Tannaka duality for stereotype algebras

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Reconstruction theorem

Let G be a group and ${}_G\mathrm{Vect}$ the category of actions of G on vector spaces $X \in \mathrm{Vect}$. Consider the “forgetting” functor $F : {}_G\mathrm{Vect} \to \mathrm{Vect}$ that assigns to every action $X \in {}_G\mathrm{Vect}$ of G the same space X, but viewed as a vector space (without the structure of G-action). An endomorphism of the functor F is a family of linear mappings

$$\rho = \{\rho_X : F(X) \overset{\mathrm{Vect}}{\to} F(X);\ X \in {}_G\mathrm{Vect}\}$$

such that for each G-morphism $\varphi : X \overset{{}_G\mathrm{Vect}}{\to} Y$

$$\begin{array}{ccc} F(X) & \overset{\rho_X}{\dashrightarrow} & F(X) \\ {\scriptstyle F(\varphi)}\downarrow & & \downarrow{\scriptstyle F(\varphi)} \\ F(Y) & \underset{\rho_Y}{\dashrightarrow} & F(Y) \end{array} \qquad (1)$$

An endomorphism ρ is an automorphism $\Leftrightarrow$ each mapping $\rho_X : F(X) \overset{\mathrm{Vect}}{\to} F(X)$ is a bijection.

Theorem (on reconstruction)

The class Aut*F* *of all automorphisms of the functor F is a group isomorphic to G.*

Proof. For each element $a \in G$ and for each G-space $X \in {}_G\mathrm{Vect}$ we consider the mapping $\widetilde{a}_X : X \to X$

$$\widetilde{a}_X(x) = a \cdot x, \qquad X \in {}_G\mathrm{Vect}, \quad x \in X. \tag{2}$$

1. For every $a \in G$, the family of mappings $\widetilde{a} = \{\widetilde{a}_X;\ X \in {}_G\mathrm{Vect}\}$ is a natural transformation of the functor F into itself, since for any morphism $\varphi : X \xrightarrow{{}_G\mathrm{Vect}} Y$

$$\widetilde{a}(\varphi(x)) = a \cdot \varphi(x) = \varphi(a \cdot x) = \varphi(\widetilde{a}(x)).$$

2. For any $a \in G$, the family of mappings $\widetilde{a} = \{\widetilde{a}_X;\ X \in {}_G\mathrm{Vect}\}$ is an isomorphism of the functor F into itself, since every mapping $\widetilde{a}_X$ is a bijection. As a corollary, $\widetilde{a} \in \mathrm{Aut}F$.

3. If $a, b \in G$, then

$$\widetilde{a \cdot b} = \widetilde{a} \circ \widetilde{b}$$

since

$$\widetilde{a \cdot b}_X(x) = (a \cdot b) \cdot x = a \cdot (b \cdot x) =$$
$$= \widetilde{a}_X(\widetilde{b}_X(x)) = (\widetilde{a} \circ \widetilde{b})_X(x)$$

Hence the mapping $a \in G \mapsto \widetilde{a} \in \mathrm{AutF}$ is a group homomorphism.

4. It is injective, because on the group algebra $\mathbb{C}_G$ we have the chain

$$a \neq b \quad \Rightarrow \quad \widetilde{a}_{\mathbb{C}_G}(\delta^1) = \delta^a \cdot \delta^1 = \delta^a \neq \delta^b = \delta^b \cdot \delta^1 = \widetilde{b}_{\mathbb{C}_G}(\delta^1)$$

5. Let us show that it is surjective. Take $\rho \in \mathrm{Aut}F$. Consider the group algebra $\mathbb{C}_G \in {}_G\mathrm{Vect}$ and the mapping $\rho_{\mathbb{C}_G} : \mathbb{C}_G \to \mathbb{C}_G$. This mapping is an automorphism of the functor F, hence it commutes with each mapping $\varphi : \mathbb{C}_G \to \mathbb{C}_G$ that commutes with the left multiplication by elements of G:

$$\Big(\forall b, x \in \mathbb{C}_G \quad \varphi(b \cdot x) = b \cdot \varphi(x)\Big) \quad \Rightarrow$$
$$\Rightarrow \quad \Big(\forall x \in \mathbb{C}_G \quad \rho_{\mathbb{C}_G}(\varphi(x)) = \varphi(\rho_{\mathbb{C}_G}(x))\Big).$$

This implies that $\rho_{\mathbb{C}_G}$ commutes with the right multiplications $\varphi(x) = x \cdot y$ by elements $y \in G$:

$$\forall x, y \in \mathbb{C}_G \quad \rho_{\mathbb{C}_G}(x \cdot y) = \rho_{\mathbb{C}_G}(x) \cdot y.$$

If we now put $a = \rho_{\mathbb{C}_G}(\delta^1)$, where 1 is the unit in G, then we get

$$\rho_{\mathbb{C}_G}(y) = \rho_{\mathbb{C}_G}(\delta^1 \cdot y) = \rho_{\mathbb{C}_G}(\delta^1) \cdot y = a \cdot y = \widetilde{a}_{\mathbb{C}_G}(y), \qquad y \in \mathbb{C}_G.$$

That is

$$\rho_{\mathbb{C}_G} = \widetilde{a}_{\mathbb{C}_G} \tag{3}$$

Let us now show that for any object $X \in {}_G\text{Vect}$ the equality $\rho_X = \widetilde{a}_X$ holds. We fix $x \in X$ and consider the mapping

$$\iota : \mathbb{C}_G \to X \quad \Big| \quad \iota(\delta^g) = g \cdot x \tag{4}$$

This is a morphism of G-modules. On the other hand, ρ is a natural transformation of $F : {}_G\text{Vect} \to \text{Vect}$ into itself. Hence, by (1),

$$\begin{array}{ccc} F(\mathbb{C}_G) & \xrightarrow{\ \rho_{\mathbb{C}_G}\ } & F(\mathbb{C}_G) \\ {\scriptstyle F(\iota)}\downarrow & & \downarrow{\scriptstyle F(\iota)} \\ F(X) & \xrightarrow[\ \rho_X\ ]{} & F(X) \end{array}$$

So

$$\rho_X(x) = \rho_X(F(\iota)(\delta^1)) = F(\iota)(\rho_{\mathbb{C}_G}(\delta^1)) = (3) = F(\iota)(\widetilde{a}_{\mathbb{C}_G}(\delta^1)) =$$
$$= (2) = F(\iota)(\delta^a \cdot \delta^1) = F(\iota)(\delta^a) = \iota(\delta^a) = (4) = a \cdot x = \widetilde{a}_X(x)$$

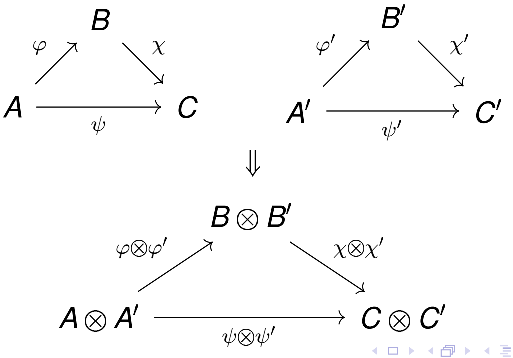
All this means that the mapping $a \in G \mapsto \widetilde{a} \in \text{AutF}$ is a group isomorphism.

Monoidal categories

A *monoidal category* is a list of six objects $\mathtt{M}, \otimes, I, \square, \lhd, \rhd$ with the following properties:

1) $\mathtt{M}$ is a category,
2) $\otimes$ is a covariant furctor $\otimes : \mathtt{M} \times \mathtt{M} \to \mathtt{M}$; the requirement of functoriality means the fulfillment of the identities

$$1_{A\otimes B} = 1_A \otimes 1_B, \qquad (\chi\otimes\chi')\circ(\varphi\otimes\varphi') = (\chi\circ\varphi)\otimes(\chi'\circ\varphi') \quad (5)$$



(6)

3) $\square$ is an isomorphism of functors

$$\square : \Big((A,B,C) \mapsto (A \otimes B) \otimes C\Big) \rightarrowtail \Big((A,B,C) \mapsto A \otimes (B \otimes C)\Big),$$

i.e. a family of isomorphisms in $\mathtt{M}$

$$\square = \{\square_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C);\ A, B, C \in \mathtt{M}\}$$

such that for each $\varphi : A \stackrel{\mathtt{M}}{\to} A'$, $\chi : B \stackrel{\mathtt{M}}{\to} B'$, $\psi : C \stackrel{\mathtt{M}}{\to} C'$

$$\begin{array}{ccc}
(A \otimes B) \otimes C & \xrightarrow{\ \square_{A,B,C}\ } & A \otimes (B \otimes C) \\
{\scriptstyle (\varphi\otimes\chi)\otimes\psi}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi\otimes(\chi\otimes\psi)}; \\
(A' \otimes B') \otimes C' & \xrightarrow[\ \square_{A',B',C'}\ ]{} & A' \otimes (B' \otimes C')
\end{array} \qquad (7)$$

and in addition for any $A, B, C, D \in \mathbb{M}$

$$\begin{array}{ccc}
(A \otimes (B \otimes C)) \otimes D & \xrightarrow[\square_{A,B\otimes C,D}]{} & A \otimes ((B \otimes C) \otimes D) \\
\square_{A,B,C} \otimes 1_D \uparrow & & \downarrow 1_A \otimes \square_{B,C,D} \\
((A \otimes B) \otimes C) \otimes D & & A \otimes (B \otimes (C \otimes D)) \\
& \searrow^{\square_{A\otimes B,C,D}} \quad (A \otimes B) \otimes (C \otimes D) \quad \nearrow^{\square_{A,B,C\otimes D}} &
\end{array}$$

4) I is an object in the category $\mathbb{M}$, and $\lhd$ and $\rhd$ are two isomorphisms of functors $\lhd : \Big(A \mapsto I \otimes A\Big) \rightarrowtail \Big(A \mapsto A\Big)$, and $\rhd : \Big(A \mapsto A \otimes I\Big) \rightarrowtail \Big(A \mapsto A\Big)$, i.e. two families of isomorphisms in $\mathbb{M}$

$$\lhd = \{\lhd_A : I \otimes A \to A;\ A \in \mathbb{M}\}, \qquad \rhd = \{\rhd_A : A \otimes I \to A;\ A \in \mathbb{M}\}$$

so that for each $\varphi : A \xrightarrow{\mathtt{M}} A'$

$$
\begin{array}{ccc}
I \otimes A & \xrightarrow{\lhd_A} & A \\
\Big\downarrow{\scriptstyle 1_I \otimes \varphi} & & \Big\downarrow{\scriptstyle \varphi} \\
I \otimes A' & \xrightarrow[\lhd_{A'}]{} & A'
\end{array}
\qquad\qquad
\begin{array}{ccc}
A \otimes I & \xrightarrow{\rhd_A} & A \\
\Big\downarrow{\scriptstyle \varphi \otimes 1_I} & & \Big\downarrow{\scriptstyle \varphi} \\
A' \otimes I & \xrightarrow[\rhd_{A'}]{} & A'
\end{array}
$$

It is additionally reqired that

— when the object I is substituted into the argument of these transformations, they must coincide

$$(\lhd_I : I \otimes I \to I) = (\rhd_I : I \otimes I \to I),$$

— for each $A, B \in \mathtt{M}$

$$
\begin{array}{ccc}
(A \otimes I) \otimes B & \xrightarrow[\square_{A,I,B}]{} & A \otimes (I \otimes B) \\
& {\scriptstyle \rhd_A \otimes 1_B} \searrow \qquad \swarrow {\scriptstyle 1_A \otimes \lhd_B} & \\
& A \otimes B &
\end{array}
\qquad (8)
$$

Symmetric categories

A monoidal category $(\mathbb{M}, \circledast, I, \square, \lhd, \rhd)$ is *symmetric* if a morphism of functors $\diamond : \Big((A,B) \mapsto A \circledast B\Big) \rightarrowtail \Big((A,B) \mapsto B \circledast A\Big)$ is defined, i.e. a family of isomorphisms in $\mathbb{M}$

$$\diamond = \{\diamond_{A,B} : A \circledast B \to B \circledast A;\ \ A, B \in \mathbb{M}\}$$

such that for each $\varphi : A \xrightarrow{\mathbb{M}} A'$, $\chi : B \xrightarrow{\mathbb{M}} B'$

$$\begin{array}{ccc} A \circledast B & \xrightarrow{\ \diamond_{A,B}\ } & B \circledast A \\ {\scriptstyle \varphi\circledast\chi}\downarrow & & \downarrow{\scriptstyle \chi\circledast\varphi} \\ A' \circledast B' & \xrightarrow[\ \diamond_{A',B'}\ ]{} & B' \circledast A' \end{array} ;$$

and in addition

— for each $A \in \mathtt{M}$

$$\begin{array}{ccc} I \otimes A & \xrightarrow{\ \diamond_{I,A}\ } & A \otimes I \\ & {\scriptstyle \lhd_A} \searrow \quad \swarrow {\scriptstyle \rhd_A} & \\ & A & \end{array}$$

— for each $A, B \in \mathtt{M}$

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\ 1_{A \otimes B}\ } & A \otimes B \\ & {\scriptstyle \diamond_{A,B}} \searrow \quad \nearrow {\scriptstyle \diamond_{B,A}} & \\ & B \otimes A & \end{array}$$

— for each $A, B, C \in \mathtt{M}$

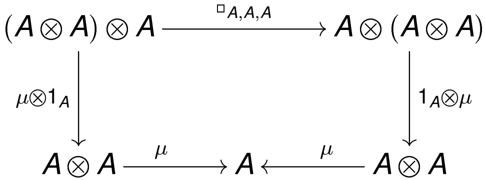
$$
\begin{array}{ccc}
A \otimes (B \otimes C) & \xrightarrow[\diamond_{A,B\otimes C}]{} & (B \otimes C) \otimes A \\
\uparrow {\scriptstyle \square_{A,B,C}} & & \downarrow {\scriptstyle \square_{B,C,A}} \\
(A \otimes B) \otimes C & & B \otimes (C \otimes A) \\
\downarrow {\scriptstyle \diamond_{A,B}\otimes 1_C} & & \uparrow {\scriptstyle 1_B \otimes \diamond_{A,C}} \\
(B \otimes A) \otimes C & \xrightarrow[\square_{B,A,C}]{} & B \otimes (A \otimes C)
\end{array}
$$

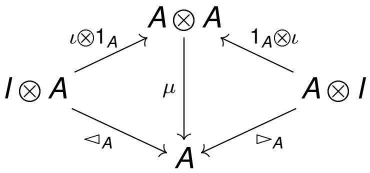
Examples:
1) The category Set of sets.
2) The category ${}_k$Vect of vector spaces over a given field k.

An *algebra* (or a *monoid*) in a monoidal category $\mathtt{M}$ is a triple (A, μ, ι), where $A \in \mathtt{M}$, and

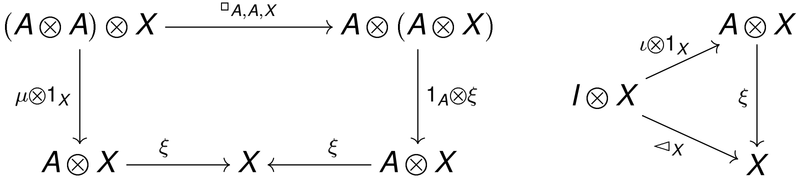
$$\mu : A \otimes A \to A \quad \text{(multiplication)}, \qquad \iota : I \to A \quad \text{(unit)}$$

are morphisms such that





A *left module* over an algebra (A, μ, ι) in a monoidal category $\mathbb{M}$ is an arbitrary pair (X, ξ), consisting of an object X and a morphism $\xi : A \otimes X \to X$ in $\mathbb{M}$ such that the following diagrams are commutative:



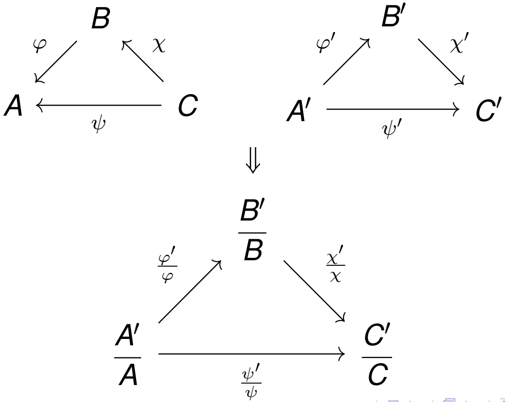
Examples:
1) In the category Set of sets the algebras are monoids, and the modules over them are actions of monoids on sets.
2) In the category ${}_k$Vect of vector spaces over a given field k the algebras are usual algebras over k, and the modules are usual modules over them.

Closed categories

A monoidal category $\mathbb{M}$ is *closed* if we have:

1) a bifunctor $(A,B)\mapsto \frac{B}{A}:\mathbb{M}\times\mathbb{M}\to\mathbb{M}$, contravariant in the first variable and covariant in the second:

$$\frac{1_B}{1_A}=1_{\frac{B}{A}},\qquad \frac{\chi'\circ\varphi'}{\varphi\circ\chi}=\frac{\chi'}{\chi}\circ\frac{\varphi'}{\varphi}$$



2) a system of bijections

$$C \boxslash (A \circledast B) \xrightarrow{\#_{A,B,C}} \frac{C}{B} \boxslash A$$

which for any three objects A, B, C establishes a bijection between the morphisms $A \circledast B \to C$ and the morphisms $A \to \frac{C}{B}$:

$$\left(A \circledast B \xrightarrow{\varphi} C \right) \quad \overset{\#_{A,B,C}}{\longrightarrow} \quad \left(A \xrightarrow{\#_{A,B,C}(\varphi)} \frac{C}{B} \right)$$

and this correspondence is natural in $A, B, C \in \mathbb{M}$: for each $A \xleftarrow{\alpha} A'$, $B \xleftarrow{\beta} B'$ and $C \xrightarrow{\gamma} C'$

$$\begin{array}{ccc} C \boxslash (A \otimes B) & \xrightarrow{\#_{A,B,C}} & \dfrac{C}{B} \boxslash A \\ \gamma^{\alpha\otimes\beta} \Big\downarrow & & \Big\downarrow \left(\frac{\gamma}{\beta}\right)^{\alpha} \\ C' \boxslash (A' \otimes B') & \xrightarrow[\#_{A',B',C'}]{} & \dfrac{C'}{B'} \boxslash A' \end{array}$$

of course, this defines an isomorphism of functors

$$\Big((A,B,C) \mapsto C \boxslash (A \otimes B)\Big) \overset{\#}{\rightarrowtail} \Big((A,B,C) \mapsto \frac{C}{B} \boxslash A\Big),$$

Examples:

1) The category Set of sets.

2) The category ${}_k\mathrm{Vect}$ of vector spaces over a given field k.

Enriched categories

Let $(\mathbb{M}, \circledast, I, \alpha, \lhd, \rhd)$ be a monoidal category. A class $\mathbb{E}$ is called an *enriched category over* $\mathbb{M}$, or an $\mathbb{M}$-*category*, if

E1: each two objects $A, B \in \mathbb{E}$ are associated with an object $B \oslash A \in \mathbb{M}$, called the *space of morphisms from A into B*,

E2: each three objects $A, B, C \in \mathbb{E}$ are associated with a morphism $\bullet_{A,B,C} : C \oslash B \circledast B \oslash A \to C \oslash A$ in the category $\mathbb{M}$, called a *morphism of composition in* $\mathbb{E}$,

E3: each object $A \in \mathbb{E}$ is associated with a morphism $\varepsilon_A : I \to A \oslash A$ in the category $\mathbb{M}$, called the *unit morphism in A*,

so that the following conditions are fulfilled:

— for each $A, B, C, D \in \mathbb{E}$

$$\big(D \oslash C \otimes C \oslash B\big) \otimes B \oslash A \xrightarrow{\ \square_{D\oslash C,\, C\oslash B,\, B\oslash A}\ } D \oslash C \otimes \big(C \oslash B \otimes B \oslash A\big)$$

$$\big(D \oslash C \otimes C \oslash B\big) \otimes B \oslash A \xrightarrow{\ \bullet_{B,C,D} \otimes 1_{B\oslash A}\ } D \oslash B \otimes B \oslash A$$

$$D \oslash C \otimes \big(C \oslash B \otimes B \oslash A\big) \xrightarrow{\ 1_{D\oslash C} \otimes \bullet_{A,B,C}\ } D \oslash C \otimes C \oslash A$$

$$D \oslash B \otimes B \oslash A \xrightarrow{\ \bullet_{A,B,D}\ } D \oslash A \xleftarrow{\ \bullet_{A,C,D}\ } D \oslash C \otimes C \oslash A$$

— for each $A, B \in \mathbb{E}$

$$I \otimes B \oslash A \xrightarrow{\ \varepsilon_B \otimes 1_{B\oslash A}\ } B \oslash B \otimes B \oslash A$$

$$I \otimes B \oslash A \xrightarrow{\ \lhd_{B\oslash A}\ } B \oslash A \xleftarrow{\ \bullet_{A,B,B}\ } B \oslash B \otimes B \oslash A$$

$$B \oslash A \otimes I \xrightarrow{\ \rhd_{B\oslash A}\ } B \oslash A \xleftarrow{\ \bullet_{A,A,B}\ } B \oslash A \otimes A \oslash A$$

$$B \oslash A \otimes I \xrightarrow{\ 1_{B\oslash A} \otimes \varepsilon_A\ } B \oslash A \otimes A \oslash A$$

Examples:

1) Usual categories are enriched over Set.

2) A closed monoidal category is enriched over itself. In particular, the category Set of sets, and the category ${}_k\mathrm{Vect}$ of vector spaces over a given field k are enriched over themselves.

3) If $\mathtt{M}$ is a closed monoidal category with equalizers, then for each algebra A in $\mathtt{M}$ the category ${}_A\mathtt{M}$ of left modules over A is enriched over $\mathtt{M}$.

Support of an enriched category

If $\mathbb{E}$ is an enriched category over $\mathbb{M}$, then its *support* supp$\mathbb{E}$ is the usual category with the same objects as in $\mathbb{E}$,

$$X \in \operatorname{supp}\mathbb{E} \quad \Leftrightarrow \quad X \in \mathbb{E},$$

and the morphisms from I into $Y \oslash X$ (in the category $\mathbb{M}$) as the morphisms from X into Y:

$$\varphi : X \overset{\operatorname{supp}\mathbb{E}}{\to} Y \quad \Leftrightarrow \quad \varphi : I \overset{\mathbb{M}}{\to} Y \oslash X.$$

In other words,

$$Y \overset{\operatorname{supp}\mathbb{E}}{\boxslash} X := (Y \oslash X) \overset{\mathbb{M}}{\boxslash} I$$

Local identities

$$1_X : X \overset{\operatorname{supp}\mathbb{E}}{\to} X \quad \Leftrightarrow \quad 1_X = \varepsilon_X : I \overset{\mathbb{M}}{\to} X \oslash X,$$

Composition of morphisms $\varphi : X \overset{\operatorname{supp}\mathbb{E}}{\to} Y$ and $\chi : Y \overset{\operatorname{supp}\mathbb{E}}{\to} Z$:

$$\begin{array}{ccc} I & \overset{\chi\circ\varphi}{\dashrightarrow} & Z \oslash X \\ {\scriptstyle \lhd_I^{-1}}\downarrow & & \uparrow{\scriptstyle \bullet_{X,Y,Z}} \\ I \otimes I & \underset{\chi\otimes\varphi}{\longrightarrow} & Z \oslash Y \otimes Y \oslash X \end{array}$$

Theorem

The support $\operatorname{supp}\mathbb{E}$ *of any enriched category* $\mathbb{E}$ *is a usual category (over* Set*).*

Fraction of morphisms

Let $\mathbb{E}$ be an enriched category over $\mathbb{M}$ and suppose $\psi : A \overset{\operatorname{supp}\mathbb{E}}{\to} B$ and $\chi : C \overset{\operatorname{supp}\mathbb{E}}{\to} D$ are two morphisms in its support:

$$\psi : I \overset{\mathbb{M}}{\to} B \oslash A, \qquad \chi : I \overset{\mathbb{M}}{\to} D \oslash C.$$

Then

$$\chi \oslash \psi : C \oslash B \overset{\mathbb{M}}{\to} D \oslash A$$

$$\begin{array}{ccc}
C \oslash B & \overset{\chi \oslash \psi}{\dashrightarrow} & D \oslash A \\
\Big\downarrow (\lhd_{C \oslash B})^{-1} & & \Big\uparrow \bullet_{A,B,D} \\
I \otimes C \oslash B & & D \oslash B \otimes B \oslash A \\
\Big\downarrow (\rhd_{I \otimes C \oslash B})^{-1} & & \Big\uparrow \bullet_{B,C,D} \otimes 1_{B \oslash A} \\
I \otimes C \oslash B \otimes I & \underset{\chi \otimes 1_{C \oslash B} \otimes \psi}{\longrightarrow} & D \oslash C \otimes C \oslash B \otimes B \oslash A
\end{array}$$

Functors between enriched categories

Suppose we have two enriched categories $\mathbb{K}$ and $\mathbb{L}$ over a monoidal category $(\mathbb{M}, \otimes, I, \alpha, \lhd, \rhd)$ and suppose we have a complex mapping which

— to each object $X \in \mathrm{Ob}(\mathbb{K})$ assigns an object $F(X) \in \mathrm{Ob}(\mathbb{L})$,

— to each pair of objects $X, Y \in \mathrm{Ob}(\mathbb{K})$ assigns a morphism $F_X^Y : Y \overset{\mathbb{K}}{\oslash} X \overset{\mathbb{M}}{\to} F(Y) \overset{\mathbb{L}}{\oslash} F(X)$,

and the following conditions are fulfilled:

(i) the unit morphisms turn into unit morphisms, i.e. for each $X \in \mathrm{Ob}(\mathbb{K})$

$$\begin{array}{ccc} & I & \\ {\scriptstyle \varepsilon_X^{\mathbb{K}}} \swarrow & & \searrow {\scriptstyle \varepsilon_{F(X)}^{\mathbb{L}}} \\ X \overset{\mathbb{K}}{\oslash} X & \xrightarrow[F_X^X]{} & F(X) \overset{\mathbb{L}}{\oslash} F(X) \end{array}$$

(ii) the multiplication operation turns into the multiplication operation, i.e. for each $X, Y, Z \in \mathrm{Ob}(\mathbb{K})$

$$\begin{array}{ccc}
Z \overset{\mathbb{K}}{\oslash} Y \otimes Y \overset{\mathbb{K}}{\oslash} X & \xrightarrow{F^Z_Y \otimes F^Y_X} & F(Z) \overset{\mathbb{L}}{\oslash} F(Y) \otimes F(Y) \overset{\mathbb{L}}{\oslash} F(X) \\
{\scriptstyle \bullet_{X,Y,Z}} \downarrow & & \downarrow {\scriptstyle \bullet_{F(X),F(Y),F(Z)}} \\
Z \overset{\mathbb{K}}{\oslash} X & \xrightarrow[F^Z_X]{} & F(Z) \overset{\mathbb{L}}{\oslash} F(X)
\end{array}$$

Then we say that we have a *covariant functor* F from the category $\mathbb{K}$ into the category $\mathbb{L}$ *over the category* $\mathbb{M}$, and denote this as follows:

$$F : \mathbb{K} \overset{\mathbb{M}}{\to} \mathbb{L}$$

Support of a functor

To each functor

$$F : \mathtt{K} \xrightarrow{\mathtt{M}} \mathtt{L}$$

between enriched categories $\mathtt{K}$ and $\mathtt{L}$ over a closed monoidal category $\mathtt{M}$ one can assign a usual functor

$$\mathsf{supp}F : \mathsf{supp}\mathtt{K} \xrightarrow{\mathsf{Set}} \mathsf{supp}\mathtt{L}$$

between the supports of these categories:

— to each object $X \in \mathrm{Ob}(\mathtt{K})$ the functor $\mathsf{supp}F$ assigns the very same object $F(X) \in \mathrm{Ob}(\mathtt{L})$, as the functor F does,

$$\mathsf{supp}F(X) = F(X),$$

— to each morphism $\varphi : X \overset{\text{supp}\mathbb{K}}{\to} Y$, i.e. $\varphi : I \overset{\mathbb{M}}{\to} Y \overset{\mathbb{K}}{\oslash} X$, the functor F assigns the morphism

$$F_X^Y \circ \varphi : I \overset{\mathbb{M}}{\to} F(Y) \overset{\mathbb{L}}{\oslash} F(X)$$

which is a morphism in supp$\mathbb{L}$

$$\mathrm{suppF}(\varphi) = \mathrm{F}_\mathrm{X}^\mathrm{Y} \circ \varphi : \mathrm{F}(\mathrm{X}) \overset{\mathrm{supp}\mathbb{L}}{\to} \mathrm{F}(\mathrm{Y}).$$

Theorem

If $F : \mathbb{K} \overset{\mathbb{M}}{\to} \mathbb{L}$ is a functor between enriched categories $\mathbb{K}$ and $\mathbb{L}$ over a closed monoidal category $\mathbb{M}$, then $\mathrm{suppF} : \mathrm{supp}\mathbb{K} \overset{\mathrm{Set}}{\to} \mathrm{supp}\mathbb{L}$ *is a functor between the supports of these categories.*

Natural transformations of functors

Let $F : \mathbb{K} \overset{\mathbb{M}}{\to} \mathbb{L}$ and $G : \mathbb{K} \overset{\mathbb{M}}{\to} \mathbb{L}$ be two covariant functors between enriched categories over a monoidal category $\mathbb{M}$. We say that a *natural transformation* or a *morphism* σ of a functor F into the functor G is given, and denote this as

$$\sigma : F \rightarrowtail G,$$

if each object $X \in \mathbb{K}$ is associated to a morphism $\sigma_X : I \overset{\mathbb{M}}{\to} G(X) \overset{\mathbb{L}}{\oslash} F(X)$ in such a way that for any two objects $X, Y \in \mathbb{K}$ the following diagram is commutative in $\mathbb{M}$:

$$
\begin{array}{ccccc}
 & & Y \overset{K}{\oslash} X & & \\
 & \swarrow^{\lhd^{-1}_{Y\overset{K}{\oslash}X}} & & \searrow^{\rhd^{-1}_{Y\overset{K}{\oslash}X}} & \\
I \otimes Y \overset{K}{\oslash} X & & & & Y \overset{K}{\oslash} X \otimes I \\
\downarrow{\scriptstyle \sigma_Y \otimes F^Y_X} & & & & \downarrow{\scriptstyle G^Y_X \otimes \sigma_X} \\
G(Y) \overset{L}{\oslash} F(Y) \otimes F(Y) \overset{L}{\oslash} F(X) & & & & G(Y) \overset{L}{\oslash} G(X) \otimes G(X) \overset{L}{\oslash} F(X) \\
 & \searrow_{\bullet_{F(X),F(Y),G(Y)}} & & \swarrow_{\bullet_{F(X),G(X),G(Y)}} & \\
 & & G(Y) \overset{L}{\oslash} F(X) & &
\end{array}
$$

A composition of two natural transformations $\sigma : F \rightarrowtail G$ and $\tau : G \rightarrowtail H$ is the natural transformation $\sigma \circ \tau : F \rightarrowtail H$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
I & \xrightarrow{\quad (\tau\circ\sigma)_X \quad} & H(X) \overset{\mathrm{L}}{\oslash} F(X) \\
{\scriptstyle \lhd_I^{-1}} \big\downarrow & & \big\uparrow {\scriptstyle \bullet_{F(X),G(X),H(X)}} \\
I \circledast I & \xrightarrow[\tau_X \circledast \sigma_X]{\qquad} & H(X) \overset{\mathrm{L}}{\oslash} G(X) \circledast G(X) \overset{\mathrm{L}}{\oslash} F(X)
\end{array}
$$

Space of natural transformations $\mathrm{Nat}(F, G)$

The system of natural transformations of the covariant functor $F : \mathtt{K} \overset{\mathtt{M}}{\to} \mathtt{L}$ into the covariant functor $G : \mathtt{K} \overset{\mathtt{M}}{\to} \mathtt{L}$ of enriched categories over $\mathtt{M}$ can be represented as an object of the category $\mathtt{M}$. This is done in two steps.

First of all, a *wedge* of the functor F into the functor G with a vertex $B \in \mathtt{M}$ is defined as an arbitrary system of morphisms

$$\beta_X : B \overset{\mathtt{M}}{\to} G(X) \overset{\mathtt{L}}{\oslash} F(X), \qquad X \in \mathrm{Ob}(\mathtt{K}),$$

such that for any morphism $\varphi : X \overset{\mathrm{supp}\mathtt{K}}{\to} Y$ the following diagram is commutative

$$\begin{array}{ccc}
B & \xrightarrow{\quad \beta_X \quad} & G(X) \overset{\mathtt{L}}{\oslash} F(X) \\
\beta_Y \Big\downarrow & & \Big\downarrow G(\varphi)\overset{\mathtt{L}}{\oslash}F(1_X) \\
G(Y) \overset{\mathtt{L}}{\oslash} F(Y) & \xrightarrow[G(1_Y)\overset{\mathtt{L}}{\oslash}F(\varphi)]{} & G(Y) \overset{\mathtt{L}}{\oslash} F(X)
\end{array}$$

In other words,

$$G(\varphi) \overset{\mathbb{L}}{\oslash} F(1_X) \circ \beta_X = G(1_Y) \overset{\mathbb{L}}{\oslash} F(\varphi) \circ \beta_Y, \qquad X, Y \in \mathrm{Ob}(\mathbb{K}).$$

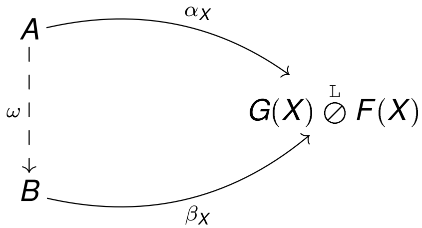
Further, if we have two wedges

$$\alpha_X : A \to G(X) \overset{\mathbb{L}}{\oslash} F(X), \qquad X \in \mathrm{Ob}(\mathbb{K})$$

and

$$\beta_X : B \to G(X) \overset{\mathbb{L}}{\oslash} F(X), \qquad X \in \mathrm{Ob}(\mathbb{K})$$

then a morphism $\omega : A \stackrel{\mathbb{M}}{\to} B$ is called a *morphism of wedges*, if for each object $X \in \mathrm{Ob}(\mathbb{K})$ the following diagram is commutative



Finally, the *space of natural transformations* or the *space of morphisms* of the functor F into the functor G is a universal attracting wedge from F to G, that is, such a wedge

$$\upsilon_X : U \xrightarrow{\mathrm{M}} G(X) \overset{\mathrm{L}}{\oslash} F(X), \qquad X \in \mathrm{Ob}(\mathrm{K}),$$

that for any other wedge

$$\beta_X : B \xrightarrow{\mathrm{M}} G(X) \overset{\mathrm{L}}{\oslash} F(X), \qquad X \in \mathrm{Ob}(\mathrm{K}),$$

there is a unique morphism of wedges

$$\omega : B \xrightarrow{\mathrm{M}} U.$$

Notation: $U = \mathrm{Nat}(\mathrm{F}, \mathrm{G})$.

Integral unit

We say that the unit I in the monoidal category $\mathtt{M}$ is *integral*, if for any two parallel morphisms $\varphi, \psi : X \overset{\mathtt{M}}{\rightrightarrows} Y$, which do not coincide, there is a morphism $\iota : I \overset{\mathtt{M}}{\to} X$ such that the compositions $\varphi \circ \iota$ and $\psi \circ \iota$ do not coinside as well:

$$\varphi \neq \psi : X \to Y \quad \Rightarrow \quad \exists \iota : I \to X \quad \varphi \circ \iota \neq \psi \circ \iota.$$

Tannaka duality

Theorem

Let $\mathbb{M}$ be a closed symmetric monoidal category with equalizers, where the unit I is an integral object. If A is a monoid in $\mathbb{M}$ and ${}_A\mathbb{M}$ the (enriched over $\mathbb{M}$) category of left modules over A in $\mathbb{M}$, then

(i) *the forgetful functor $F : {}_A\mathbb{M} \to \mathbb{M}$ has the space of endomorphisms $\operatorname{End}(F)$, which is a monoid in $\mathbb{M}$;*

(ii) *the monoid A can be reconstructed from the forgetful functor $F : {}_A\mathbb{M} \to \mathbb{M}$ by the formula*

$$A \cong \operatorname{End}(F),$$

and this is not only isomorphism of objects of the category $\mathbb{M}$, but also of monoids in $\mathbb{M}$.

Stereotype spaces

A *stereotype space* is a topological vector space X over $\mathbb{C}$ such that the natural map

$$i_X : X \to X^{\star\star}, \qquad i_X(x)(f) = f(x), \qquad x \in X,\ f \in X^\star$$

is an isomorphism of topological vector spaces (i.e. a linear and a homeomorphic map). Here the dual space $X^\star$ is defined as the space of all linear continuous functionals $f : X \to \mathbb{C}$ endowed with the topology of *uniform convergence on totally bounded sets in* X, and the second dual space $X^{\star\star}$ is the space dual to $X^\star$ in the same sense.

Theorem

The category Ste *of stereotype spaces is a closed symmetric monoidal category with equalizers, where the unit* $\mathbb{C}$ *is an integral object.*

Tannaka theorem for stereotype algebras

Corollary

In the category Ste *of stereotype spaces the Tannaka theorem holds: each stereotype algebra A can be reconstructed from the forgetful functor $F : {}_A\mathrm{Ste} \to \mathrm{Ste}$.*

Stereotype group algebras

Theorem

For every locally compact group G and for every stereotype space X the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ \ \delta\ \ } & \mathcal{C}^\star(G) \\ & {\scriptstyle\pi}\searrow \qquad \swarrow{\scriptstyle\varphi} & \\ & \mathcal{L}(X) & \end{array}$$

sets a bijection between the continuous representations of G in the space X and the morphisms of stereotype algebras $\varphi : \mathcal{C}^\star(G) \to \mathcal{L}(X)$, and G is reconstructed from $\mathcal{C}^\star(G)$ as its involutive group part:

$$G \cong \mathrm{G}(\mathcal{C}^\star(G))$$

Reconstruction theorem for locally compact groups

Consider the category

$$ {}_G\mathrm{Ste} = {}_{\mathcal{C}^\star(G)}\mathrm{Ste} $$

of G-actions on stereotype spaces, or what is the same the left stereotype $\mathcal{C}^\star(G)$-modules.

Corollary

Each locally compact group G can be reconstructed from the forgetful functor $F : {}_G\mathrm{Ste} \to \mathrm{Ste}$.

Literature

S. S. Akbarov. Stereotype spaces and algebras. Berlin. De Gruyter, 2022.