumference, the equality became automatic.

chords, which we shall come to later, the ancient works do not seem to have engaged with such a question (see [5], p. 130). Similarly formulae for volumes, other than for certain simple rectilinear shapes, seem to have made an appearance only later.

3 Jaina Geometry in later phases

There seems to have been a lull for a while in the interest in mathematics in the early centuries of CE, after Umāsvāti, until around the 8th century when the earlier geometric understanding was vigorously brought forth and carried forward by various scholars.[14] During the period from the 8th to 10th or 11th centuries we come across important works, due to Śrīdhara,, Vīrasēna, Mahāvīra and Nemicandra. Again from somewhat later, late 13th and early 14th centuries, we have the work of Ṭhakkura Phērū which made an impact. In this section we shall discuss some aspects of these works focusing on geometry.

Before going into the mathematical details, a few words putting the development in an historical perspective would be in order. The period in question succeeds the heyday of the Siddhānta tradition of mathematical astronomy, with the works of Āryabhaṭa, Brahmagupta, and Bhāskara the first; the name of Varāhamihira may also be added here for the broader context, though his contributions relate primarily astronomy (and astrology) and with no significant mathematical component, unlike in the case of the others mentioned; see [25] for a general reference. Āryabhaṭa's *Āryabhaṭīya,* composed in 499 CE, set the tone, serving as pioneering work, for a tradition that lasted over a thousand years. It was also inspirational to the Kerala school of Mādhava that flourished from the middle of the 14th century for almost 3 centuries, making remarkable contributions. Āryabhaṭa was followed in a little over a century by the works of Bhāskara the first, in the early part of the 7th century, which not only elaborated on Āryabhaṭa's concise presentation and clarified various matters, but also introduced new techniques in astronomical as well as mathematical aspects. Around the same time in 628 CE Brahmagupta composed his *Brāhmasphuṭasiddhānta,* which is an extensive work in the tradition, though critical of Āryabhaṭa in various respects. Both Bhāskara the first and Brahmagupta show influence of the earlier Jaina works in some ways, including the use of $\sqrt{10}$ for π (which is absent in *Āryabhaṭīya*).

As to be expected, the *Siddhānta* tradition did sustain a broad mathematical learning. However its *raison d'etr*e remained pinned to astronomy, with applications around astrology and issues involved in producing almanacs, predicting eclipses etc., and the mathematical learning associated with it remained confined to the community engaged with these pursuits. It was not until Bhāskarāchārya's *Līlāvatī* that an independent book on mathematics seems to have emerged in that tradition; *Līlāvatī* and its successor *Bījagaṇita*, were also meant to be parts of the larger treatise *Siddhāntaśiromaṇi* in the then prevailing model of *Siddhānta* works. However by this period the general interest in the mathematical topics discussed in the wok had got extended well beyond the community involved with astronomy, as a result of which the *Līlāvatī* part came to be copied numerous times and studied on a much wider scale than the larger treatise, thus giving *Līlāvatī* an identity as a mathematical book on its own. A similar development occurred, though on a smaller scale, with regard to *Bījagaṇita,* which is more technical than *Līlāvatī* but nevertheless of considerable independent interest, outside the ambit of astronomy.

[14] A few names and works from the interim period have been mentioned in literature (see [19], p. 20, in particular) but no mathematical details have been prominently known.

Accumulation of basic mathematical understanding emerging from the Jaina philosophical tradition, the *Siddhānta* tradition of mathematical astronomy, and possibly also imports from other cultures, from within the country and also from outside, seem to have gradually percolated to a wider populace, outside the limited circles of scholars and practitioners engaged with it, creating an appetite for mathematical knowledge. Mathematics by this time seems to have come into greater and active contact with trade, artisanry and a variety of practical activities, and sources of independent learning of mathematics would have been sought after. The Jaina scholars noted above catered substantially to the emerging appetite for mathematical learning, and were also instrumental in popularizing the use of mathematics in a range of areas. They also shaped the mathematical activity of the time with new contributions. Their works inevitably played a pedagogical role, incorporating as they did, various features engendering interest in the subject. We shall now discuss the works of some select authors in respect of some geometric aspects involved in the work. As many of the basic geometric ideas are common to all of them, and have been discussed in current literature especially in the context of the pedagogical role, our emphasis here will be on the features that distinguish the individual works.

3.1 Śrīdhara and the volume of a sphere

There has been a debate in among historians on whether Śrīdhara belongs to the Jaina tradition or not, but thankfully it now seems to be settled, confirming that to be the case. At any rate the overall academic profile of his work is consistent with his being from the Jaina tradition, providing adequate testimony in that respect. Similarly, there had also been some confusion about his period, and in particular whether he preceded or succeeded Mahāvīra. It is now confirmed that he flourished sometime around the middle of the 8th century, *before* Mahāvīra; see [11] for various details in this respect - see also [20] and [32].

Two of his works have come down to us, *Triśatikā* and *Pāṭīgaṇita*, the latter only in a single copy, which is also incomplete. Both the works deal with a variety of topics in arithmetic and geometry.

Pāṭīgaṇita (see [30]) in particular contains a large number of examples and illustrations from everyday life. On account of this feature it also throws light on various aspects of life during the author's time. After a discussion of the formulae for areas of rectangles, trapezia, triangles etc. the circumference and area of the circle is described, with $\sqrt{10}$ as the value for π, following the Jaina tradition. Though Āryabhaṭa had introduced a more accurate value 3.1416 (the original is in terms of the circumference of a circle of diameter 20000, which in the decimal notation corresponds to this value of π), it did not attain currency even in the scholarly circles, and use of the Jaina value was preferred by many authors.

For the area of a quadrilateral Śrīdhara first recalls the crude rule, as the product of the average lengths of the pairs of opposite sides, namely $\frac{a+c}{2}$, $\frac{b+d}{2}$, where *a, b, c* and *d* are the sides of the quadrilateral, labeled cyclically. Such an expression for the area goes back to *Barahmagupta's Brāhmasphuṭasiddhānta* (verse XII, 21, where it is referred to as *sthūlaphalaṃ*) and has been repeated in various sources.[15] After recalling the formula Śrīdhara points out in *Pāṭīgaṇita* specifically that it is applicable only when the differences in the sizes are small, and proceeds to a discussion on more exact formulae, giving in particular the formula for the areas of trapezia (see [30]).

[15] I was struck, and rather dismayed, to find that the overseer involved in the construction of my apartment building also used the formula!

In *Triśatikā* the formula $\sqrt{s(s-a)(s-b)(s-c)}$ for the area of a triangle, where *a, b, c* are the sides and *s* the semi-perimeter of the triangle, which is generally known after Heron of Alexandria is described. The corresponding formula $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where *a*, *b, c, d* are the sides and *s* semi-perimeter of the quadrilateral, known after *Brahmagupta*, is mentioned; the author however misses, like various other Indian authors around the period[16], stipulating the condition of cyclicity that is required for its validity.

Let me now come to a notable feature in Śrīdhara's work which does not seem to have received due attention. In the Indian context the credit for having been the first to give the correct formula for the volume of a sphere, is normally given to Bhāskarāchārya (12th c.); see [28], p. 210.[17] However, Śrīdhara's *Triśatikā* (verse 56) describes the volume (in our notation) to be $\frac{d^3}{2}\left(1+\frac{1}{18}\right)$, where d is the diameter; see [12] for a detailed discussion. It would seem that Śrīdhara had the correct formula $\frac{4}{3}\pi\left(\frac{d}{2}\right)^3$ in mind, but for the factor $\frac{\pi}{3}$ he put down only the approximate value $1+\frac{1}{18}$. This ought to be seen in context by recalling that unlike now there was no standard notation for π at that time; as $\sqrt{10}$ was the routinely adopted value for π the factor could have been expressed as $\sqrt{10/9}$, in which case we could have been more certain that he means it to be $\frac{\pi}{3}$. However, as *Triśatikā* was meant to be a short introduction, and the formulae were written down to facilitate computation by potential users of them, a simpler expression in the form of a fraction may have been preferred; the issue of versification of the statement may also have contributed to choosing the format with the fraction, which in this instance is easy to split and describe, compared to $\sqrt{10/9}$. It may be noted in this respect that 1 + 1/18 was the standard approximation that time for the factor $\sqrt{10/9}=\sqrt{1+1/9}$ involved (by the square root formula which we alluded to earlier, in Section 2.1). Unfortunately as there is no indication of how the volume formula was arrived at, and the reference to the issue is not found in the extant part of Pāṭīgaṇita, it would not be possible to ascertain such a conjecture.

Notwithstanding whether the formula may be identified with the correct one as we now know, it is quite a good formula in terms of accuracy, involving an error of less than 1 percent. It is a peculiar quirk of history that despite such a good and usable formula having been discovered, various later authors even in the same tradition, including some celebrated ones, did not adopt it; the formulae described by Mahāvīra about a century later, and even by Pherū as late as the 14th century, are substantially cruder than this formula.

Curiously Śrīdhara does not seem to have considered the surface area of the sphere, which in the Jaina tradition appears in the work of Mahāvīra (see infra); see [11] for details on various volume computations in ancient India. It may be mentioned however that as in classical Jaina mathematics Śrīdhara considers regions cut off from the circle by a chord and the minor arc (the smaller of the arcs resulting from the division); however, the formulae for the arclength etc. are not recalled, but a formula is given for the area of the region between a chord and minor arc, to be $\frac{\sqrt{10}}{3}.\frac{c+h}{2}.h$ (*c* and *h* are the chord and arrow respectively, as before); the formula is

[16] While apparently Brahmagupta meant the formula to be for cyclic quadrilaterals (see [21] for details) it is not adequately clear from the text, which seems to have led to perpetual confusion on the issue in India, until it was refuted by Āryabhaṭa II (11th century) and Bhāskaracārya (12th century); see [4].

[17] It may be recalled in this respect that *Āryabhaṭīya* purports to give a formula, but it is incorrect; see [28], p. 197; other early Indian authors have not discussed the issue.

somewhat crude, and what the general idea in the derivation might have been is not clear.[18] For the same area Mahāvīra in *Gaṇitasārasaṅgraha* gives the expression $\sqrt{10}$ch/4 which is quite crude (Datta describes it as "wrong"; see[5], p. 145). Nemichandra gives the latter value as gross (*sthūla*) value, mentioning Śrīdhara's value as neat (*sūkṣma*) one; see [6].

3.2 *Vīrasena* and the volume of a conical frustrum

Vīrasena, the author of *Dhavalā Tīkā* on the *Ṣaṭkhanḍāgama* apparently flourished in the eighth or ninth century (the year 816 is listed as his period in [19]; [28] however mentioned him as being from the 8th century). He is often quoted for a good approximation to the value of π given in the verse from *Ṣaṭkhanḍāgama,* Vol. IV.

व्यासं षोडशगुणितं षोडशसहितं त्रिरूपरूपैर्भक्तम् ।
व्यासं त्रिगुणितं सूक्ष्मादपि तद्भवेत् सूक्ष्मम् ।

Vyāsaṁ ṣōḍaśaguṇitaṁ ṣōḍaśasahitaṁ trirūparūpairbhaktam ।
Vyāsaṁ triguṇitaṁ sūkṣmādapi tadbhavet sūkṣmam ।

"The diameter multiplied by 16, together with 16, divided by 113, and three times the diameter becomes finest of fine (value of the perimeter)" (translation as in [28]). The part "together with 16," is a rather puzzling feature in this, since one could not be adding a fixed number independent of the size of the diameter in the computation of the circumference. When the part is dropped out of consideration we see that what is described corresponds to an approximation to π as $3\frac{16}{113} = \frac{355}{113} \approx$ 3.1415929 . . . , coinciding with the true value for π upto 6 decimal places, the latter being 3.1415926 . . . . Such a value was earlier proposed in China by Zu Chongzhi (5th century), who determined the value of π to be between 3.1415926 and 3.1415927 and deduced from it in some way the fractional approximation as above (see [22], p. 50). In the overall context it seems plausible that the value mentioned by Vīrasena may ultimately have its origin in the work of Zu Chongzhi.

That leaves us with the issue of the "together with 16" part mentioned above. Generally the response in the literature on the topic to this peculiarity has been not to pay attention to it. In this regard I would like to record here a suggestion, of linguistic nature, which may explain the point. The suggestion is that "together with 16" relates to the same 16 appearing in the earlier part. Thus a small emendation in the interpretation, somewhat like "The diameter to be multiplied by 16 and the product together with that 16 to be divided by 113 ...", would set things right (thus, while *sahitaṃ* indeed suggests addition, that need not be the only interpretation, especially when it is seen to lead to a manifestly wrong inference); in a scholarly statement one does not normally expect such a repetition of the kind being suggested, but in colloquial communication it seems well imaginable, and hence also may be involved as a valid form in some linguistic practice. Moreover, if it is true that the statement has a Chinese precursor then it is possible the nature of the original statement could have prompted the repetition, even though it is uncalled for and misleading in the Sanskrit rendering.

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[18] It may be recalled here that Metrica of Hero of Alexandria offers $\frac{c+h}{2}\,h + \frac{1}{14}\left(\frac{c}{2}\right)^2$ as the formula in this respect, and the same was adopted in the ancient Hebrew text (period uncertain) Mishnatoha-Middot; see [23], p. 163.

Vīrasena actually deserves to be better known for another of his work which does not seem to have received much attention; this consists of his formula for the volume of a conical frustrum; see [28], p. 203. In his *Dhavalā Ṭīkā* the volume of such a frustrum with diameters *a* and *b* at the base and the top respectively, and height *h* is stated to be $\frac{\pi h}{4} . \frac{a2 + ab + b2}{3}$. It is worth mentioning that, unlike in much of the extant ancient and medieval mathematical literature in India, we find in this work a description of the method of determining the volume. Moreover, the method involves summation of an infinite series and the idea of infinitesimals, akin to Calculus; see [28], pages 203-205 for details of the computation. While neither the formula nor the general idea of using infinitesimals as involved here may be considered unprecedented in the global context, and are reminiscent of computations going back to Archimedes and Liu Hui, in the Greek and Chinese traditions respectively, the details are arguably new, and seem to be a notable first, especially in mathematics in India.[19]

3.3 Mahāvīra on the quadrilaterals

Gaṇitasārasaṅgraha of *Mahāvīra*, from around 850 CE, has been one of the very influential books in mathematics and mathematical education, especially in South India. It served as a textbook of mathematics over a broad geographical region for some centuries, quite likely until the other popular medieval Indian book on arithmetic and geometry, *Līlāvatī*, took the place sometime in the 12th century. *Gaṇitasārasaṅgraha* is an extensive and leisurely exposition of arithmetic, combinatorics, and geometry, with numerous examples in the form of exercises.

Kṣetragaṇitavyavahāraḥ, the Chapter on geometry, is the second largest of the nine chapters in the book, with well over 200 verses, that include a large number of numerical examples. An interesting feature of the exposition is that there is a separate section in the chapter devoted to formulae for practical usage, described as *vyāvahārika gaṇitaṃ*. Many relations described here are approximate, *sthūla*, in the nature of thumb rules facilitating quick computation; on the other hand they cover various geometric forms not commonly dealt with in other works, e.g. the shape of a conch or elephant tusk etc. Better formulae, often involving somewhat more intricate expressions, are described separately later. These are described as *sūkṣma* (literally meaning "fine") values; they are meant to be the better values as perceived by the author, typically involving a little more intricate computation compared to the *sthūla* values, but not necessarily exact in general; in some instances the formulae from both the groups are only surmised ones (and were apparently not derived) and are inexact.

Mahāvīra also recalls the formula for the area of quadrilaterals that goes back to Brahmagupta, which we recalled earlier. In his Introduction to *Gaṇitasārasaṅgraha,* edited by M. Raṅgācārya [26], p. xxiii, David Eugene Smith seems to make a point that it is not observed that the formula holds only for a cyclic figure.[20] In this respect Sarasvati Amma points out ([28], p. 92) that the relevant verse (GSS VII.50) excludes "*viṣama caturarśra*" from application of the formula, and follows it with the comment "which makes it probable that he was aware of the restriction in the

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[19] In particular it seems plausible that it may have influenced Bhāskaracārya, who apparently was familiar with the work of Śrīdhara. It would however take a closer analysis of the of the two to come to any definitive conclusion.

[20] His observation extends to Brahmagupta as well which, as mentioned earlier, has been clarified in [21].

formula". Whether this properly clarifies the issue, and a similar point arising in respect of the formula for the diagonals of the quadrilaterals is debatable. We shall however not go into further discussion on it here. Mahāvīra also deals with a variety of problems concerning construction of quadrilaterals, involving ideas from the general areas of geometric algebra and Diophantine equations. A comprehensive analysis and exposition of these parts of *Gaṇitasārasaṅgraha* would be worthwhile, but we shall not go into it here.

3.4 Mahāvīra, on curved figures and the *āyatavṛtta*

One of the interesting features of Mahāvīra's *Gaṇitasārasaṅgraha* with regard to geometry is the consideration of issues of mensuration of certain figures that are generally not found elsewhere in literature. Among these are various planar figures involving combinations on the theme of circle and semicircle : a conchiform area (*kambukāvṛtta*, thought of as formed of two semicircles of different diameters joined along the diameters, on one side of the longer one). Then there are surfaces with spatial curvature. He considered also sections of the sphere cut off by planes, in the form of concave and convex circular areas (called *nimnavṛtta* and *unnatavṛtta* respectively), as well as outward and inward curved annular shapes (*bahiṣcakravālavṛtta* and *antaṣcakravālavṛtta*). While on the one hand this is in the nature of generalization of ideas from ancient Jaina literature, about the interrelation of the arc and the chord, there seems to be a strong motivation here in the form of practical applications; the examples to which the formulae are applied include various interesting practical shapes, such as the tortoise-back or concave sacrificial pit. The formula for the area of such a surface is given, in GSS-VII-25, as $\frac{1}{4}$th of the product of the circumference of the circular boundary and what is referred to as the "*viṣkambha*". The latter term has been interpreted in much of the historical literature on the topic, including [26] and [24], as meaning the diameter of the same circle. Such an interpretation is erroneous, as has been pointed out by R.C. Gupta (see [13]); the same product was given as the gross (*sthūla*) value for the area of the planar disc through the boundary circle; actually they were aware that the latter was smaller than the fine (*sūkṣma*) value for the same figure, and would surely have realized that the area of the curved surface would be even more than that. An interpretation of the term *viṣkambha* as the diameter along the surface, viz. length of the curve on the surface joining diametrically opposite points contained in a plane passing through the centre of the sphere, is more satisfactory, and has been justified in [13] based on a historical perspective on such empirical formulae.

The formula applied in particular to a hemisphere yields the area to be $C^2/8$, where C is the circumference, and thus the area of the sphere may readily be inferred to be $C^2/4$. As recalled earlier in this article Mahāvīra seems to have been the first one in the Jaina tradition to have had a formula (though implicitly) for the surface for the area of a sphere. The reader may note that the true value of the area of the sphere is C^2/π, so the value here is rather crude, but is of historical significance as an empirical value. The formula was later improved by Pherū to $\frac{10}{9}C^2/4$, perhaps on experimental grounds, bringing it a little closer to the true value; see [27].

Along with these there is also another figure mentioned, which has been called *āyatavṛtta*. The term has been translated in current literature (e.g. [26], [28], [24], [13], and others) on the topic as "ellipse". This however does not seem to be a valid choice, when the latter term is understood in the mathematical sense, originating from the Greek tradition.

Unfortunately the work contains no description of how the figure was drawn, nor any clue about it. The term *āyatavṛtta* literally means stretched or elongated circle, and a variety of geometric shapes would qualify for such a description in terms of an heuristic interpretation of the phrase. As we shall see below the formulae proposed for the circumference are based on specific familiar "models" for "elongated circles", as shown in Figure 2 (see details below), neither of which is an ellipse in the usual sense of the term. Thus the identification of the "*āyatavṛtta*" with ellipse lacks proper justification.

We recall that the formulae for circumference are given in two versions, *sthūla* and *sūkṣma* ("gross" and "fine" respectively). They involve two parameters, called *āyāma* and *vyāsa,* corresponding to the longer and shorter (understood to be axial) dimensions, respectively. With *a* for *āyāma* and *b* for *vyāsa* the *sthūla* value of the circumference is given to be $2\left(a+\frac{b}{2}\right)$. This may be readily seen to correspond to the circumference of the "elongated circle" consisting of two semicircular arcs of a circle of diameter b, joined by two straight line segments of length $a - b$, as in the top part of Figure 2, with the circumference of the circle taken to be three times the diameter, adopting the *sthūla* value for the factor; the total circumference is then $3 \times b + 2(a - b) = 2(a + \frac{b}{2})$.

For the *sūkṣhma* value the author evidently falls back on the traditional Jaina formula for lengths of arcs of circles cut of by chords (discussed in §1.4); see also the discussion in [13] in this respect. The *āyatavṛtta* is now visualized as two such identical arcs joined symmetrically (as in the bottom part of Figure 2), ignoring the cusps formed at the two ends. For each of the arcs the corresp-

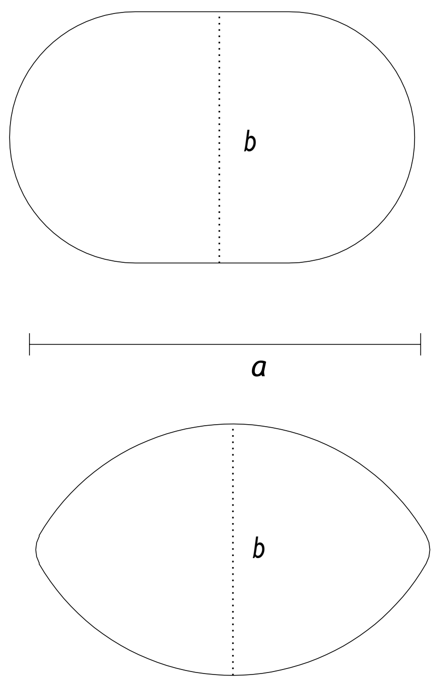

Figure 2: Āyatavṛtta

onding chord is the *āyāma* of the *āyatavṛtta*, viz. *a* as above, and the "arrow" is half of the *vyāsa*, which is $\frac{b}{2}$. Hence by Formula (6) the length of each of them is $\sqrt{\frac{3}{2}b^2+a^2}$, and the combined length of the two arcs is $\sqrt{4a^2+6b^2}$, which is given as the *sūkṣhma* value of the circumference of the *āyatavṛtta*.

The area of the *āyatavṛtta* in either of the *sthūla* and *sūkṣhma* versions is given to be half the product of the respective circumference with what is called "*viṣkambha*"; in most of the current literature, including [24] and [26] the latter term has been interpreted as synonymous with *vyāsa,* while [13] makes a convincing case for a different interpretation. With the modified interpretation the value assigned turns out to be substantially closer to the true value, even though it is still a crude empirical assignment, meant for practical day to day use.

In the Greek tradition the notion of conic sections arose out of the problem of the means, and the notion of an ellipse was a natural outcome of the studies in this respect. In India there was no such specific context for introduction of a geometrical shape that we now call the ellipse, and one does not also find any unambiguous reference to such a notion. The planetary models also did not involve ellipses at that time, but rather were based on epicycles. It would thus be more appropriate to suppose that *āyatavr̥tta* was meant to be a more general figure fitting the overall idea of an elongated circle, and the formulae were meant for approximate practical computations, which is very much the spirit of exposition as a whole.

3.5 Ṭhakkura Pherū

Ṭhakkura Pherū is another Jaina mathematician to have made a significant impact on promotion of mathematical understanding, in North India. He was a polymath, with significant contributions in a variety of branches of knowledge including, Gemmology, Astronomy, Architecture, Metallurgy, besides mathematics, in the court of Khaljī Sultāns of Delhi during the early decades of the 14th century. In mathematics Pherū contributed the *Gaṇitasārakaumudi* (see [27]) which apparently had been very influential during his time. He is also known to have been the first to provide general constructions of magic squares, though there has been much interest in them in India over a long period; see [27]. Unlike the works of Śrīdhara and Mahāvīra, which are in Sanskrit, *Gaṇitasārakaumudi* is in *Apabhraṁṣa Prākr̥ta*. Perhaps due to this, the work has not been much studied by the modern scholarship. The work [27] has made a good beginning in this respect, by providing a translation and putting the contents in perspective. The new context of the onset of Islamic culture in India during the period is manifest in Pherū's work in various ways. With regard to geometry for instance, computations relating to minars, arches etc. may be mentioned in this respect. Apart from having been influential in introducing mathematical knowledge in various activities, he seems to have been perceptive on issues of accuracy of the (empirical) formulae that were in use and is known to have introduced some corrections as we noted earlier. Hopefully more detailed further studies will throw more light on the specific mathematical significance of the work.

4 In place of Conclusion

Much of the literature on mathematics from the Jaina tradition, as also with various other ancient cultures, has been in the nature of cataloguing various mathematical notions and observations found in the texts, coupled with some claims on historical priority etc. with relatively little focus on aspects of understanding and anticipating the development of ideas from an intrinsic perspective. While emphasis on the first part is inevitable to an extent in the early stages of any developing studies in any historical tradition, there is now a need to direct efforts towards progress in the latter. With this in view we have analyzed various formulae, ideas involved, and concepts introduced. It is hoped that the discussion in the preceding pages will serve as a small step towards such an endeavor.

5 Acknowledgements

The author would like to thank Professors Anupam Jain and Surender K. Jain, and Mr. Pankajkumar K. Shah for encouragement and help in preparing this paper. Thanks are due to Dr. Qijun Yan and Dr. Lovy Singhal for making available a copy of the book of Yoshio Mikami. Thanks are also due to Dr. Manoj Choudhuri for producing the figures used in the article, and to Mr. Anil Kumar Jain for help in preparation of the manuscript in word format.

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