Differential Operators with Singular Coefficients

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Let (a,b) be a finite or infinite interval on $\mathbb{R}$. Consider

$$l(y) := \sum_{k=0}^{n} (-1)^{n-k}(p_k(x)y^{(n-k)})^{(n-k)} +$$
$$+i\sum_{k=0}^{n-1}(-1)^{n-k-1}\left[(q_k(x)y^{(n-k-1)})^{(n-k)} + (q_k(x)y^{(n-k)})^{(n-k-1)}\right]. \tag{1}$$

Here we consider even order expressions, odd order ones require a separate consideration.

How to understand this expression? Smooth coefficients, no problems. Classical results give a chance to define this operator, provided that the conditions

$$(p_0)^{-1},\ p_k,\ q_{k-1} \in L^1_{loc}(a,b), \quad k = 1,\dots,n;\ q_0 \in \mathrm{AC}_{loc}(a,b), \tag{2}$$

hold.

The first our objective is to weaken these conditions, we let the coefficients to be distributions of finite order singularity (depending on the indices k).

The second objective is to find how to work with arbitrary non-self-adjoint expressions, not only with formally self-adjoint ones.

Namely, we will work with differential expressions of the general form

$$\tau(y) = \sum_{r,s=0}^{n} \left(r_{ks} y^{(n-k)} \right)^{(n-s)}, \tag{3}$$

where r_{ks} are complex valued functions.

Results:

Conditions (2) can be replaced by

$$\frac{1}{\sqrt{|p_0|}}, \quad \frac{P_k}{\sqrt{|p_0|}}, \quad \frac{Q_{k-1}}{\sqrt{|p_0|}} \in L^2_{loc}(a,b), \quad k = 1, 2, \ldots, n. \tag{4}$$

Here P_k and Q_k are $k-$th antiderivatives of p_k and q_k in the sense of distributions, i.e., $P_k^{(k)} = p_k(x)$ и $Q_k^{(k)} = q_k(x)$, where the equalities are understood in $\mathcal{D}'(a,b)$.

In the case of non-self-adjoint expression $\tau(y)$ we assume that

$$\frac{1}{\sqrt{|r_0|}}, \quad \frac{1}{\sqrt{|r_0|}} R_{ks} \in L^2_{loc}(a,b) \qquad k,s = 0,1,\dots,n, \tag{5}$$

where R_{ks} are functions defined by

$$R_{ks} = r_{ks}^{(l)}, \quad l = \min(k,s),$$

For giving a correct definition of differential operators associated with the above expressions we will apply the regularization procedure. The main idea is to rewrite these expressions in a nicer form. To do this we have to associate with these expressions the so-called Shin-Zettl matrices.

REGULARIZATION PROCEDURE. Let us consider

$$\mathcal{F}y = -y'' + u'(x)y, \quad u(x) = \int q(x)\,dx.$$

Here the antiderivative is understood in the sense of distributions.

$$u(x) \in L_{2,loc}(a,b).$$

Denote

$$y^{[1]} = y'(x) - u(x)y.$$

Then

$$\mathcal{F}y = -y'' + u'(x)y = -(y^{[1]})' - u(x)y^{[1]} - u^2(x)y.$$

Denote $y_1 = y, \quad y_2 = y^{[1]}$. Then

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} u & 1 \\ -\lambda - u^2 & -u \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

Here the first row is the definition of the quasi-derivative and the second row is the resolvent equation $Ay - \lambda y = f$. The matrix

$$\mathbf{F} = \begin{pmatrix} u & 1 \\ -u^2 & -u \end{pmatrix}$$

is said to compatible with the differential expression $\mathcal{F}y$.

Theorem. *Let* $\mathbf{F}(x) - m \times m$ *be a matrix function with entries* $f_{ij}(x) \in L_{1,loc}(a,b)$. *Then* $\forall f \in L_{1,loc}(a,b)$ *the equation*

$$\mathbf{y}' = \mathbf{F}(x)\mathbf{y} + \mathbf{f}, \quad \mathbf{y}(c) = \xi \in \mathbb{C}^m, \quad c \in (a,b),$$

has a unique solution which is absolutely continuous on any subinterval in (a,b). *If*

$$\|\mathbf{F}_1(x) - \mathbf{F}(x)\|_{L_1} \leqslant \varepsilon,$$

then

$$\|\mathbf{y}_1 - \mathbf{y}\|_{W^{1,1}} \leqslant C\varepsilon(\|\mathbf{f}\| + \|\xi\|).$$

Now we can define the operators

$$\mathcal{D}(F_{max}) = \{y \in \mathcal{D}(\mathcal{F}) |\ y,\, \mathcal{F}y \in L_2(a,b)\},$$
$$\mathcal{D}(F^0_{min}) = \{y \in \mathcal{D}(\mathcal{F}_{max}) |\ \operatorname{supp} y \subset K_y \subset (a,b)\},$$

The operator F_{min} which is the closure of F^0_{min} is called minimal operator, and F_{max} is the maximal one.

Denote by F^+_{max} and F^+_{min} maximal and minimal operators which correspond to the differential expression with conjugate potential $\bar{q}$.

Theorem. *$\mathcal{D}(F_{min})$ is dense in $L_2(a,b)$. The following relations hold $F^*_{max} = F^+_{min}, F^*_{min} = F_{max}$. All these operators are Fredholm with the indices $\leqslant 2$. If the problem is regular then any correct restriction of then operator F_{max} is given by two boundary conditions*

$$U_j(y) = \alpha_j y^{[1]}(a) + \beta_j y^{[1]}(b) + \gamma_j y(a) + \delta_j y(b) = 0, \qquad j = 1,2.$$

Let us remind the concepts of quasi-derivatives. We say that the matrix function $\mathbf{F} = \{f_{kj}(x)\}_{k,j=1}^{m}$ belongs to the class $\mathcal{S}_m(a,b)$ (the class of Shin-Zettl matrices) if all its entries $f_{k,j} \in L^1_{loc}(a,b)$ and

$$f_{k,k+1}(x) \neq 0 \text{ a.e. on}(a,b), \quad f_{k,j}(x) \equiv 0 \text{ if } j > k+1, \; k = 1,\dots,m.$$

Let $\mathbf{F} = \{f_{kj}(x)\}_{k,j=1}^{m} \in \mathcal{S}_m(a,b)$. Let us define the quasi-derivatives

$$y^{[0]} := y, \quad y^{[k]} = \left(f_{k,k+1}(x)\right)^{-1}\left[\left(y^{[k-1]}\right)' - \sum_{j=1}^{k} f_{k,j}(x) y^{[j-1]}\right], \quad k = 1,\dots,m-1.$$

Consider the operator

$$\mathcal{F}y = \left(y^{[m-1]}\right)' - \sum_{j=1}^{m} f_{m,j}(x) y^{[j-1]}$$

on the domain

$$\mathcal{D}(\mathcal{F}) = \{y| \;\; y^{[k]} \in AC_{loc}(a,b), \; k = 0,1,\dots,m-1\}.$$

This operator is correctly defined in the space $L^1_{loc}(a,b)$. Consider' the restrictions F_{max} and F^0_{min} of this operator on the space $L_2(a,b)$. We take the following domains for these operators

$$\mathcal{D}(F_{max}) = \{y \in \mathcal{D}(\mathcal{F}) |\ \ y,\, \mathcal{F}y \in L_2(a,b)\},$$
$$\mathcal{D}(F^0_{min}) = \{y \in \mathcal{D}(\mathcal{F}_{max}) |\ \ \operatorname{supp} y \subset K_y \subset (a,b)\},$$

The operator F_{min} which is the closure of F^0_{min} is called minimal operator, and F_{max} is the maximal one.

Now, how does the classical theory treat the differential expressions? Let us correspond to dif. expression (1) the matrix function

$$
\mathbf{F} = \begin{pmatrix}
0 & 1 & 0 & 0 & . & . & 0 & 0 & 0 & . & . & . & 0 \\
0 & 0 & 1 & 0 & . & . & 0 & 0 & 0 & . & . & . & 0 \\
0 & 0 & 0 & 1 & . & . & 0 & 0 & 0 & . & . & . & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & . & . & 0 & 1 & 0 & 0 & . & . & . & 0 \\
0 & 0 & 0 & . & . & 0 & \beta & -p_0^{-1} & 0 & . & . & . & 0 \\
. & . & . & . & . & iq_1 & \alpha & \beta & 1 & . & . & . & 0 \\
. & . & . & . & . & p_2 & iq_1 & 0 & 0 & . & . & . & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & 0 \\
0 & iq_{m-2} & p_{m-2} & . & . & . & 0 & . & . & . & . & 1 & 0 \\
iq_{m-1} & p_{m-1} & iq_{m-2} & . & . & . & . & . & . & . & . & 0 & 1 \\
p_m & iq_{m-1} & 0 & . & . & . & . & . & . & . & . & 0 & 0
\end{pmatrix}
$$

$$
\alpha = p_1 + q_0^2 p_0^{-1}, \quad \beta = -q_0 p_0^{-1}.
$$

This is the Shin-Zettl type matrix, provided that the classical conditions hold.

The crucial moment is that one can check $\mathcal{F}y = l(y)$, provided that the coefficients p_k and q_k are smooth enough. But the left hand side is defined for non-smooth coefficients.

This follows from the equation

$$-\begin{pmatrix} y \\ y^{[1]} \\ \cdot \\ \cdot \\ y^{[2n-1]} \end{pmatrix}' + \mathbf{F}\begin{pmatrix} y \\ y^{[1]} \\ \cdot \\ \cdot \\ y^{[2n-1]} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ f \end{pmatrix}, f \in L_2(a,b).$$

Therefore this equation is equivalent to $\mathcal{F}y = l(y) = f$.

Deffinition 1. *The Shin-Zettl matrix* $\mathbf{F}$ *is said to be associated with differential expression (1) (or (3)) if*

$$l(y) = \mathcal{F}y \quad \text{for all } y \in C_0^\infty(a,b)$$

and all infinitely smooth functions p_k *and* q_k*. In the case of the existence of the associated matrix* $\mathbf{F}$ *we define* $l(y) = \mathcal{F}y$*.*

Definition 2. *The matrix* $\mathbf{F} \in \mathcal{S}_{2n}(a,b)$ *is said to be compatible with differential expression (3), if for all* $y \in \mathcal{D}(F_{max})$ *the functions* $r_{ks}(x)y^{(n-k)}(x)$ *in the parenthesis of the expression (3) belong to the space of distributions* $\mathcal{D}'(a,b)$ *and the equality*

$$\tau(y) = \mathcal{F}y \quad \forall y \in \mathcal{D}(\mathcal{F}), \tag{6}$$

holds in the sense of distributions.

It turns out that these definitions are equivalent.

Let us formulate the main results.

Differential expression (1) *with complex valued coefficients* p_k, q_k *obeying the conditions* (13) *we call normal. It is formally symmetric iff all these coefficients are real.*

Theorem1. *Any differential expression of the form* (3) *with complex valued coefficients* r_{ks} *subject to conditions* (5) *can be reduced to a normal form* (1) *subject to conditions* (13).

The proof follows from

$$\left(ry^{(k)}\right)^{(s)} = \left(ry^{(k-1)}\right)^{(s+1)} - \left(r'y^{(k-1)}\right)^{(s)}, \quad \text{если } k > s,$$

$$\left(ry^{(k)}\right)^{(s)} = \left(r'y^{(k)}\right)^{(s-1)} + \left(ry^{(k+1)}\right)^{(s-1)}, \quad \text{если } k < s.$$

Theorem 2. *Let the conditions* (5) *hold. Then there exists a compatible matrix function with differential expression* (3).

For normal differential expression (1) the compatible matrix function $\mathbf{F}$ *can be presented explicitly. Namely, let us denote*

$$\varphi_k := P_k + iQ_{k-1}, \quad \widetilde{\varphi}_k := P_k - iQ_{k-1}, \quad k = 1, 2, \ldots, n.$$

Let us define the matrices $\mathbf{F}^1$ *and* $\mathbf{F}^2$ *as follows. All the entries of* $\mathbf{F}^1$ *which are staying in the first* $n-1$ *rows and in the last* $n-1$ *columns we define to be zero, but the entries* $f^1_{k,k+1}$*, which we put to be equal 1 for all* $k = 1, \ldots, n-1, n+1, \ldots, 2n$*. All the other entries* $\mathbf{F}_1$ *define as follows:*

$$f^1_{nj} := (-1)^{n-j}\varphi_{n+1-j}p_0^{-1}, \;\; j = 1, 2, \ldots, n; \;\; f^1_{n,n+1} := p_0^{-1};$$

$$f^1_{kj} := (-1)^{n+1-j}\varphi_{n+1-j}\widetilde{\varphi}_{k-n}p_0^{-1}; \;\; f^1_{k,n+1} := -\widetilde{\varphi}_{k-n}p_0^{-1}, \\ k = n+1, \ldots, 2n, \;\; j = 1, 2, \ldots, n.$$

All the entries of the matrix $\mathbf{F}^2$ *in the first* n *rows and the last* n *columns we define to be equal 0 as well as the entries* $f^2_{n+k,s}$ *for* $k = 1, \ldots, n, s \leqslant k$*. All the other entries of the matrix* $\mathbf{F}^2$

define as follows:

$$f^2_{n+k,n+k-j} := (-1)^{j+k} C^k_{j+1} \left[P_{j+1} + i \left(1 - \frac{2k}{j+1} \right) Q_j \right], \qquad k = 1, \dots, j, \;\; j = 1, \dots, n-1,$$

where C^k_{j+1} *are the binomial coefficients. Then the compatible with differential expression* (1) *matrix* $\mathbf{F}$ $(=: \mathbf{F}_{2n})$ *is the sum* $\mathbf{F}^1 + \mathbf{F}^2$.

It is easy to show that $\mathbf{F}_{2n}$ is the Shin-Zettl matrix, provided that the conditions (13) hold. To check the compatibility is much more difficult task.

In the cases $n = 1, 2, 3$ the compatible matrices $\mathbf{F}_2, \mathbf{F}_4, \mathbf{F}_6$ have the representation

$$F_2 = \begin{pmatrix} \dfrac{\varphi_1}{p_0} & \dfrac{1}{p_0} \\ -\dfrac{\varphi_1 \widetilde{\varphi}_1}{p_0} & -\dfrac{\widetilde{\varphi}_1}{p_0} \end{pmatrix},$$

$$F_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\varphi_2}{p_0} & \frac{\varphi_1}{p_0} & \frac{1}{p_0} & 0 \\ \frac{\varphi_2\widetilde{\varphi}_1}{p_0} & 2P_2 - \frac{\varphi_1\widetilde{\varphi}_1}{p_0} & -\frac{\widetilde{\varphi}_1}{p_0} & 1 \\ \frac{\varphi_2\widetilde{\varphi}_2}{p_0} & -\frac{\varphi_1\widetilde{\varphi}_2}{p_0} & -\frac{\widetilde{\varphi}_2}{p_0} & 0 \end{pmatrix},$$

$$
F_6 =
$$

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\frac{\varphi_3}{p_0} & -\frac{\varphi_2}{p_0} & \frac{\varphi_1}{p_0} & \frac{1}{p_0} & 0 & 0 \\
-\frac{\varphi_3 \widetilde{\varphi}_1}{p_0} & -3P_3 - iO_2 + \frac{\varphi_2 \widetilde{\varphi}_1}{p_0} & 2P_2 - \frac{\varphi_1 \widetilde{\varphi}_1}{p_0} & -\frac{\widetilde{\varphi}_1}{p_0} & 1 & 0 \\
-\frac{\varphi_3 \widetilde{\varphi}_2}{p_0} & \frac{\varphi_2 \widetilde{\varphi}_2}{p_0} & 3P_3 - iQ_2 - \frac{\varphi_1 \widetilde{\varphi}_2}{p_0} & -\frac{\widetilde{\varphi}_2}{p_0} & 0 & 1 \\
\frac{\varphi_3 \widetilde{\varphi}_3}{p_0} & \frac{\varphi_2 \widetilde{\varphi}_3}{p_0} & -\frac{\varphi_1 \widetilde{\varphi}_3}{p_0} & -\frac{\widetilde{\varphi}_3}{p_0} & 0 & 0
\end{pmatrix}.
$$

При нечетных $n = 2m+1$ имеем (по-прежнему $\varphi_k := p_k + iq_k$).

1. При $n=3$ $(m=1)$ матрица F имеет вид

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2q_0}} & 0 \\ -\frac{i\overline{\varphi_1}}{\sqrt{2q_0}} & \frac{ip_0}{2q_0} & \frac{1}{\sqrt{2q_0}} \\ 0 & -\frac{i\varphi_1}{\sqrt{2q_0}} & 0 \end{pmatrix},$$

а квазипроизводные $y^{[k]}$ $(k=0,1,2)$ и квазидифференциальное выражение $\tau[y]$ выглядят так:

$$y^{[0]} = y,\ y^{[1]} = \sqrt{2q_0}y',\ y^{[2]} = (q_0y')' + q_0y'' - ip_0y' + i\overline{\varphi_1}y.$$

$$\tau[y] = -i((p_0y')' - p_1'y + i(q_0y')'' + (q_0y'' + q_1y)' + q_1y')).$$

2. При $n = 5$ $(m = 2)$ матрица F имеет вид

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2q_0}} & 0 & 0 \\ \frac{i\overline{\varphi_2}}{\sqrt{2q_0}} & -\frac{i\overline{\varphi_1}}{\sqrt{2q_0}} & \frac{ip_0}{2q_0} & \frac{1}{\sqrt{2q_0}} & 0 \\ 0 & -2ip_2 & \frac{i\varphi_1}{\sqrt{2q_0}} & 0 & 1 \\ 0 & 0 & \frac{i\varphi_2}{\sqrt{2q_0}} & 0 & 0 \end{pmatrix}.$$

3. Пусть $n = 7$ $(m = 3)$ матрица F имеет вид

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2q_0}} & 0 & 0 & 0 \\ -\frac{i\overline{\varphi_3}}{\sqrt{2q_0}} & \frac{\overline{\varphi_2}}{\sqrt{2q_0}} & -\frac{\overline{\varphi_1}}{\sqrt{2q_0}} & \frac{ip_0}{2q_0} & \frac{1}{\sqrt{2q_0}} & 0 & 0 \\ 0 & q_3 + 3ip_3 & -2ip_2 & \frac{i\varphi_1}{\sqrt{2q_0}} & 0 & 1 & 0 \\ 0 & 0 & q_3 - 3ip_3 & \frac{i\varphi_2}{\sqrt{2q_0}} & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{i\varphi_3}{\sqrt{2q_0}} & 0 & 0 & 0 \end{pmatrix}.$$

4. матрица F_{2m+1} имеет вид

$$
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & f_{n,n+1} & 0 & 0 & \cdots & 0 \\
f_{n+1,1} & f_{n+1,2} & f_{n+1,3} & \cdots & f_{n+1,n} & f_{n+1,n+1} & f_{n+1,n+2} & 0 & \cdots & 0 \\
0 & f_{n+2,2} & f_{n+2,3} & \cdots & f_{n+2,n} & f_{n+2,n+1} & 0 & 1 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 0 & \cdots & f_{2n,n} & f_{2n,n+1} & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 & f_{2n+1,n+1} & 0 & 0 & \cdots & 0
\end{pmatrix}
$$

$$f_{n,n+1} = f_{n+1,n+2} = \frac{1}{\sqrt{2q_0}}$$

$$f_{n+1,n+1-j} = (-1)^j \frac{i\overline{\varphi_j}}{\sqrt{2q_0}},\ j = 1, 2, \ldots, n$$

$$f_{n+1+j,n+1} = \frac{i\varphi_j}{\sqrt{2q_0}},\ j = 1, 2, \ldots, n$$

$$f_{n+1+k,n+k-j} = (-1)^{j+k+1} C_{j+1}^{k} \left[ip_{j+1} + \left(1 - \frac{2k}{j+1}\right) q_{j+1}\right]$$

Denote by $\tau^{+}(y)$ the differential expression, which is formally adjoint to (3) (the functions r_{ks} are replaced by $(-1)^{k+s}\overline{r}_{sk}$). By virtue of Theorem 2 there is a compatible matrix with the expression $\tau^+(y)$. Denote by F^+_{max} and F^+_{min} the maximal and minimal operators generated by the matrix $\mathbf{F}^+$.

Theorem 3. *Let the conditions* (5) *are valid and the matrices* $\mathbf{F}$ *and* $\mathbf{F}^+$ *are compatible with differential expressions* (3) *and formally adjoint* $\tau^+(y)$, *respectively. Let* F_{max}, F^+_{max} *and* F_{min}, F^+_{min} *be maximal and minimal operators generated by the matrices* $\mathbf{F}$ *and* $\mathbf{F}^+$, *respectively. Then all 4 operators have dense domains of definition in the space* $L_2(a,b)$ *and are Fredholm operators. The image codimensions of the maximal operators and the kernel dimensions of the minimal ones are equal to zero. The defects of the minimal operators and the kernel dimensions of the maximal operators do not exceed* $2n$. *The following equalities hold*

$$F^+_{max} = F^*_{min}, \quad F^+_{min} = F^*_{max},$$

The differential expression (3) is said to be *regular* if the interval (a, b) is finite and all the functions in conditions (5) belong to the space $L^2(a, b)$ (instead $L^2_{loc}(a, b)$). Further let us deal with regular expressions. Consider the operators F_U, which are the extensions of the minimal operator and the restrictions of the maximal one, i.e. $F_{min} \subset F_U \subset F_{max}$.

Theorem 4. *Let the expression (3) be regular. Then any operator F_U subject the condition $F_{min} \subset F_U \subset F_{max}$, has the domain*

$$\mathcal{D}(F_U) = \{y \in \mathcal{D}(F_{max})| \;\; U_j(y) = 0, j = 1, \ldots s\},$$

where $U_j(y)$ are linear forms (boundary conditions) having the representation

$$U_j(y) = \sum_{k=1}^{2n-1} \left(\alpha_{jk} y^{[k-1]}(a) + \beta_{jk} y^{[k-1]}(b)\right) = 0, \quad j = 1, \ldots, s. \tag{7}$$

The operator F_U is Fredholm and its index $\geqslant 2n - s$. In particular, if the index equals to zero, then the domain id defined by $2n$ boundary conditions of the above form. In this case the spectrum of the operator F_U is discrete iff its resolvent set is not empty.

Theorem 5. *Let the expression* (1) *be regular and its coefficients p_k и q_k are real. Then the operator F_U which is defined in Theorem 4 is self-adjoint iff the boundary conditions* (7) *are linearly independent, $s = 2n$, and the coefficients in* (7) *satisfy the relations*

$$\sum_{k=1}^{2n} (-1)^k \alpha_{l,k} \bar{\alpha}_{j,2n+1-k} = \sum_{k=1}^{2n} (-1)^k \beta_{l,k} \bar{\beta}_{j,2n+1-k}, \quad l, j = 1, 2, \ldots, 2n. \tag{8}$$

For any such conditions the spectrum of F_U is discrete.

Theorem 6. *Let the operator F_U is generated by a regular differential expression* (3) *and boundary conditions of the form*

(8). *Assume* $c_1 \leqslant r_{00}(x) \leqslant c_2$ *where* c_1, c_2 *are constants. Assume that a family* r^{ε}_{ks} *of infinitely smooth functions and their* l*-antiderivatives* ($l = \min(k,s)$) *are such that*

$$\|R^{\varepsilon}_{ks} - R_{ks}\|_{L_2} \to 0 \quad \textit{as } \varepsilon \to 0, \quad k, s = 0, 1, \ldots, n.$$

Let the quasi-derivatives $y^{[k]}_{\varepsilon}$ *be defined as before but the functions* R_{ks} *are replaced by* R^{ε}_{ks}. *Then the family of the corresponding operators* F^{ε}_U *is norm resolvent convergent to the operator* $\mathcal{F}_U$. *This means that for all* $\mu \in \rho(F_U)$ *holds*

$$\|(F^{\varepsilon}_U - \mu)^{-1} - (F_U - \mu)^{-1}\| \to 0 \qquad \textit{as } \varepsilon \to 0.$$

The results can be generalized for dealing with spectral problems of the form

$$l(y) = w(x)\lambda y$$

or for operators generated by differential expressions $\widetilde{l}(y) = [w(x)]^{-1} l(y)$, where w is a positive function and $w \in L^1_{loc}(a,b)$. Certainly, in this case one should work in the weighted space $L^2((a,b);w)$ instead of $L^2(a,b)$.

The fundamental role in the theory of ordinary differential operators plays Birkhoff theorem on asymptotic solutions for differential equations with spectral parameter. To move further one should have an analogue of this theorem for differential equations with singular coefficients.

Theorem (Savchuk-Shkalikov, 2003). *Assume* $q \in W_2^{-1}[a, b]$. *Then the equation* $-y'' + q(x)y = \lambda^2 y$ *has a pair of solutions of the form*

$$y_s(x, \lambda) = \sin \lambda x + \varphi(x, \lambda), \quad y_c(x, \lambda) = \cos \lambda x + \psi(x, \lambda),$$

where $|\varphi(x, \lambda)| + |\psi(x, \lambda)| = o(1)$ *provided that* $|\lambda| \to \infty$ *in a strip* $|Im\lambda| \leqslant r$ *uniformly with respect to* $x \in [a, b]$. *The number* r *can be taken arbitrary large. These asymptotic representations generally do not admit the differentiation. However, if* $y^{[1]} = y' - u(x)y$ *with* $u = \int q\, dt$, *then*

$$y_s^{[1]}(x, \lambda) = \lambda(\cos \lambda x + \varphi_1(x, \lambda)),$$
$$y_c^{[1]}(x, \lambda) = -\lambda(\sin \lambda x + \psi_1(x, \lambda))$$

with preserving of the property

$$|\varphi_1(x, \lambda)| + |\psi_1(x, \lambda)| = o(1) \quad \text{при } |\lambda| \to \infty$$

in the strip $|Im\lambda| \leqslant r$ *uniformly with respect to* $x \in [a, b]$.

The Birkhoff idea for proving of this theorem does not work. We used the Prüfer substitution and worked with the corresponding Prúfer non-linear equation.

A recent result. **Theorem.** (Shkalikov-Vladykina, 2015). *Consider the equation*

$$-y'' + p(x)y' + q(x)y = \lambda^2 \rho(x)y, \qquad x \in [a,b] \subset \mathbb{R},$$

provided that

$$p(x) \in L_2[a,b], \quad q \in W^{-1,2}[a,b], \quad \rho \in W^{1,1}[a,b],$$

Moreover, $\rho(x) > 0$ *for* $x \in [a,b]$*, and*

$$\rho'(x)u(x) \in L_1[a,b], \quad \textit{where } u(x) = \int q(x)\,dx.$$

Then there exists a pair of fundamental solutions of this equation of the form

$$y_\pm(x,\lambda) = [\rho(x)]^{-1/4} \exp\left(\frac{1}{2}P(x) \pm i\lambda \int_a^x \sqrt{\rho(\tau)}\,d\tau\right)(1 + \varphi_\pm(x,\lambda)). \quad (9)$$

Here

$$P(x) = \int_a^x p(\tau)\,d\tau,$$

and the functions $\varphi_\pm$ *vanish asymptotically*

$$|\varphi_+(x,\lambda)| + |\varphi_-(x,\lambda)| = o(1) \quad \text{as } |\lambda| \to \infty$$

in the half plane $Im\lambda \geqslant -r$, *uniformly with respect to* $x \in [a,b]$, *and the number* r *can be taken arbitrary large. These asymptotics are preserved if we take the quasi-derivative* $y^{[1]} = y' - u(x)y$.

ДИФФЕРЕНЦИАЛЬНЫЕ ОПЕРАТОРЫ ВЫСШИХ ПОРЯДКОВ.

Для определенности рассматриваем четный порядок и операторы вида

$$l_{2n}(y) := (\sum_{k=0}^{n}(-1)^{n-k}(p_k y^{(n-k)})^{(n-k)} +$$
$$+ i\sum_{k=0}^{n-1}(-1)^{n-k-1}\{(q_k y^{(n-k-1)})^{(n-k)} + (q_k y^{(n-k)})^{(n-k-1)}\}), \tag{10}$$

Предполагаем, что функция $p_0(x)$ отлична от нуля почти всюду на (a,b), причем

$$p_0^{-1}(x) \in L^1_{loc}(a,b), \tag{11}$$

т.е. функция p_0^{-1} локально суммируема на (a,b). Теория дифференциальных операторов, порожденых формально симметрическими выражениями вида (10), в которых вещественные функции

$$p_1, p_2, \ldots, p_n; \; q_1, q_2, \ldots, q_{n-1} \tag{12}$$

локально суммируемы на (a, b), хорошо развита (Д.Шин, М.Крейн, Ахиезер, Everitt, Marcus, Zettl). Мы хотим расширить рамки этой классической теории, а именно, дать корректное определение и привести аналоги классических теорем для оператора, в котором коэффициенты являются не классическими фукнкиями, а распределениями. Более точно, мы покажем, что расширение классической теории дифференциальных операторов можно провести, если вместо условий локальной суммируемости для коэффициентов дифференциального выражения ограничиться условиями

$$\frac{1}{\sqrt{p_0(x)}} P_k(x) \in L^2_{loc}(a, b) \qquad k = 1, 2, \ldots, n;$$

$$\frac{1}{\sqrt{p_0(x)}} Q_k(x) \in L^2_{loc}(a, b), \qquad k = 1, 2, \ldots, n-1. \tag{13}$$

Здесь P_k и Q_k $k-$е первообразные функций p_k и q_k в смысле теории распределений, т.е. $P_k^{(k)} = p_k(x)$ и $Q_k^{(k)} = q_k(x)$, где равенства понимаются в пространстве распределений $\mathcal{D}'(a,b)$. Конечно, если p_k и q_k локально суммируемы, то P_k и Q_k абсолютно непрерывные функции и условия (13) следуют из (11). Заметим также, что выбор первообразных неоднозначен, но выполнение условий (13) в силу условия (11) не зависят от их выбора. В случае, если функция p_0 отделена от нуля на на любом компакте из (a,b), т.е. для каждого компакта $K \subset (a,b)$ найдется зависящее от K число $\delta > 0$, такое, что $|p_0(x)| > \delta$, то условия (13) упрощаются. В этом случае достаточно потребовать $P_k(x), Q_k(x) \in L^2_{loc}(a,b)$ или $p_k, q_k \in W^{-k,2}[c,d]$ для любого отрезка $[c,d] \subset (a,b)$. Здесь $W^{-k,2}[c,d]$ - соболевское пространство с негативным индексом гладкости. Построить оператор, отвечающий дифференциальному выражению (10), непросто даже в случае выполнения условий (12). Дело в том, что в случае негладких коэффициентов

p_k, q_k мы не имеем права дифференцировать слагаемые в (10), т.е. само выражение определено только формально.

Основной прием для проведения регуляризации в классической теории - введение так называемых квазипроизводных и построение *согласованной с дифференциальным выражением* (10) *матрицы* $\mathcal{A}$ размера $2n \times 2n$ Построение такой матрицы - важнейший этап, поскольку позволяет свести разрешимость скалярного квазидифференциального уравнения $l_{2n}(y) = h$ с произвольной функцией $h \in L^1_{loc}(a, b)$ к разрешимости матричного дифференциального уравнения первого порядка ссуммируемыми коэффициентами.

Построение согласованной матрицы $\mathcal{A}$.

Положим теперь

$$\varphi_k := P_k + iQ_{k-1}, \quad k = 1, 2, \ldots, n$$

и определим квадратные матрицы $\mathcal{F}_1$ и $\mathcal{F}_2$ размерности $2n$, полагая

$$\mathcal{F}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 1 & 0 & 0 & . & . & . & 0 \\ f^1_{n,1} & f^1_{n,2} & f^1_{n,3} & f^1_{n,4} & . & . & . & f^1_{n,n} & f^1_{n,n+1} & 0 & . & . & . & 0 \\ f^1_{n+1,1} & f^1_{n+1,2} & f^1_{n+1,3} & f^1_{n+1,4} & . & . & . & f^1_{n+1,n} & f^1_{n+1,n+1} & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ f^1_{2n-1,1} & f^1_{2n-1,2} & f^1_{2n-1,3} & f^1_{2n-1,4} & . & . & . & f^1_{2n-1,n} & f^1_{2n-1,n+1} & 0 & . & . & . & 1 \\ f^1_{2n,1} & f^1_{2n,2} & f^1_{2n,3} & f^1_{2n,4} & . & . & . & f^1_{2n,n} & f^1_{2n,n+1} & 0 & . & . & . & 0 \end{pmatrix}$$

и

$$
\mathcal{F}_2 = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & f^2_{n+1,2} & f^2_{n+1,3} & \cdots & f^2_{n+1,n-1} & f^2_{n+1,n} & 0 & \cdots & 0 \\
0 & 0 & f^2_{n+2,3} & \cdots & f^2_{n+2,n-1} & f^2_{n+2,n} & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 0 & \cdots & f^2_{2n-2,n-1} & f^2_{2n-2,n} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & f^2_{2n-1,n} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
$$

где элементы f^1_{kj} $(k = n, n+1\ldots, 2n, j = 1, 2\ldots n+1)$ матрицы $\mathcal{F}_1$ определяются равенствами

$$
f^1_{nj} := (-1)^{n-j}\varphi_{n+1-j}p_0^{-1}, \;\; j = 1, 2, \ldots, n; \;\; f^1_{n,n+1} := p_0^{-1}
$$

$$
f^1_{kj} := (-1)^{n+1-j}\varphi_{n+1-j}\overline{\varphi}_{k-n}p_0^{-1};
$$
$$
f^1_{k,n+1} := -\overline{\varphi}_{k-n}p_0^{-1}, \;\; k = n+1, \ldots, 2n, j = 1, 2, \ldots n,
$$

а элементы $f^2_{n+k,n+k-j}$ матрицы $\mathcal{F}_2$

$$f^2_{n+k,n+k-j} = (-1)^{j+k} C^k_{j+1} \left[p_{j+1} + i \left(1 - \frac{2k}{j+1} \right) q_j \right] =: \psi_{kj} \tag{14}$$

где j принимает целочисленные значения от 1 до $n-1$ и $k = 1, 2, \ldots, j$, а C^k_{j+1} - биномиальные коэффициенты.

Таким образом, $\mathcal{F}_1$ -это квадратная матрица размерности $2n$, такая, что все наддиагональные элементы равны единице, кроме элемента с номером $(n, n+1)$, элементы квадратного блока размерности $n+1$, расположенного в нижнем левом угле, равны $f^1_{n+j,k}$, $j = 0, 1, \ldots, n$, $k = 1, 2, \ldots, n+1$, а структура матрицы $\mathcal{F}_2$ такова, что, блок размерности $(n-1) \times (n-1)$ с номерами строк от $n+1$ до $2n-1$ и столбцов с номерами от 2 до n состоит из верхней треугольной матрицы с элементами $f^2_{n+k,n+k-j}$ (см. (14)).

Определим, наконец, матрицу $\mathcal{A}_{2n} = \mathcal{A}$, полагая

$$\mathcal{A} := \mathcal{F}_1 + \mathcal{F}_2.$$

Для наглядности приведём явный вид матриц $\mathcal{A}_2, \mathcal{A}_4$ и $\mathcal{A}_6$:

$$\mathcal{A}_2 = \begin{pmatrix} \dfrac{\varphi_1}{p_0} & \dfrac{1}{p_0} \\ -\dfrac{|\varphi_1|^2}{p_0} & -\dfrac{\overline{\varphi}_1}{p_0} \end{pmatrix},$$

$$\mathcal{A}_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\varphi_2}{p_0} & \frac{\varphi_1}{p_0} & \frac{1}{p_0} & 0 \\ \frac{\varphi_2\overline{\varphi}_1}{p_0} & 2p_2 - \frac{|\varphi_1|^2}{p_0} & -\frac{\overline{\varphi}_1}{p_0} & 1 \\ \frac{|\varphi_2|^2}{p_0} & -\frac{\varphi_1\overline{\varphi}_2}{p_0} & -\frac{\overline{\varphi}_2}{p_0} & 0 \end{pmatrix}$$

$$\mathcal{A}_6 =$$
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\varphi_3}{p_0} & -\frac{\varphi_2}{p_0} & \frac{\varphi_1}{p_0} & \frac{1}{p_0} & 0 & 0 \\ -\frac{\varphi_3\overline{\varphi}_1}{p_0} & -3p_3 - iq_2 + \frac{\varphi_2\overline{\varphi}_1}{p_0} & 2p_2 - \frac{|\varphi_1|^2}{p_0} & -\frac{\overline{\varphi}_1}{p_0} & 1 & 0 \\ -\frac{\varphi_3\overline{\varphi}_2}{p_0} & \frac{|\varphi_2|^2}{p_0} & 3p_3 - iq_2 - \frac{\varphi_1\overline{\varphi}_2}{p_0} & -\frac{\overline{\varphi}_2}{p_0} & 0 & 1 \\ \frac{|\varphi_3|^2}{p_0} & \frac{\varphi_2\overline{\varphi}_3}{p_0} & -\frac{\varphi_1\overline{\varphi}_3}{p_0} & -\frac{\overline{\varphi}_3}{p_0} & 0 & 0 \end{pmatrix}.$$

Основная теорема. (Мирзоев-Шкаликов, 2016) *Матриц-функция $\mathcal{A}$ согласована с диференциальным выражением* (10). *Со всеми вытекающими из этого факта последствиями.*