Orbits of Algebraic Groups and Classification Problems

Vladimir L. Popov

Steklov Mathematical Institute
Russian Academy of Sciences, Moscow
popovvl@mi-ras.ru

International Conference
Algebraic Groups, Their Friends and Relations
September 21, 2022
St. Petersburg Department of Steklov Mathematical Institute
Russian Academy of Sciences

Notation

Notation

- k an algebraically closed ground field k of arbitrary characteristic.

Notation

Notation

- k an algebraically closed ground field k of arbitrary characteristic.
- G a connected linear algebraic group.

Notation

Notation

- k an algebraically closed ground field k of arbitrary characteristic.
- G a connected linear algebraic group.
- V a finite-dimensional algebraic kG-module. The action of G of V is denoted as

$$G \times V \to V, \quad (g, v) \mapsto g \cdot v.$$

Main problem

The following problem permanently arises in algebraic transformation group theory and its applications to various classification problems:

Main problem

The following problem permanently arises in algebraic transformation group theory and its applications to various classification problems:

Problem ($*$)

Given two points a and $b \in V$, how can one find out whether or not they lie in the same G-orbit?

Decidability

Elucidation

By “finding out” we mean **decidability** of Problem $(*)$, i.e., the existence of an **algorithm** providing a correct yes-or-no answer to Problem $(*)$ by means of finitely many effectively feasible operations.

Elucidation

By "finding out" we mean **decidability** of Problem $(*)$, i.e., the existence of an **algorithm** providing a correct yes-or-no answer to Problem $(*)$ by means of finitely many effectively feasible operations.

Warning

Finding an algorithm with the best parameters (running time and memory used) is a **separate topic** lying beyond the scope of this talk.

Example 1: Classification of algebras

Example 1: Classification of algebras

L a finite dimensional vector space over k.

Example 1: Classification of algebras

L a finite dimensional vector space over k.
$V = L^* \otimes L^* \otimes L$.

Example 1: Classification of algebras

L a finite dimensional vector space over k.
$V = L^* \otimes L^* \otimes L$.

Points of V are the **structures of (not necessarily associative)** $\boldsymbol{k}$**-algebras on the vector space** $\boldsymbol{L}$:

Example 1: Classification of algebras

L a finite dimensional vector space over k.
$V = L^* \otimes L^* \otimes L$.

Points of V are the **structures of (not necessarily associative) $\boldsymbol{k}$-algebras on the vector space $\boldsymbol{L}$**:

$$v = \sum f \otimes h \otimes \ell \in V = L^* \otimes L^* \otimes L \text{ determines}$$

the algebra $\boldsymbol{\mathcal{A}(v)}$ with the multiplication

$$st := \sum f(s)h(t)\ell \quad \text{for all } s, t \in L.$$

Example 1: Classification of algebras

L a finite dimensional vector space over k.
$V = L^* \otimes L^* \otimes L$.

Points of V are the **structures of (not necessarily associative) $\boldsymbol{k}$-algebras on the vector space $\boldsymbol{L}$**:

$$v = \sum f \otimes h \otimes \ell \in V = L^* \otimes L^* \otimes L \text{ determines}$$

the algebra $\boldsymbol{\mathcal{A}(v)}$ with the multiplication

$$st := \sum f(s)h(t)\ell \quad \text{for all } s, t \in L.$$

$G = \mathrm{GL}(L)$ naturally acts on $V = L^* \otimes L^* \otimes L$.

Example 1: Classification of algebras

L a finite dimensional vector space over k.
$V = L^* \otimes L^* \otimes L$.

Points of V are the **structures of (not necessarily associative) k-algebras on the vector space L**:

$$v = \sum f \otimes h \otimes \ell \in V = L^* \otimes L^* \otimes L \text{ determines}$$

the algebra $\mathcal{A}(v)$ with the multiplication

$$st := \sum f(s)h(t)\ell \quad \text{for all } s, t \in L.$$

$G = \mathrm{GL}(L)$ naturally acts on $V = L^* \otimes L^* \otimes L$.

$\mathcal{A}(a)$ **and** $\mathcal{A}(b)$ **are isomorphic algebras** $\iff$ $G \cdot a = G \cdot b$.

Example 1: Classification of algebras (continued)

If $G \cdot a$ lies in the closure of $G \cdot b$, then $\mathcal{A}_a$ is called a **degeneration** of $\mathcal{A}_b$.

Example 1: Classification of algebras (continued)

If $G \cdot a$ lies in the closure of $G \cdot b$, then $\mathcal{A}_a$ is called a **degeneration** of $\mathcal{A}_b$.

In general case, to find out whether $\mathcal{A}_a$ is a degeneration of $\mathcal{A}_b$ is considered a difficult problem. In some special cases degeneration are classified by means of ad hoc methods.

Example 2: Classification of modules

Example 2: Classification of modules

L a finite dimensional vector space over k.

Example 2: Classification of modules

L a finite dimensional vector space over k.
A an associative k-algebra.

Example 2: Classification of modules

L a finite dimensional vector space over k.
A an associative k-algebra. Every k-algebra homomorphism $\varphi\colon A \to \mathrm{End}(L)$ determines an A-module structure on L by

$$a \cdot \ell := \varphi(a)(\ell) \quad \text{for all } a \in A,\ \ell \in L.$$

Example 2: Classification of modules

L a finite dimensional vector space over k.
A an associative k-algebra. Every k-algebra homomorphism $\varphi\colon A \to \mathrm{End}(L)$ determines an A-module structure on L by

$$a \cdot \ell := \varphi(a)(\ell) \quad \text{for all } a \in A,\ \ell \in L.$$

This yields a bijection between the sets of such homomorphisms and A-module structures.

Example 2: Classification of modules

L a finite dimensional vector space over k.
A an associative k-algebra. Every k-algebra homomorphism $\varphi\colon A \to \mathrm{End}(L)$ determines an A-module structure on L by

$$a \cdot \ell := \varphi(a)(\ell) \quad \text{for all } a \in A,\ \ell \in L.$$

This yields a bijection between the sets of such homomorphisms and A-module structures.
In turn, if A is generated by the elements $a_1, \ldots, a_d$, then φ is determined by the point

$$s = \big(\varphi(a_1), \ldots, \varphi(a_d)\big) \in V = \mathrm{End}(L) \times \cdots \times \mathrm{End}(L) \quad (d \text{ factors}).$$

Example 2: Classification of modules

L a finite dimensional vector space over k.
A an associative k-algebra. Every k-algebra homomorphism $\varphi\colon A \to \mathrm{End}(L)$ determines an A-module structure on L by

$$a \cdot \ell := \varphi(a)(\ell) \quad \text{for all } a \in A,\ \ell \in L.$$

This yields a bijection between the sets of such homomorphisms and A-module structures.
In turn, if A is generated by the elements $a_1, \ldots, a_d$, then φ is determined by the point

$$s = \big(\varphi(a_1), \ldots, \varphi(a_d)\big) \in V = \mathrm{End}(L) \times \cdots \times \mathrm{End}(L) \quad (d \text{ factors}).$$

Denote $\mathcal{M}_s$ the A-module corresponding to this φ.

Example 2: Classification of modules (continued)

Example 2: Classification of modules (continued)

When φ runs over all homomorphisms, s runs over a closed algebraic subset $\mathrm{Mod}_A(L)$ in V called the **variety of $\boldsymbol{A}$-module structures on $\boldsymbol{L}$**.

Example 2: Classification of modules (continued)

When φ runs over all homomorphisms, s runs over a closed algebraic subset $\mathrm{Mod}_A(L)$ in V called the **variety of A-module structures on L**.

$G = \mathrm{GL}(L)$ diagonally acts on V by conjugation.

$\mathcal{M}_a$ and $\mathcal{M}_b$ are isomorphic A-modules $\iff$ $G \cdot a = G \cdot b$.

Example 2: Classification of modules (continued)

If $G \cdot a$ lies in the closure of $G \cdot b$, then $\mathcal{M}_a$ is called a **degeneration** of $\mathcal{M}_b$.

Example 2: Classification of modules (continued)

If $G \cdot a$ lies in the closure of $G \cdot b$, then $\mathcal{M}_a$ is called a **degeneration** of $\mathcal{M}_b$.

In general case, to find out whether $\mathcal{M}_a$ is a degeneration of $\mathcal{M}_b$ is considered a difficult problem. In some cases it is solved by means of ad hoc methods (for instance, if A is the path algebra of an oriented extended Dynkin graph of a root system of type A_l, D_l, E_6, E_7, or E_8 (Bongartz, 1995)).

Example 3: Classification of representations of quivers

Example 3: Classification of representations of quivers

Q a quiver (i.e., a finite directed graph).

Example 3: Classification of representations of quivers

Q a quiver (i.e., a finite directed graph).
$v_1, \ldots, v_m$ the set of all vertices of Q.

Example 3: Classification of representations of quivers

Q a quiver (i.e., a finite directed graph).
$v_1, \ldots, v_m$ the set of all vertices of Q.
E the set of all edges of Q.

Example 3: Classification of representations of quivers

Q a quiver (i.e., a finite directed graph).
$v_1, \ldots, v_m$ the set of all vertices of Q.
E the set of all edges of Q.
$t\alpha$ (resp. $h\alpha$) the tail (resp. head) of $\alpha \in E$.

Example 3: Classification of representations of quivers

Q a quiver (i.e., a finite directed graph).
$v_1, \ldots, v_m$ the set of all vertices of Q.
E the set of all edges of Q.
$t\alpha$ (resp. $h\alpha$) the tail (resp. head) of $\alpha \in E$.
U_{v_i} a finite-dimensional vector space over k assigned to every v_i.

Example 3: Classification of representations of quivers

Q a quiver (i.e., a finite directed graph).
$v_1, \dots, v_m$ the set of all vertices of Q.
E the set of all edges of Q.
$t\alpha$ (resp. $h\alpha$) the tail (resp. head) of $\alpha \in E$.
U_{v_i} a finite-dimensional vector space over k assigned to every v_i.
The k-vector space

$$V = \bigoplus_{\alpha \in E} \mathrm{Hom}_k(U_{t\alpha}, U_{h\alpha}).$$

Example 3: Classification of representations of quivers

Q a quiver (i.e., a finite directed graph).
$v_1, \ldots, v_m$ the set of all vertices of Q.
E the set of all edges of Q.
$t\alpha$ (resp. $h\alpha$) the tail (resp. head) of $\alpha \in E$.
U_{v_i} a finite-dimensional vector space over k assigned to every v_i.
The k-vector space

$$V = \bigoplus_{\alpha \in E} \mathrm{Hom}_k(U_{t\alpha}, U_{h\alpha}).$$

is the set of all representations of Q of dimension $\big(\dim U_{v_1}, \ldots, \dim U_{v_m}\big) \in \mathbb{N}^m$.

Example 3: Classification of representations of quivers (continued)

Example 3: Classification of representations of quivers (continued)

The group

$$G = \mathrm{GL}(U_{v_1}) \times \cdots \times \mathrm{GL}(U_{v_m})$$

Example 3: Classification of representations of quivers (continued)

The group

$$G = \mathrm{GL}(U_{v_1}) \times \cdots \times \mathrm{GL}(U_{v_m})$$

acts on V by

$$(g_{v_1}, \ldots, g_{v_n}) \cdot \bigoplus_{\alpha \in E} h_{t\alpha, h\alpha} = \bigoplus_{\alpha \in E} g_{h\alpha} h_{t\alpha, h\alpha} g_{t\alpha}^{-1}.$$

Example 3: Classification of representations of quivers (continued)

The group

$$G = \mathrm{GL}(U_{v_1}) \times \cdots \times \mathrm{GL}(U_{v_m})$$

acts on V by

$$(g_{v_1}, \ldots, g_{v_n}) \cdot \bigoplus_{\alpha \in E} h_{t\alpha, h\alpha} = \bigoplus_{\alpha \in E} g_{h\alpha} h_{t\alpha, h\alpha} g_{t\alpha}^{-1}.$$

Representations $a, b \in V$ are equivalent $\iff$ $G \cdot a = G \cdot b$.

Example 4: Classification of smooth hypersurfaces

Example 4: Classification of smooth hypersurfaces

h_1 and h_2 irreducible forms of the same positive degree in the homogeneous coordinates of $\mathbb{P}^n$.

Example 4: Classification of smooth hypersurfaces

h_1 and h_2 irreducible forms of the same positive degree in the homogeneous coordinates of $\mathbb{P}^n$.

H_i the hypersurface in $\mathbb{P}^n$ defined by the equation $h_i = 0$.

Example 4: Classification of smooth hypersurfaces

h_1 and h_2 irreducible forms of the same positive degree in the homogeneous coordinates of $\mathbb{P}^n$.

H_i the hypersurface in $\mathbb{P}^n$ defined by the equation $h_i = 0$.

Theorem (Severy–Lefschetz–Andreotti)

Every positive divisor on any smooth hypersurface in $\mathbb{P}^n$, where $n \geqslant 4$, is cut out by a hypersurface in $\mathbb{P}^n$.

Example 4: Classification of smooth hypersurfaces

h_1 and h_2 irreducible forms of the same positive degree in the homogeneous coordinates of $\mathbb{P}^n$.
H_i the hypersurface in $\mathbb{P}^n$ defined by the equation $h_i = 0$.

Theorem (Severy–Lefschetz–Andreotti)

Every positive divisor on any smooth hypersurface in $\mathbb{P}^n$, where $n \geqslant 4$, is cut out by a hypersurface in $\mathbb{P}^n$.

This implies:

If $n \geqslant 4$ and both H_1 and H_2 are smooth, then

H_1 and H_2 are isomorphic varieties $\iff \mathrm{GL}_{n+1} \cdot h_1 = \mathrm{GL}_{n+1} \cdot h_2$.

Example 5: Classification of algebraic curves of genus $g \geqslant 2$

Example 5: Classification of algebraic curves of genus $g \geqslant 2$

There are other types of algebraic varieties for which the isomorphism problem is reduced to finding out whether some forms lie in the same orbit of a linear algebraic group. For instance, this is so for algebraic curves.

Example 5: Classification of algebraic curves of genus $g \geqslant 2$

There are other types of algebraic varieties for which the isomorphism problem is reduced to finding out whether some forms lie in the same orbit of a linear algebraic group. For instance, this is so for algebraic curves.

Theorem

Example 5: Classification of algebraic curves of genus $g \geqslant 2$

There are other types of algebraic varieties for which the isomorphism problem is reduced to finding out whether some forms lie in the same orbit of a linear algebraic group. For instance, this is so for algebraic curves.

Theorem

- *Every irreducible smooth projective curve of a genus $g \geqslant 2$ is embedded into $\mathbb{P}^{5g-6}$ be means of its tripled canonical class.*

Example 5: Classification of algebraic curves of genus $g \geqslant 2$

There are other types of algebraic varieties for which the isomorphism problem is reduced to finding out whether some forms lie in the same orbit of a linear algebraic group. For instance, this is so for algebraic curves.

Theorem

- *Every irreducible smooth projective curve of a genus $g \geqslant 2$ is embedded into $\mathbb{P}^{5g-6}$ be means of its tripled canonical class.*
- *The following properties are equivalent:*

Example 5: Classification of algebraic curves of genus $g \geqslant 2$

There are other types of algebraic varieties for which the isomorphism problem is reduced to finding out whether some forms lie in the same orbit of a linear algebraic group. For instance, this is so for algebraic curves.

Theorem

- *Every irreducible smooth projective curve of a genus $g \geqslant 2$ is embedded into $\mathbb{P}^{5g-6}$ be means of its tripled canonical class.*
- *The following properties are equivalent:*
  - *Two such curves X and Y are isomorphic.*

Example 5: Classification of algebraic curves of genus $g \geqslant 2$

There are other types of algebraic varieties for which the isomorphism problem is reduced to finding out whether some forms lie in the same orbit of a linear algebraic group. For instance, this is so for algebraic curves.

Theorem

- *Every irreducible smooth projective curve of a genus $g \geqslant 2$ is embedded into $\mathbb{P}^{5g-6}$ be means of its tripled canonical class.*
- *The following properties are equivalent:*
  - *Two such curves X and Y are isomorphic.*
  - *Their images $\widetilde{X}$ and $\widetilde{Y}$ in $\mathbb{P}^{5g-6}$ are transformed one into another by a projective transformation of $\mathbb{P}^{5g-6}$.*

Example 5: Classification of algebraic curves of genus $g \geqslant 2$

There are other types of algebraic varieties for which the isomorphism problem is reduced to finding out whether some forms lie in the same orbit of a linear algebraic group. For instance, this is so for algebraic curves.

Theorem

- *Every irreducible smooth projective curve of a genus $g \geqslant 2$ is embedded into $\mathbb{P}^{5g-6}$ be means of its tripled canonical class.*
- *The following properties are equivalent:*
  - *Two such curves X and Y are isomorphic.*
  - *Their images $\widetilde{X}$ and $\widetilde{Y}$ in $\mathbb{P}^{5g-6}$ are transformed one into another by a projective transformation of $\mathbb{P}^{5g-6}$.*
  - *The Chow forms of $\widetilde{X}$ and $\widetilde{Y}$ lie in the same orbit of the corresponding GL_N.*

Main problem

Main problem

The key property of orbits of algebraic group actions related to Problem $(*)$ is given by the following classical

Main problem

The key property of orbits of algebraic group actions related to Problem $(*)$ is given by the following classical

Theorem

Every orbit of algebraic group action is open in its closure.

Main problem

The key property of orbits of algebraic group actions related to Problem $(*)$ is given by the following classical

Theorem

Every orbit of algebraic group action is open in its closure.

This yields

Corollary

$$G\cdot a = G\cdot b \iff G\cdot a \subseteq \overline{G\cdot b} \text{ and } G\cdot b \subseteq \overline{G\cdot a}.$$

Main problem

This leads to a more general

Problem ($**$**)**

Given two points a and $b \in V$, *how can one find out whether or not* $G \cdot a$ *lies in* $\overline{G \cdot b}$*?*

Example: Orbit closures and Geometric Complexity Theory

Example: Orbit closures and Geometric Complexity Theory

Using the sizes of arithmetic circuits computing multivariate polynomials over $k = \mathbb{C}$, one defines certain classes **VP** and **VNP** $\supseteq$ **VP** of sequences of such polynomials.

Example: Orbit closures and Geometric Complexity Theory

Using the sizes of arithmetic circuits computing multivariate polynomials over $k = \mathbb{C}$, one defines certain classes **VP** and **VNP** $\supseteq$ **VP** of sequences of such polynomials.

Valiant's conjecture:
VP $\neq$ **VNP**

Example: Orbit closures and Geometric Complexity Theory

Using the sizes of arithmetic circuits computing multivariate polynomials over $k = \mathbb{C}$, one defines certain classes **VP** and **VNP** $\supseteq$ **VP** of sequences of such polynomials.

Valiant's conjecture:
VP $\neq$ **VNP**

This conjecture can be seen as an algebraic version of Cook's famous $P \neq NP$ hypothesis.

Example: Orbit closures and Geometric Complexity Theory (continued)

The following major result led to a conjecture about orbit closures that implies Valiant's conjecture.

Example: Orbit closures and Geometric Complexity Theory (continued)

The following major result led to a conjecture about orbit closures that implies Valiant's conjecture.
Let

$$\mathrm{perm}_n := \text{permanent of} \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \cdots & \cdots & \cdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \text{ where } x_{ij} \text{ are variables}$$

Example: Orbit closures and Geometric Complexity Theory (continued)

The following major result led to a conjecture about orbit closures that implies Valiant's conjecture.
Let

$$\mathrm{perm}_n := \text{permanent of} \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \dots & \dots & \dots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \text{ where } x_{ij} \text{ are variables}$$

Theorem (Valiant)

The following properties are equivalent:

Example: Orbit closures and Geometric Complexity Theory (continued)

The following major result led to a conjecture about orbit closures that implies Valiant's conjecture.
Let

$$\mathrm{perm}_n := \text{permanent of} \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \ldots & \ldots & \ldots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} \text{ where } x_{ij} \text{ are variables}$$

Theorem (Valiant)

The following properties are equivalent:

- *The sequence* $\mathrm{perm}_1, \mathrm{perm}_2, \ldots$ *does not lie in* **VP**.

Example: Orbit closures and Geometric Complexity Theory (continued)

The following major result led to a conjecture about orbit closures that implies Valiant's conjecture.

Let

$$\mathrm{perm}_n := \text{permanent of } \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \multicolumn{3}{c}{\dotfill} \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}$$ where x_{ij} are variables

Theorem (Valiant)

The following properties are equivalent:

- *The sequence* $\mathrm{perm}_1, \mathrm{perm}_2, \ldots$ *does not lie in* **VP**.
- **VP** $\neq$ **VNP**.

Example: Orbit closures and Geometric Complexity Theory (continued)

Example: Orbit closures and Geometric Complexity Theory (continued)

M_m the vector space of $m \times m$-matrices with coefficients in k.

Example: Orbit closures and Geometric Complexity Theory (continued)

M_m the vector space of $m \times m$-matrices with coefficients in k.
$x_{i,j}$ the standard coordinate functions on M_m.

Example: Orbit closures and Geometric Complexity Theory (continued)

M_m the vector space of $m \times m$-matrices with coefficients in k.
$x_{i,j}$ the standard coordinate functions on M_m.
n a positive integer, $n < m$.

Example: Orbit closures and Geometric Complexity Theory (continued)

M_m the vector space of $m \times m$-matrices with coefficients in k.
$x_{i,j}$ the standard coordinate functions on M_m.
n a positive integer, $n < m$.

$$d_m \colon \mathrm{M}_m \to k \quad A \mapsto \det A,$$
$$p_n \colon \mathrm{M}_m \to k \quad A \mapsto x_{1,1}^{m-n} \cdot \mathrm{permanent\ of\ } A_n,$$

where A_n is the right down $n \times n$-corner of A.

Example: Orbit closures and Geometric Complexity Theory (continued)

M_m the vector space of $m \times m$-matrices with coefficients in k.
$x_{i,j}$ the standard coordinate functions on M_m.
n a positive integer, $n < m$.

$$\begin{aligned} d_m &\colon \mathrm{M}_m \to k \quad A \mapsto \det A, \\ p_n &\colon \mathrm{M}_m \to k \quad A \mapsto x_{1,1}^{m-n} \cdot \text{permanent of } A_n, \end{aligned}$$

where A_n is the right down $n \times n$-corner of A.
GL_{m^2} naturally acts on the space of polynomial functions $\mathrm{M}_m \to k$.

Example: Orbit closures and Geometric Complexity Theory (continued)

M_m the vector space of $m \times m$-matrices with coefficients in k.
$x_{i,j}$ the standard coordinate functions on M_m.
n a positive integer, $n < m$.

$$
\begin{aligned}
d_m \colon \mathrm{M}_m \to k \quad & A \mapsto \det A, \\
p_n \colon \mathrm{M}_m \to k \quad & A \mapsto x_{1,1}^{m-n} \cdot \text{permanent of } A_n,
\end{aligned}
$$

where A_n is the right down $n \times n$-corner of A.
GL_{m^2} naturally acts on the space of polynomial functions $\mathrm{M}_m \to k$.

Theorem

Example: Orbit closures and Geometric Complexity Theory (continued)

M_m the vector space of $m \times m$-matrices with coefficients in k.
$x_{i,j}$ the standard coordinate functions on M_m.
n a positive integer, $n < m$.

$$d_m\colon \mathrm{M}_m \to k \quad A \mapsto \det A,$$
$$p_n\colon \mathrm{M}_m \to k \quad A \mapsto x_{1,1}^{m-n} \cdot \mathrm{permanent\ of\ } A_n,$$

where A_n is the right down $n \times n$-corner of A.
GL_{m^2} naturally acts on the space of polynomial functions $\mathrm{M}_m \to k$.

Theorem

There is n_0 such that for all $n \geqslant n_0$ and m large enough compared to n, namely, $m = O(n^2 2^n)$,

$$p_n \text{ lies in the closure of } \mathrm{GL}_{m^2} \cdot d_m. \qquad (!)$$

Example: Orbit closures and Geometric Complexity Theory (continued)

Example: Orbit closures and Geometric Complexity Theory (continued)

Let $m(n)$ be the **minimal** m such that property (!) holds for every $n \geqslant n_0$.

Example: Orbit closures and Geometric Complexity Theory (continued)

Let $m(n)$ be the **minimal** m such that property (!) holds for every $n \geqslant n_0$.

Conjecture (Mulmuley, Sohoni)

The function $n \mapsto m(n)$ grows faster than any polynomial in n.

Example: Orbit closures and Geometric Complexity Theory (continued)

Let $m(n)$ be the **minimal** m such that property (!) holds for every $n \geqslant n_0$.

Conjecture (Mulmuley, Sohoni)

The function $n \mapsto m(n)$ grows faster than any polynomial in n.

Theorem

Mulmuley–Sohoni's conjecture $\Rightarrow$ *Valiant's conjecture.*

Plan

Plan

The talk is aimed to explain that

Problem $(**)$ is decidable if $\mathrm{char}\, k = 0$ and G is reductive.

Plan

The talk is aimed to explain that

Problem $(**)$ is decidable if $\mathrm{char}\, k = 0$ and G is reductive.

The plan:

Plan

The talk is aimed to explain that

Problem $(**)$ is decidable if $\mathrm{char}\, k = 0$ and G is reductive.

The plan:

- Describing the input of the algorithm.

Plan

The talk is aimed to explain that

Problem (∗∗) is decidable if $\mathrm{char}\, k = 0$ and G is reductive.

The plan:

- Describing the input of the algorithm.
- Reducing to the case of conical orbits $G \cdot a$ and $G \cdot b$.

Plan

The talk is aimed to explain that

Problem (∗∗) is decidable if $\mathrm{char}\, k = 0$ and G is reductive.

<u>The plan</u>:

- Describing the input of the algorithm.
- Reducing to the case of conical orbits $G \cdot a$ and $G \cdot b$.
- Describing the algorithm.

Describing the input of the algorithm

The input of the algorithm consists of **two components**.

Describing the input of the algorithm

The input of the algorithm consists of **two components**.

The **first component** is a certain
set of n^2 functions describing the G-action on V.

Describing the input of the algorithm

The input of the algorithm consists of **two components**.

The **first component** is a certain
set of n^2 functions describing the G-action on V.

This set is constructed using the following

rational parametrization of G.

On the input: Rational parametrization of G

Notation

On the input: Rational parametrization of G

Notation

- $\mathbb{A}^{r,s} := \{(\varepsilon_1, \ldots, \varepsilon_{r+s}) \in \mathbb{A}^{r+s} \mid \varepsilon_1 \cdots \varepsilon_r \neq 0\}, \qquad r, s \in \mathbb{N}.$

On the input: Rational parametrization of G

Notation

- $\mathbb{A}^{r,s} := \{(\varepsilon_1, \ldots, \varepsilon_{r+s}) \in \mathbb{A}^{r+s} \mid \varepsilon_1 \cdots \varepsilon_r \neq 0\}, \qquad r, s \in \mathbb{N}.$
- $x_1, \ldots, x_{r+s}$ the standard coordinate functions on $\mathbb{A}^{r,s}$:

$$x_i\big((\varepsilon_1, \ldots, \varepsilon_{r+s})\big) = \varepsilon_i.$$

On the input: Rational parametrization of G

Notation

- $\mathbb{A}^{r,s} := \{(\varepsilon_1, \ldots, \varepsilon_{r+s}) \in \mathbb{A}^{r+s} \mid \varepsilon_1 \cdots \varepsilon_r \neq 0\}, \qquad r, s \in \mathbb{N}.$
- $x_1, \ldots, x_{r+s}$ the standard coordinate functions on $\mathbb{A}^{r,s}$:

$$x_i\big((\varepsilon_1, \ldots, \varepsilon_{r+s})\big) = \varepsilon_i.$$

The (ordered) set of all Laurent monomials

$$x_1^{i_1} \cdots x_{r+s}^{i_{r+s}}, \quad \text{where} \quad i_1, \ldots, i_r \in \mathbb{Z} \text{ and } i_{r+1}, \ldots, i_{r+s} \in \mathbb{N},$$

On the input: Rational parametrization of G

Notation

- $\mathbb{A}^{r,s} := \{(\varepsilon_1, \ldots, \varepsilon_{r+s}) \in \mathbb{A}^{r+s} \mid \varepsilon_1 \cdots \varepsilon_r \neq 0\}, \qquad r, s \in \mathbb{N}.$
- $x_1, \ldots, x_{r+s}$ the standard coordinate functions on $\mathbb{A}^{r,s}$:

$$x_i\big((\varepsilon_1, \ldots, \varepsilon_{r+s})\big) = \varepsilon_i.$$

The (ordered) set of all Laurent monomials

$$x_1^{i_1} \cdots x_{r+s}^{i_{r+s}}, \quad \text{where} \quad i_1, \ldots, i_r \in \mathbb{Z} \text{ and } i_{r+1}, \ldots, i_{r+s} \in \mathbb{N},$$

is a basis of the k-vector space

$$k[\mathbb{A}^{r,s}] = k[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}, x_{r+1}, \ldots, x_{r+s}]$$

of all regular functions on $\mathbb{A}^{r,s}$.

On the input: Rational parametrization of G

Lemma

There is an open embedding

$$\iota\colon \mathbb{A}^{r,s} \hookrightarrow G$$

with $r = \operatorname{rk} G$.

On the input: Example of rational parametrization

Example

Let $G = \mathrm{SL}_2$.

Example

Let $G = \mathrm{SL}_2$.

The map

$$\iota\colon \mathbb{A}^{1,2} \to \mathrm{SL}_2,$$

On the input: Example of rational parametrization

Example

Let $G = \mathrm{SL}_2$.
The map

$$\iota\colon \mathbb{A}^{1,2} \to \mathrm{SL}_2,$$

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mapsto \begin{bmatrix} 1 & \varepsilon_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_1^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \varepsilon_3 & 1 \end{bmatrix} = \begin{bmatrix} \varepsilon_1^{-1}\varepsilon_2\varepsilon_3 + \varepsilon_1 & \varepsilon_1^{-1}\varepsilon_2 \\ \varepsilon_1^{-1}\varepsilon_3 & \varepsilon_1^{-1} \end{bmatrix}$$

is an open embedding.

On the input: Data determining G-action on V

Notation

$e_1, \dots, e_n$ a basis in V and

$$\rho_{i,j}\colon G \to k, \quad i,j \in \{1,\dots,n\}$$

the regular functions on G (**matrix coefficients**) such that the action of G on V is given in the basis $e_1, \dots, e_n$ by the matrix representation

$$\rho\colon G \to \mathrm{GL}_n, \quad \rho(g) = \begin{bmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,n}(g) \\ \multicolumn{3}{c}{\dotfill} \\ \rho_{n,1}(g) & \cdots & \rho_{n,n}(g) \end{bmatrix}, \quad g \in G,$$

On the input: Data determining G-action on V

In other words,

$$g \cdot \Big(\sum_{i=1}^{n} \gamma_i e_i\Big) = \sum_{j=1}^{n}\Big(\sum_{i=1}^{n} \rho_{j,i}(g)\gamma_i\Big)e_j \quad \text{for all } g \in G \text{ and } \gamma_i \in k.$$

On the input: Classical example of binary forms

The next example is the main object of the pre-Hilbertian classical invariant theory.

On the input: Classical example of binary forms

The next example is the main object of the pre-Hilbertian classical invariant theory.

Example

$\operatorname{char} k = 0$.

On the input: Classical example of binary forms

The next example is the main object of the pre-Hilbertian classical invariant theory.

Example

$\mathrm{char}\, k = 0$.

$G = \mathrm{SL}_2$.

On the input: Classical example of binary forms

The next example is the main object of the pre-Hilbertian classical invariant theory.

Example

$\mathrm{char}\, k = 0$.
$G = \mathrm{SL}_2$.
$V = \mathscr{B}_h$ the $(h+1)$-dimensional space of binary forms of degree h in variables z_1, z_2 over k with the G-action given by

$$g \cdot z_1 = \alpha z_1 + \gamma z_2, \quad g \cdot z_2 = \beta z_1 + \delta z_2 \qquad \text{for } g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in G.$$

On the input: Classical example of binary forms

The next example is the main object of the pre-Hilbertian classical invariant theory.

Example

$\operatorname{char} k = 0$.
$G = \mathrm{SL}_2$.
$V = \mathscr{B}_h$ the $(h+1)$-dimensional space of binary forms of degree h in variables z_1, z_2 over k with the G-action given by

$$g \cdot z_1 = \alpha z_1 + \gamma z_2, \;\; g \cdot z_2 = \beta z_1 + \delta z_2 \qquad \text{for } g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in G.$$

As a basis $e_1, \ldots, e_n$ (with $n = h+1$) in $\mathscr{B}_h$ take

$$e_i = z_1^{h+1-i} z_2^{i-1}.$$

Example (continued)

Then

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \cdot e_j = (\alpha z_1 + \gamma z_2)^{h+1-j}(\beta z_1 + \delta z_2)^{j-1}$$

$$= \sum_{i=1}^{h+1} \rho_{i,j}(g) z_1^{h+1-i} z_2^{i-1}$$

Example (continued)

Then

$$\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\cdot e_j = (\alpha z_1+\gamma z_2)^{h+1-j}(\beta z_1+\delta z_2)^{j-1}$$
$$=\sum_{i=1}^{h+1}\rho_{i,j}(g)z_1^{h+1-i}z_2^{i-1}$$

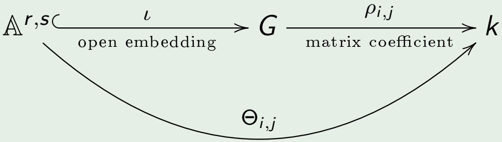
For instance, if $h=2$, then

$$\rho_{2,3}\colon \mathrm{SL}_2\to k,\quad \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\mapsto 2\beta\delta,$$
$$\rho_{2,2}\colon \mathrm{SL}_2\to k,\quad \begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\mapsto \alpha\delta+\gamma\beta$$

The first component of the input

The first component of the input is the following system of n^2 regular functions on $\mathbb{A}^{r,s}$:

$$\Theta_{i,j} := \rho_{i,j} \circ \iota \in k[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}, x_{r+1}, \ldots, x_{r+s}]$$



The first component of the input: Classical example of binary forms

Example

The first component of the input: Classical example of binary forms

Example

char $k = 0$, $G = \mathrm{SL}_2$,
ι, $V = \mathscr{B}_h$, $e_i = z_1^{h+1-i} z_2^{i-1}$ as in the above examples.

The first component of the input: Classical example of binary forms

Example

char $k = 0$, $G = \mathrm{SL}_2$,
ι, $V = \mathscr{B}_h$, $e_i = z_1^{h+1-i} z_2^{i-1}$ as in the above examples.

Then $\Theta_{i,j}$ is the coefficient of $z_1^{h+1-i} z_2^{i-1}$ in the decomposition of

$$\big((x_1 + x_1^{-1}x_2x_3)z_1 + (x_1^{-1}x_3)z_2\big)^{h+1-j}\big((x_1^{-1}x_2)z_1 + (x_1^{-1})z_2\big)^{j-1}$$

as a linear combination of monomials in z_1, z_2 with the coefficients in $k[x_1, x_1^{-1}, x_2, x_3]$.

The first component of the input: Classical example of binary forms

Example (continued)

For instance, if $h = 2$, then

The first component of the input: Classical example of binary forms

Example (continued)

For instance, if $h = 2$, then

$$\Theta_{2,2}\colon \mathbb{A}^{1,2} \to k,$$
$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mapsto \alpha\delta + \gamma\beta\Big|_{\substack{\alpha=\varepsilon^{-1}\varepsilon_2\varepsilon_3+\varepsilon_1,\\ \beta=\varepsilon_1^{-1}\varepsilon_2,\\ \gamma=\varepsilon_1^{-1}\varepsilon_3,\\ \delta=\varepsilon_1^{-1}}} = 1 + 2\varepsilon_1^{-2}\varepsilon_2\varepsilon_3.$$

The first component of the input: Classical example of binary forms

Example (continued)

For instance, if $h = 2$, then

$$\begin{aligned}&\Theta_{2,2}\colon \mathbb{A}^{1,2} \to k,\\ &(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mapsto \alpha\delta + \gamma\beta\big|_{\substack{\alpha=\varepsilon^{-1}\varepsilon_2\varepsilon_3+\varepsilon_1,\\ \beta=\varepsilon_1^{-1}\varepsilon_2,\\ \gamma=\varepsilon_1^{-1}\varepsilon_3,\\ \delta=\varepsilon_1^{-1}}} = 1 + 2\varepsilon_1^{-2}\varepsilon_2\varepsilon_3.\end{aligned}$$

Therefore,

$$\Theta_{2,2} = 1 + 2x_1^{-2}x_2x_3.$$

The second component of the input: $\deg \rho(G)$

The second component of the input: $\deg \rho(G)$

X a locally closed irreducible subset of $\mathbb{A}^m$.

The second component of the input: $\deg \rho(G)$

X a locally closed irreducible subset of $\mathbb{A}^m$.
$\mathscr{L}$ the set of all $(m - \dim X)$-dimensional linear subspaces of $\mathbb{A}^m$.
$\mathscr{L}$ has a structure of irreducible algebraic variety.

The second component of the input: $\deg \rho(G)$

X a locally closed irreducible subset of $\mathbb{A}^m$.
$\mathscr{L}$ the set of all $(m - \dim X)$-dimensional linear subspaces of $\mathbb{A}^m$.
$\mathscr{L}$ has a structure of irreducible algebraic variety.

We use the following classical

Theorem

For every L in a dense open subsubset of $\mathscr{L}$, the intersection $X \cap L$ is a finite set whose cardinality does not depend on L.

The second component of the input: $\deg \rho(G)$

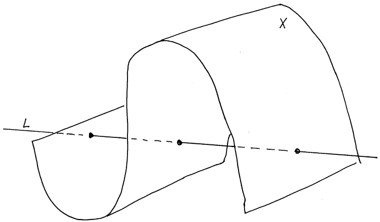
X a locally closed irreducible subset of $\mathbb{A}^m$.
$\mathscr{L}$ the set of all $(m - \dim X)$-dimensional linear subspaces of $\mathbb{A}^m$.
$\mathscr{L}$ has a structure of irreducible algebraic variety.

We use the following classical

Theorem

For every L in a dense open subsubset of $\mathscr{L}$, the intersection $X \cap L$ is a finite set whose cardinality does not depend on L.

This cardinality is called the **degree of** X and denoted by $\mathbf{deg}\, \boldsymbol{X}$.



The second component of the input: $\deg \rho(G)$

As $\rho(G)$ is a locally closed irreducible subset of the n^2-dimensional affine space $\mathrm{Mat}_{n,n}(k)$, the integer $\deg \rho(G)$ is defined.

The second component of the input: $\deg \rho(G)$

As $\rho(G)$ is a locally closed irreducible subset of the n^2-dimensional affine space $\mathrm{Mat}_{n,n}(k)$, the integer $\deg \rho(G)$ is defined.

The integer $\deg \rho(G)$ is the second component of the input of the algorithm.

The second component of the input: $\deg \rho(G)$

As $\rho(G)$ is a locally closed irreducible subset of the n^2-dimensional affine space $\mathrm{Mat}_{n,n}(k)$, the integer $\deg \rho(G)$ is defined.

The integer $\deg \rho(G)$ is the second component of the input of the algorithm.

The reason to consider $\deg \rho(G)$ known is that

if $\mathrm{char}\, k = 0$ and G is **reductive**, then there is **a formula for $\deg \rho(G)$** .

The second component of the input: $\deg \rho(G)$

As $\rho(G)$ is a locally closed irreducible subset of the n^2-dimensional affine space $\mathrm{Mat}_{n,n}(k)$, the integer $\deg \rho(G)$ is defined.

The integer $\deg \rho(G)$ is the second component of the input of the algorithm.

The reason to consider $\deg \rho(G)$ known is that

if $\mathrm{char}\, k = 0$ and G is **reductive**, then there is **a formula for $\deg \rho(G)$** .

In this formula, the following notation is used.

Formula for $\deg \rho(G)$: Notation

Notation

Notation

- T a maximal torus of G.

Formula for $\deg \rho(G)$: Notation

Notation

- T a maximal torus of G.
- $r = \dim T = \mathrm{rk}\, G$.

Formula for $\deg \rho(G)$: Notation

Notation

- T a maximal torus of G.
- $r = \dim T = \mathrm{rk}\, G$.
- $X(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ in additive notation and naturally embedded in $E = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

Formula for $\deg \rho(G)$: Notation

Notation

- T a maximal torus of G.
- $r = \dim T = \mathrm{rk}\, G$.
- $X(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ in additive notation and naturally embedded in $E = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.
- E and $\mathbb{R}^r$ are identified by means of an isomorphism $E \xrightarrow{\simeq} \mathbb{R}^r$ such that $X(T)$ is identified with $\mathbb{Z}^r$.

Formula for $\deg \rho(G)$: Notation

Notation

- T a maximal torus of G.
- $r = \dim T = \mathrm{rk}\, G$.
- $X(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ in additive notation and naturally embedded in $E = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.
- E and $\mathbb{R}^r$ are identified by means of an isomorphism $E \xrightarrow{\simeq} \mathbb{R}^r$ such that $X(T)$ is identified with $\mathbb{Z}^r$.
- $d\nu$ the standard volume form on E.

Formula for $\deg \rho(G)$: Notation

Notation

- T a maximal torus of G.
- $r = \dim T = \mathrm{rk}\, G$.
- $X(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ in additive notation and naturally embedded in $E = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.
- E and $\mathbb{R}^r$ are identified by means of an isomorphism $E \xrightarrow{\simeq} \mathbb{R}^r$ such that $X(T)$ is identified with $\mathbb{Z}^r$.
- $d\nu$ the standard volume form on E.
- $W := N_G(T)/T \subset \mathrm{GL}(E)$ the Weyl group.

Formula for $\deg \rho(G)$: Notation

Notation

- T a maximal torus of G.
- $r = \dim T = \mathrm{rk}\, G$.
- $X(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ in additive notation and naturally embedded in $E = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.
- E and $\mathbb{R}^r$ are identified by means of an isomorphism $E \xrightarrow{\simeq} \mathbb{R}^r$ such that $X(T)$ is identified with $\mathbb{Z}^r$.
- $d\nu$ the standard volume form on E.
- $W := N_G(T)/T \subset \mathrm{GL}(E)$ the Weyl group.
- $\mathcal{P}_V \subset E$ the convex envelope of 0 and all T-weights of V.

Formula for $\deg \rho(G)$: Notation

Notation (continued)

Notation (continued)

- We assume $\dim \mathcal{P}_V = r$. This is attained by replacing G with $G/(\ker \rho)^0$.

Formula for $\deg \rho(G)$: Notation

Notation (continued)

- We assume $\dim \mathcal{P}_V = r$. This is attained by replacing G with $G/(\ker \rho)^0$.
- R_+ a system of positive roots of G with respect to T.

Formula for $\deg \rho(G)$: Notation

Notation (continued)

- We assume $\dim \mathcal{P}_V = r$. This is attained by replacing G with $G/(\ker \rho)^0$.
- R_+ a system of positive roots of G with respect to T.
- $\alpha^\vee$ the coroot corresponding to a root $\alpha \in R_+$, i.e.,

$$\alpha^\vee : E \to \mathbb{R}, \quad v \mapsto 2\langle \alpha \,|\, v \rangle / \langle \alpha \,|\, \alpha \rangle,$$

  where $\langle \ | \ \rangle$ is a W-invariant inner product on E.

Formula for $\deg \rho(G)$: Notation

Notation (continued)

- We assume $\dim \mathcal{P}_V = r$. This is attained by replacing G with $G/(\ker \rho)^0$.
- R_+ a system of positive roots of G with respect to T.
- $\alpha^\vee$ the coroot corresponding to a root $\alpha \in R_+$, i.e.,

  $$\alpha^\vee \colon E \to \mathbb{R}, \quad v \mapsto 2\langle \alpha \,|\, v\rangle / \langle \alpha \,|\, \alpha\rangle,$$

  where $\langle \ | \ \rangle$ is a W-invariant inner product on E.
- $m_1, \ldots, m_r$ the exponents of W (i.e., $m_1 + 1, \ldots, m_r + 1$ are the degrees of homogeneous free generators of $\mathbb{R}[E]^W$).

Notation (continued)

- We assume $\dim \mathcal{P}_V = r$. This is attained by replacing G with $G/(\ker \rho)^0$.
- R_+ a system of positive roots of G with respect to T.
- $\alpha^\vee$ the coroot corresponding to a root $\alpha \in R_+$, i.e.,

$$\alpha^\vee : E \to \mathbb{R}, \quad v \mapsto 2\langle \alpha \,|\, v\rangle / \langle \alpha \,|\, \alpha\rangle,$$

  where $\langle \ | \ \rangle$ is a W-invariant inner product on E.
- $m_1, \ldots, m_r$ the exponents of W (i.e., $m_1 + 1, \ldots, m_r + 1$ are the degrees of homogeneous free generators of $\mathbb{R}[E]^W$).

Particular case: $G = T$. Then

Formula for $\deg \rho(G)$: Notation

Notation (continued)

- We assume $\dim \mathcal{P}_V = r$. This is attained by replacing G with $G/(\ker \rho)^0$.
- R_+ a system of positive roots of G with respect to T.
- $\alpha^\vee$ the coroot corresponding to a root $\alpha \in R_+$, i.e.,

  $$\alpha^\vee : E \to \mathbb{R}, \quad v \mapsto 2\langle \alpha \,|\, v\rangle / \langle \alpha \,|\, \alpha \rangle,$$

  where $\langle \ | \ \rangle$ is a W-invariant inner product on E.
- $m_1, \ldots, m_r$ the exponents of W (i.e., $m_1 + 1, \ldots, m_r + 1$ are the degrees of homogeneous free generators of $\mathbb{R}[E]^W$).

Particular case: $G = T$. Then

$$R_+ = \varnothing, \quad |W| = 1, \qquad m_1 = \cdots = m_r = 0.$$

Formula for $\deg \rho(G)$: Theorem

Theorem (B. Kazarnovskiĭ)

Let $\operatorname{char} k = 0$, *let* G *be reductive, and let* $\ker \rho$ *be finite. Then*

$$\deg \rho(G) := \frac{\dim G\,!}{|W|(m_1! \cdots m_r!)^2 |\ker \rho|} \int_{\mathcal{P}_V} \prod_{\alpha \in R_+} (\alpha^\vee)^2 d\nu.$$

Formula for $\deg \rho(G)$: Classical example of binary forms

Example

$\operatorname{char} k = 0$, $G = \mathrm{SL}_2$, $V = \mathscr{B}_h$ the kG-module of binary forms of degree h in variables z_1, z_2.

Formula for $\deg \rho(G)$: Classical example of binary forms

Example

$\operatorname{char} k = 0$, $G = \mathrm{SL}_2$, $V = \mathscr{B}_h$ the kG-module of binary forms of degree h in variables z_1, z_2.
Then

$$\dim G = 3, \quad r = 1, \quad E = \mathbb{R}, \quad X(T) = \mathbb{Z}, \quad |W| = 2, \quad m_1 = 1,$$
$$R_+ = \{\alpha = 2\},$$
$$\alpha^\vee \text{ is the standard coordinate function } x\colon \mathbb{R} \to \mathbb{R},\ a \mapsto a,$$
$$\ker_V = \begin{cases} \text{trivial} & \text{if } h \text{ is odd,} \\ \text{cyclic group } \big(\mathrm{diag}(-1,-1)\big) \text{ of order } 2 & \text{if } h \text{ is even.} \end{cases}$$

Formula for $\deg \rho(G)$: Classical example of binary forms

Example (continued)

Take

$$T = \{\mathrm{diag}(t, t^{-1}) \mid t \in k \setminus \{0\}\}.$$

Formula for $\deg \rho(G)$: Classical example of binary forms

Example (continued)

Take

$$T = \{\mathrm{diag}(t, t^{-1}) \mid t \in k \setminus \{0\}\}.$$

Then $e_i = z_1^{h+1-i} z_2^{i-1}$ is the T-weight vector of the weight $h - 2i + 2$.

Formula for $\deg \rho(G)$: Classical example of binary forms

Example (continued)

Take

$$T = \{\mathrm{diag}(t, t^{-1}) \mid t \in k \setminus \{0\}\}.$$

Then $e_i = z_1^{h+1-i} z_2^{i-1}$ is the T-weight vector of the weight $h - 2i + 2$. Hence $\{h, h-2, \ldots, -h+2, -h\}$ is the T-weight system of V.

Formula for $\deg \rho(G)$: Classical example of binary forms

Example (continued)

Take

$$T = \{\mathrm{diag}(t, t^{-1}) \mid t \in k \setminus \{0\}\}.$$

Then $e_i = z_1^{h+1-i} z_2^{i-1}$ is the T-weight vector of the weight $h - 2i + 2$. Hence $\{h, h-2, \ldots, -h+2, -h\}$ is the T-weight system of V. Whence

$$\mathcal{P}_V = [-h, h].$$

Formula for $\deg \rho(G)$: Classical example of binary forms

Example (continued)

Take

$$T = \{\operatorname{diag}(t, t^{-1}) \mid t \in k \setminus \{0\}\}.$$

Then $e_i = z_1^{h+1-i} z_2^{i-1}$ is the T-weight vector of the weight $h - 2i + 2$. Hence $\{h, h-2, \ldots, -h+2, -h\}$ is the T-weight system of V. Whence

$$\mathcal{P}_V = [-h, h].$$

This yields

$$\deg \rho(\mathrm{SL}_2) = \frac{3!}{2|\ker \rho|} \int_{-h}^{h} x^2 dx = \begin{cases} 2h^3 & \text{if } h \text{ is odd}, \\ h^3 & \text{if } h \text{ is even}. \end{cases}$$

Reducing Problem $(**)$ to the case of conical orbits

Now we explain that when searching for an algorithmic solution to

Problem $(**)$

Given two points a and $b \in V$, how can one find out whether or not $G \cdot a$ lies in $\overline{G \cdot b}$?

one can assume that the orbits $G \cdot a$ and $G \cdot b$ are **conical**.

Reducing Problem $(**)$ to the case of conical orbits

Definition

A subset C of a vector space L over k is called **conical** if it is stable with respect to scalar multiplication by every nonzero element of k:

$$\lambda C = C \quad \text{for every } \lambda \in k \setminus \{0\}.$$

Reducing Problem $(**)$ to the case of conical orbits: Step 1

The reduction is performed in **two steps**.

Reducing Problem $(**)$ to the case of conical orbits: Step 1

The reduction is performed in **two steps**.

Step 1

Consider the G-action on the projective space $\mathbb{P}^n$ given by

$$g \cdot (\alpha_0 : \alpha_1 : \cdots : \alpha_n) := \left(\alpha_0 : \sum_{i=1}^{n} \rho_{1,i}(g)\alpha_i : \cdots : \sum_{i=1}^{n} \rho_{n,i}(g)\alpha_i\right).$$

Reducing Problem $(**)$ to the case of conical orbits: Step 1

This yields

$$\textbf{the group embedding } \rho(G) \hookrightarrow \mathbf{Aut}\,\mathbb{P}^n$$

Reducing Problem $(**)$ to the case of conical orbits: Step 1

This yields

$$\textbf{the group embedding } \rho(G) \hookrightarrow \mathrm{Aut}\,\mathbb{P}^n$$

and

$$\textbf{the } G\textbf{-equivariant embedding of varieties } V \hookrightarrow \mathbb{P}^n$$

whose image is the standard principal open subset

$$\{(\alpha_0 : \alpha_1 : \cdots : \alpha_n) \mid \alpha_0 \neq 0\}.$$

Reducing Problem $(**)$ to the case of conical orbits: Step 2

Consider the natural surjections

$$\tau\colon \mathrm{GL}_{n+1} \to \mathrm{Aut}\,\mathbb{P}^n \qquad \text{and} \qquad \pi\colon k^{n+1} \setminus \{0\} \to \mathbb{P}^n.$$

Reducing Problem $(**)$ to the case of conical orbits: Step 2

Consider the natural surjections

$$\tau\colon \mathrm{GL}_{n+1} \to \operatorname{Aut}\mathbb{P}^n \qquad \text{and} \qquad \pi\colon k^{n+1}\setminus\{0\} \to \mathbb{P}^n.$$

Step 2 Replace

- G with $\widetilde{G} := \tau^{-1}(\rho(G))$,

Reducing Problem $(**)$ to the case of conical orbits: Step 2

Consider the natural surjections

$$\tau\colon \mathrm{GL}_{n+1} \to \operatorname{Aut}\mathbb{P}^n \qquad \text{and} \qquad \pi\colon k^{n+1}\setminus\{0\} \to \mathbb{P}^n.$$

Step 2 Replace

- G with $\widetilde{G} := \tau^{-1}(\rho(G))$,
- V with $\widetilde{V} := k^{n+1}$,

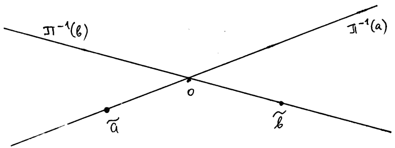
Reducing Problem $(**)$ to the case of conical orbits: Step 2

Consider the natural surjections

$$\tau\colon \mathrm{GL}_{n+1} \to \mathrm{Aut}\,\mathbb{P}^n \qquad \text{and} \qquad \pi\colon k^{n+1} \setminus \{0\} \to \mathbb{P}^n.$$

Step 2 Replace

- G with $\widetilde{G} := \tau^{-1}(\rho(G))$,
- V with $\widetilde{V} := k^{n+1}$,
- a and b with any $\widetilde{a}, \widetilde{b} \in \widetilde{V}$ such that $\pi(\widetilde{a}) = a, \pi(\widetilde{b}) = b$.



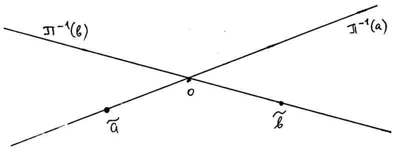
Reducing Problem $(**)$ to the case of conical orbits: Step 2

Consider the natural surjections

$$\tau\colon \mathrm{GL}_{n+1} \to \mathrm{Aut}\,\mathbb{P}^n \qquad \text{and} \qquad \pi\colon k^{n+1}\setminus\{0\} \to \mathbb{P}^n.$$

Step 2 Replace

- G with $\widetilde{G} := \tau^{-1}(\rho(G))$,
- V with $\widetilde{V} := k^{n+1}$,
- a and b with any $\widetilde{a}, \widetilde{b} \in \widetilde{V}$ such that $\pi(\widetilde{a}) = a, \pi(\widetilde{b}) = b$.



$\widetilde{G}$ is connected. $\widetilde{G}$ is reductive if G is.

Reducing Problem $(**)$ to the case of conical orbits

The following lemma shows that performing Steps 1 and 2 reduces Problem $(**)$ to the case of conical orbits.

The following lemma shows that performing Steps 1 and 2 reduces Problem (∗∗) to the case of conical orbits.

Lemma

The following properties are equivalent:

Reducing Problem $(**)$ to the case of conical orbits

The following lemma shows that performing Steps 1 and 2 reduces Problem $(**)$ to the case of conical orbits.

Lemma

The following properties are equivalent:

- $G \cdot a$ *lies in the closure of* $G \cdot b$ *in* $\mathbb{P}^n$;

Reducing Problem $(**)$ to the case of conical orbits

The following lemma shows that performing Steps 1 and 2 reduces Problem $(**)$ to the case of conical orbits.

Lemma

The following properties are equivalent:

- $G \cdot a$ *lies in the closure of* $G \cdot b$ *in* $\mathbb{P}^n$;
- $G \cdot a$ *lies in the closure of* $G \cdot b$ *in* V;

Reducing Problem $(**)$ to the case of conical orbits

The following lemma shows that performing Steps 1 and 2 reduces Problem $(**)$ to the case of conical orbits.

Lemma

The following properties are equivalent:

- $G \cdot a$ *lies in the closure of* $G \cdot b$ *in* $\mathbb{P}^n$;
- $G \cdot a$ *lies in the closure of* $G \cdot b$ *in* V;
- $\widetilde{G} \cdot \widetilde{a}$ *lies in the closure of* $\widetilde{G} \cdot \widetilde{b}$ *in* $\widetilde{V}$.

Reducing Problem $(**)$ to the case of conical orbits

The following lemma shows that performing Steps 1 and 2 reduces Problem $(**)$ to the case of conical orbits.

Lemma

The following properties are equivalent:

- $G \cdot a$ *lies in the closure of* $G \cdot b$ *in* $\mathbb{P}^n$*;*
- $G \cdot a$ *lies in the closure of* $G \cdot b$ *in* V*;*
- $\widetilde{G} \cdot \widetilde{a}$ *lies in the closure of* $\widetilde{G} \cdot \widetilde{b}$ *in* $\widetilde{V}$*.*

The orbits $\widetilde{G} \cdot \widetilde{a}$ *and* $\widetilde{G} \cdot \widetilde{b}$ *are conical.*

Application to a more general setting

Application to a more general setting

As every normal quasiprojective G-variety can be equivariantly embedded in some $\mathbb{P}^m$, it frequently arises the problem analogous to Problem $(**)$ but for a G-action on some **projective space**.

Application to a more general setting

As every normal quasiprojective G-variety can be equivariantly embedded in some $\mathbb{P}^m$, it frequently arises the problem analogous to Problem $(**)$ but for a G-action on some **projective space**.

The above lemma shows that **this problem is reduced to Problem $(**)$ for linear actions on vector spaces.**

Algorithm

<u>Algorithm for solving Problem $(**)$</u>

Algorithm for solving Problem $(**)$

We recall that

Algorithm for solving Problem $(**)$

We recall that

$$n = \dim V, \quad d = \deg \rho(G), \quad r = \operatorname{rank} G, \quad s = \dim G - r,$$

$$\Theta_{i,j} \in k[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}, x_{r+1}, \ldots, x_{r+s}]$$

(the restrictions of matrix coefficients of ρ to the open subset $\iota(\mathbb{A}^{r,s})$ of G)

Algorithm

The cases $n = 1$ and $\overline{G \cdot b} = V$ (i.e., $\dim G \cdot b = n$) are clear.

Algorithm

The cases $n = 1$ and $\overline{G \cdot b} = V$ (i.e., $\dim\, G \cdot b = n$) are clear. Therefore, in what follows we assume

$$n \geqslant 2 \qquad \text{and} \qquad \dim\, G \cdot b < n.$$

Algorithm

Step 1

Following the procedure described above, reduce Problem $(**)$ to the case of **conical orbits** $G \cdot a$ and $G \cdot b$.

Step 2

Find the coordinates of a and b in the basis $e_1, \dots, e_n$:

$$a = \alpha_1 e_1 + \cdots + \alpha_n e_n, \quad b = \beta_1 e_1 + \cdots + \beta_n e_n,$$

Algorithm

Step 2

Find the coordinates of a and b in the basis $e_1, \ldots, e_n$:

$$a = \alpha_1 e_1 + \cdots + \alpha_n e_n, \quad b = \beta_1 e_1 + \cdots + \beta_n e_n,$$

and, replacing $e_1, \ldots, e_n$ by another basis if necessary, **ensure that**

$$\beta_1 \cdots \beta_n \neq 0.$$

Algorithm

Step 3

Consider the "generic" polynomials $F_1, \ldots, F_n$ of degree $2d-2$ in the variables $y_1, \ldots, y_n$,

$$F_s := \sum_{\substack{q_1,\ldots,q_n \in \mathbb{N} \\ q_1+\cdots+q_n \leqslant 2d-2}} c_{s,q_1,\ldots,q_n} y_1^{q_1} \cdots y_n^{q_n}, \quad s = 1, \ldots, n,$$

where the coefficients $c_{s,q_1,\ldots,q_n}$ are **indeterminates** over k,

Algorithm

Step 3

Consider the "generic" polynomials $F_1, \dots, F_n$ of degree $2d-2$ in the variables $y_1, \dots, y_n$,

$$F_s := \sum_{\substack{q_1,\dots,q_n \in \mathbb{N} \\ q_1+\dots+q_n \leqslant 2d-2}} c_{s,q_1,\dots,q_n} y_1^{q_1} \cdots y_n^{q_n}, \quad s = 1, \dots, n,$$

where the coefficients $c_{s,q_1,\dots,q_n}$ are **indeterminates** over k, and then put

$$H := (y_1 - \alpha_1)F_1 + \dots + (y_n - \alpha_n)F_n - 1.$$

Step 4

Replace every y_i in H with

$$\sum_{j=1}^{n} \beta_j \Theta_{i,j}.$$

Algorithm

Step 4

Replace every y_i in H with

$$\sum_{j=1}^{n} \beta_j \Theta_{i,j}.$$

The result of this is a sum

$$\sum_{(i_1,\ldots,i_{r+s})\in M} \ell_{i_1,\ldots,i_{r+s}} x_1^{i_1} \cdots x_{r+s}^{i_{r+s}},$$

where M is a finite subset of $\mathbb{Z}^r \times \mathbb{N}^s$ and **every** $\ell_{i_1,\ldots,i_{r+s}}$ **is a linear combination of** $c_{s,q_1,\ldots,q_n}$**'s with the coefficients in** k.

Algorithm

Step 5

Consider the following **system of linear equations in variables** $c_{s,q_1,\ldots,q_n}$ **with the coefficients in** k:

$$\ell_{i_1,\ldots,i_{r+s}} = 0 \quad \text{where } (i_1,\ldots,i_{r+s}) \text{ runs over } M.$$

Algorithm

Step 5

Consider the following **system of linear equations in variables** $c_{s,q_1,\dots,q_n}$ **with the coefficients in** k:

$$\ell_{i_1,\dots,i_{r+s}} = 0 \quad \text{where } (i_1,\dots,i_{r+s}) \text{ runs over } M.$$

Denote this system by $(\bigstar)$.

Algorithm: The data-driven decision rule

Theorem

Algorithm: The data-driven decision rule

Theorem

The following properties are **equivalent***:*

Algorithm: The data-driven decision rule

Theorem

The following properties are **equivalent***:*

- *the orbit* $G \cdot a$ **lies in the closure** *of the orbit* $\overline{G \cdot b}$;

Algorithm: The data-driven decision rule

Theorem

The following properties are **equivalent***:*

- *the orbit* $G \cdot a$ **lies in the closure** *of the orbit* $\overline{G \cdot b}$;
- *the system* $(\bigstar)$ *of linear equations is* **inconsistent**.

Remark

Remark

By the Kronecker–Capelli theorem, this yields the following **numerical criterion**:

Remark

By the Kronecker–Capelli theorem, this yields the following **numerical criterion**:

Let A be the coefficient matrix of ($\bigstar$) and let $\widetilde{A}$ be the augmented matrix obtained from A by adding the column of free terms.

Algorithm: The data-driven decision rule

Remark

By the Kronecker–Capelli theorem, this yields the following **numerical criterion**:

Let A be the coefficient matrix of ($\bigstar$) and let $\widetilde{A}$ be the augmented matrix obtained from A by adding the column of free terms. Then

$$G \cdot a \subseteq \overline{G \cdot b} \iff \mathrm{rank}\, A \neq \mathrm{rank}\, \widetilde{A}.$$

Example: Binary forms

Example

Example: Binary forms

Example

$G = \mathrm{SL}_2$, $V = \mathscr{B}_h$.

Example: Binary forms

Example

$G = \mathrm{SL}_2$, $V = \mathscr{B}_h$.

The number of variables $c_{s,q_1,\ldots,q_n}$ in system $(\bigstar)$ is

$$
\begin{aligned}
&(h+1)\binom{2h^3+h-1}{h+1} && \text{if } h \text{ is even,}\\
&(h+1)\binom{4h^3+h-1}{h+1} && \text{if } h \text{ is odd.}
\end{aligned}
$$

Algorithm: Replacing d by a smaller integer

Remark

Algorithm: Replacing d by a smaller integer

Remark

It can be shown that

$$d = \deg\ \rho(G) \geqslant \deg\ G \cdot v \text{ for every } v \in V$$

Algorithm: Replacing d by a smaller integer

Remark

It can be shown that

$$d = \deg\ \rho(G) \geqslant \deg\ G \cdot v \text{ for every } v \in V$$

and that the algorithm and Theorem remain valid if d is replaced with $\deg G \cdot b$.

Algorithm: Replacing d by a smaller integer

Remark

It can be shown that

$$d = \text{deg } \rho(G) \geqslant \text{deg } G \cdot v \text{ for every } v \in V$$

and that the algorithm and Theorem remain valid if d is replaced with $\deg G \cdot b$.
Therefore, if $\deg G \cdot b$ is known, this leads to decreasing the number of variables and equations in system ($\bigstar$).

Algorithm: Replacing d by a smaller integer

Remark

It can be shown that

$$d = \deg\,\rho(G) \geqslant \deg\,G \cdot v \text{ for every } v \in V$$

and that the algorithm and Theorem remain valid if d is replaced with $\deg G \cdot b$.
Therefore, if $\deg G \cdot b$ is known, this leads to decreasing the number of variables and equations in system ($\bigstar$).

In some cases the degrees of orbits indeed have been computed.

Example: Degrees of orbits of binary forms

Example

Example: Degrees of orbits of binary forms

Example

$G = \mathrm{SL}_2,\ V = \mathscr{B}_h.$

Example: Degrees of orbits of binary forms

Example

$G = \mathrm{SL}_2$, $V = \mathscr{B}_h$.
Every nonzero $v \in \mathscr{B}_h$ decomposes as

$$v = \ell_1^{s_1} \cdots \ell_m^{s_m},$$

with pairwise nonproportional linear forms $\ell_1, \ldots, \ell_m \in \mathscr{B}_1$.

Example

$G = \mathrm{SL}_2$, $V = \mathscr{B}_h$.
Every nonzero $v \in \mathscr{B}_h$ decomposes as

$$v = \ell_1^{s_1} \cdots \ell_m^{s_m},$$

with pairwise nonproportional linear forms $\ell_1, \ldots, \ell_m \in \mathscr{B}_1$.
If $m \geqslant 3$ and $h/s_i \geqslant 2$ for every i, then the G-stabilizer G_v of v is finite.

Example: Degrees of orbits of binary forms

Example

$G = \mathrm{SL}_2$, $V = \mathscr{B}_h$.

Every nonzero $v \in \mathscr{B}_h$ decomposes as

$$v = \ell_1^{s_1} \cdots \ell_m^{s_m},$$

with pairwize nonproportional linear forms $\ell_1, \ldots, \ell_m \in \mathscr{B}_1$.

If $m \geqslant 3$ and $h/s_i \geqslant 2$ for every i, then the G-stabilizer G_v of v is finite. Then, according to Moser-Jauslin (1992),

$$|G_v| \deg G \cdot v = -2(m-1)h^3 - 4\sum_{i=1}^{m}(h-s_i)^3 + 3h^2\sum_{i=1}^{m}(h-s_i) + 3h\sum_{i=1}^{m}(h-s_i)(h-2s_i)$$

Example (continued)

Example: Degrees of orbits of binary forms

Example (continued)

In particular, if all roots of v are simple, i.e., $m = h$ and $s_1 = \cdots = s_h = 1$,

Example: Degrees of orbits of binary forms

Example (continued)

In particular, if all roots of v are simple, i.e., $m = h$ and $s_1 = \cdots = s_h = 1$,
then

$$\deg G \cdot v = \frac{2h(h-1)(h-2)}{|G_v|}.$$

Example (continued)

In particular, if all roots of v are simple, i.e., $m = h$ and $s_1 = \cdots = s_h = 1$, then

$$\deg G \cdot v = \frac{2h(h-1)(h-2)}{|G_v|}.$$

This formula can also be deduced from a calculation made by Enriques and Fano in 1897. This has been done in 1983 by Mukai and Umemura (with a gap fixed in 1992 by Moser-Jauslin).