Jaina mathematics from ancient and medieval India

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Introduction

Jainism was one of the flourishing religions in India in the ancient and medieval times, following its rejuvenation by Vardhamana Mahavira (6th c. BCE), alongside the Vedic religion and Buddhism.

Mathematics had an important role in the philosophical pursuits of the Jaina monks. Their involvement stemmed from interest in calculations involving

- Their conception of the terrestrial world, with the earth as a flat disc surrounded alternately by rings of water and land, extending indefinitely.
- Geographical model of the earth, divided by parallel mountain ranges.
- Their model for population of human souls and their individual spiritual developments (in terms of ”karma”).

Early works with mathematical content

Works with explicit mathematical content, involving numerical calculations, geometry, as also certain other mathematical topics, perhaps go back to about 500 BCE, not long after Mahavira, who himself is believed to have been a mathematician.

- Chandraprajnapti, Suryaprajnapti, Jambudvipa prajnapti (ca. 500 BCE)
- Sthananga sutra (ca.300 BCE)
- Bhagavati sutra of Sudharma Svami (ca. 300 BCE)
- Shatkhandagama of Pushpadanta and Bhutabali (between 87 and 156 CE)
- Tatvarthadhigama sutrabhashya of Umasvati (2nd c. CE, or older?)

Revived Jaina tradition

After a lull in during some early centuries of the first millennium, there was renewed interest among Jaina scholars in mathematics, and this time it was guided also by various emerging needs of practical life.

On the one hand, a body of basic mathematics had evolved, in the Aryabhatan tradition of mathematical astronomy, and was ripe for diffusion, by scholars with pedagogical interest.

On the other hand there was rise in trade and commerce generating interest in calculations of various kinds.

Many Jaina scholars, who were not involved with astronomy, contributed mathematical works that were instrumental in promotion and applications of mathematics in a wider practical context, from around the 8th century, lasting for about 6 centuries.

Later Jaina exponents

Here are some of the principal figures from the Jaina tradition from 8th to 14th centuries; we shall explore here the develoments in terms of their works.

• Sridhara; there has been much debate about his period, and whether he was Jaina or a Shaivite; the see-saw has now settled on his being Jaina, and from around 750 CE.

Two of his works Patiganita, Trishatika are extant - these seem to be parts of a much larger work which is lost.

He has been a much cited author (in later works), and in particular was cited in Bhaskaracharaya's influential work(s).

• Mahavira; he lived in the later part of the rule of Amoghavarsha Nripatunga (815 – 877 CE), of the Rashtrakuta dynasty (perhaps belonged to his court). He is renowned for the work Ganita Sara Sangraha, which served as a textbook in a wide region of South India for several centuries, as witnessed by extant copies of the work.

Later Jaina exponents - Contd.

• Virsena (816 CE); known for Dhavala, a commentary, in prakrit, on Shatkhandagama

• Nemichandra (981 CE); author of Trilokasara, Gommatasara, and Labdhisara, all in prakrit. A commentary of Trilokasara in Sanskrit, by his pupil Madhavachandra Traividya, was much in circulation.

• Thakkura Pheru; held important positions in the treasury, during the period of the successive Sultans Alauddin Khalji (1296–1316 CE), Shihabuddin Umar (1316 CE), Qutubddin Mubarak Shah (1316–1320 CE) and Ghiyasuddin Tughluq (1320–1325 CE). He is noted for incorporating emerging trends brought in by the Islamic rule in India.

Numerous other Jaina scholars also contributed during the period, especially in pedagogical terms.

Early Geometry

In the ancient world the ratio of the circumference of the circle to its diameter (π for us now) was taken to be 3.

The Jainas were among the first to depart from the practice.

(The Manava sulbasutra, which would be anterior to the Jaina works in question mentions the value $3\frac{1}{5}$ for the ratio, but not found used anywhere, or mentioned in later works.)

The Jainas adopted the value $\sqrt{10}$ for the ratio, which is about 3.16, about $\frac{2}{3}$% in excess of the true value $3.1415\ldots$.

This value for π had great longeivity and was in use for almost 2000 years, even after more accurate values, such as 3.1416 in Aryabhatiya, were known; not only in the Jaina tradition but also in the hindu tradition, including by Brahmagupta, and later by Arabs and Europeans, during some early period.

Circumference of the Earth

Prevalence of the value $\sqrt{10}$ lies in its convenience, as is seen from the ancient example.

The diameter of the Earth (Jambudvipa, as a flat disc) was supposed to be 100,000 *yojana*. Thus the circumference, according to them, would be $\sqrt{10} \times 100000$ and this would be computed as the square root of 100,000,000,000 and is described in various early Jaina texts to be

316,227 yojana, 3 gavyuti, 128 dhanu, $13\frac{1}{2}$ aṅgula and *a little over*.

Here gavyuti, dhanu and angula are smaller units of distance prevailing at the time. One gavyuti was equivalent to $\frac{1}{4}$ yojana, a dhanu was a 2000th part of a gavyuti, and an angula was 96th part of a dhanu; thus 1 yojana = 4 gavyuti = 8000 dhanu = 768000 angula.

The mention of "a little over" is especially notable, as a matter of the attention paid to detail.

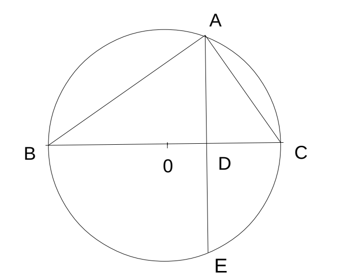
Computations with numbers

As the calculation of the circumference of the earth shows the Jaina scholars had great aptitude for computation with numbers. There are also many such other instances involving even much larger numbers. The decimal system for representation of numbers together with zero is seen to be in full use.

Their method of computation of square roots is not definitively known, but it may have incorporated the method known after Aryabhata, together with an approximation formula of the kind $\sqrt{a^2 + b} \approx a + \frac{b}{2a}$.

They also dealt with permutations and combinations, partitions of numbers, summation of (finite) series, an analogue of the logarithms, etc.

Geometry of the circle



Chord and arc

Consider a circle with diameter d, and a chord, AE in the figure. Let c be the length of AE and h that of DC, the corresponding "arrow". The formula

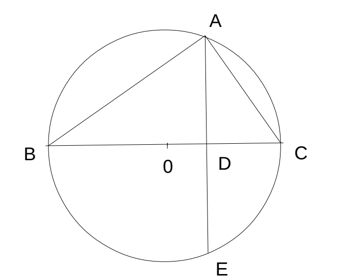
$$c = \sqrt{4h(d-h)}$$

occurs quite frequently in early Jaina works.

The formula may be deduced easily from consideration of similar triangles, together with the fact that the diameter subtends a right angle and any point on the circle.

It is however more likely the Jainas deduced it using the Pythagoras theorem, which has been known in India from even earlier than Pythagoras, in the Vedic tradition.

Chord and the arc



Chord and arc

Formulae for h and d are also derived from the above relation, thus determining from any one of them in terms of the other two.

More intriguingly, they had a formula for the arc a corresponding to the chord as

$$a = \sqrt{6h^2 + c^2}.$$

The formula is clearly an ad hoc and approximate one, and is inspired by the Pythagoras theorem.

Heron of Alexandria (2nd c) also had formulae for the arc as $\sqrt{4h^2 + c^2} + \frac{1}{4}h$ and (a more refined) $\sqrt{4h^2 + c^2}(1 + \frac{h}{c}) - h$. The following table shows that on the whole the Jaina formula fares better in comparison with Heron's formulae.

Comparison with Heron's formula

Angle	arclength true value	Jaina value	Heron's 1st value	Heron's 2nd value
15°	0.5236	0.5243 (+0.0007)	0.5306 (+0.0070)	0.5224 (-0.0012)
30°	1.0472	1.0525 (+0.0053)	1.0688 (+0.0216)	1.0400 (-0.0072)
45°	1.5708	1.5858 (+0.0150)	1.6049 (+0.0341)	1.5549 (-0.0159)
60°	2.0944	2.1213 (+0.0269)	2.1250 (+0.0306)	2.0774 (-0.0170)
75°	2.6180	2.6511 (+0.0331)	2.6203 (+0.0023)	2.6281 (+0.0101)
90°	3.1416	3.1623	3.0784	3.2426

Permutations and combinations

References are found in ancient Jaina works to number of permutations and combinations in various contexts.

A general formula for the number of ways of choosing certain number of specimen from a given collection was described by Sridhara (ca. 750), in a sutra, which corresponds to the usual formula $C^n_m = \frac{n!}{m!(n-m)!}$.

The Bhagavatisutra discusses the number of ways s souls can enter h hells/heavens (the occupancy problem) for values of s and h upto 12. It may be noted that this number corresponds to C^{s+h-1}_s. While there are no general formula found in the ancient works, it is seen in the calculations described by the commentator Abhayadeva (10/11 c.) that the inductive rule $C^{n+m}_{m+1} = C^{n+m-1}_m \times \frac{n+m}{m+1}$ was recognized.

Series summation

Summation of various (finite) series, and related problems, have been a favourite topic in ancient and medieval Indian mathematics, going back at least to Aryabhata, and the medieval Jainas are no exception.

Sridhara and Mahavira especially are seen to relish the topic, and in particullar created complicated variations of the arithmetic progressions to sum up, such at $\Sigma_1^m T_n$ where T_n itself is given in terms of partial sums of an arithmetical series, as $a_1 + a_2 + \cdots + a_n$, where for each k, $a_k = b + kc$, with given b and c.

Generalizing Aryabhata's consideration of the series $\Sigma_1^m n^2$ and $\Sigma_1^m n^3$, Sridhara and Mahavira also discuss sums of the form $\Sigma_1^m a_n^2$ and $\Sigma_1^m a_n^3$, for any arithmetic progression $a_n = b + nd$, for some b and d.

Sridhara's volume formula

The main task before the exponents in the later phase was exposition of mathematical knowledge that had accumulated over the period. In course of it, apart from pedagogical skills they contributed also new knowledge, of varying degrees of depth. We focus here only some certain select items.

In the Indian context credit for giving the first correct formula for the volume of a sphere is given to Bhaskaracharya (12th c.); Aryabhatiya has a formula for it but it is incorrect.

It turns out however that the formula described by Sridhara in this respect is correct, in spirit. He gives the volume of a sphere of diameter d to be $\frac{1}{2}d^3(1+\frac{1}{18})$. By correct in spirit I mean that the factor $1+\frac{1}{18}$ corresponds in a natural way in their context to $\frac{1}{3}\pi$, so the formula rightly corresponds to the correct formula $\frac{1}{6}\pi d^3$ for the volume.

We note that $\frac{1}{3}\pi \approx \frac{1}{3}\sqrt{10} = \sqrt{\frac{10}{9}} = \sqrt{(1+\frac{1}{9})} \approx 1+\frac{1}{18}$.

Virsena's formula for a conical frustrum

In his *Dhavala Tika* the volume Virasena gives a formula for the volume of a conical frustrum with diameters a and b at the base and top respectively, and height h as

$$\frac{\pi h}{4} \cdot \frac{a^2 + ab + b^2}{3}.$$

Unlike in much of the extant ancient and medieval mathematical literature in India, we find here a description of the method of determining the volume.

Moreover, the method involves summation of an infinite series and the idea of infinitesimals, akin to calculus. The ideas are reminiscent of computations going back to Archimedes and Liu Hui, but the details are arguably new, and seem to be a notable first in the mathematics in India. In particular the work may have influenced Bhaskaracarya.

“Logarithms”

The Jaina scholars also toyed with an idea like the logarithms. While it is perhaps rooted in quite old times in some way, a discussion on it in some detail is found in Virsena’s Dhavala commentary. It was also elaborated on by Nemichandra.

The idea occurs in the form of what is called “ardhachcheda”, essentially counting how many times the number can be divided by 2 (sometimes base 3 was also considered).

The usual laws of indices (corresponding to (log mn = log m + log n etc.) are stated in Dhavala.

Unfortunately they did not find a proper way to determine the ardhachheda for numbers which are not powers of 2, leading to a peculiar choice of taking ardhchheda of $1 + x$ to be x for “small” x; thus the ardhachheda of $\frac{256}{13}$, which is $2^4 \times \frac{16}{13}$ is given as $4\frac{3}{13}$.

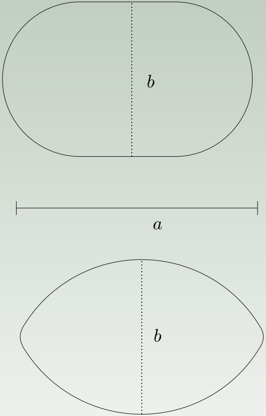
Mahavira's formulas for "ellipse" etc.

Mahavira is known for his discussing areas and volumes of various non-standard figures, such as a a conchiform, toirtoise back, curved shape of a ring, etc.:

He also gives formulae for what he refers to as "ayatavritta". "ayata" refers to rectangle and "vritta" to circle, so the figure under consideration is a rectangular, or oblong, "circle". In the literature the term is generally translated as "ellipse"; this however may not be justified; there is no description of how the figure was constructed, or any characteristic that would identify it with ellipses as in the Greek tradition.

Invoking two parameters "ayama" and "vyasa", for the longer and shorter dimensions, say a and b respectively, the circumference of the figure is given to be $2a + b$ in approximate measure, and to be $\sqrt{4a^2 + 6b^2}$ in fine measure.

"ellipse" continued



ayatavritta

The two formulae are seen to correspond to viewing "ayatavritta" (approximately) as the shapes here; the bottom figure is a join of two bow-arrow figures (with chord a and arrow $b/2$), for which the early Jaina formula yields $\sqrt{a^2 + \frac{3}{2}b^2}$ for each of the arcs.

Formulas are also given for the area, involving multiplying the circumference by half of what is referred to as "vishkambha"; the term is subject to interpretations, and we will not go into it.

Pheru and Magic squares

Pheru's work Ganita Sara Kaumudi, apart form giving an exposition of various mathematics topics of broad interest, provides improved formulae in various respects. One distintuishing feature is his treatment of Magic squares.

Though magic squares have been known in India for a long time, had religious significance and were used to convey good wishes, Pheru was the first one to give a systematic treatment construction of magic squares.

Pheru classifies magic squares into three groups according to the size of the square; those of odd order, of order divisible by 4 and of even order not divisible by 4. He gives general constructions for the first two groups. As to the last, only a magic square of order 6 is described (presumably he had difficulty for larger values).

Pheru also discusses ways of transforming a magic square into another.

A 9×9 magic square of Pheru

I convey my best wishes to you all by presenting this 9×9 magic square of Pheru!!

37	48	59	70	81	2	13	24	35
36	38	49	60	71	73	3	14	25
26	28	39	50	61	72	74	4	15
16	27	29	40	51	62	64	75	5
6	17	19	30	41	52	63	65	76
77	7	18	20	31	42	53	55	66
67	78	8	10	21	32	43	54	56
57	68	79	9	11	22	33	44	46
47	58	69	80	1	12	23	34	45

The numbers in each row, each column and along the two diagonals add to 369.

* Thank you *

for your attention.