Gabor analysis for rational functions

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Time-frequency Analysis

Let f be a function from $L^2(\mathbb{R})$.

Translation and modulation

$$T_x f(t) = f(t-x), M_\omega f(t) = e^{2\pi i \omega t} f(t).$$

Noncommutative

$$T_x M_\omega = e^{-2\pi i \omega x} M_\omega T_x.$$

If f is concentrated near 0 in time domain and frequency domain, then $T_x M_\omega f$ is concentrated near x in time domain and near ω in frequency domain.

Definition

We will say that $T_x M_\omega f$ is a time-frequency shift of f.

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Short-time Fourier transform

$$V_g f(x,\omega) := \int_{\mathbb{R}} f(t)\overline{e^{2\pi i\omega(t-x)} g(t-x)}dt = (f, T_x M_\omega g) - \text{ overdeterm.}$$

Uncertainty principle

A realizable signal occupies a region of area at least one in the time-frequency plane.

We want to reconstruct arbitrary function f from the discrete set of samples $(f, T_x M_\omega g)$, $(x,\omega) \in \Lambda \subset \mathbb{R}^2$, where g is some fixed function.

Example: $g = \chi_{[0,1]}, \Lambda = \mathbb{Z} \times \mathbb{Z}$.
We have a usual orthogonal representation in each interval $[n, n+1]$.

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Definition

The set of time-frequency shifts

$$\mathcal{G}(g,\alpha,\beta) = \{T_{\alpha k} M_{\beta m} g : k, m \in \mathbb{Z}\}, \quad \alpha, \beta > 0,$$

is called a Gabor system.

Stability

Definition

We will say that $\mathcal{G}(g,\alpha,\beta)$ is a frame if

$$A\|f\|_2^2 \le \sum_{k,m\in\mathbb{Z}} |(f, T_{\alpha k} M_{\beta m} g)|^2 \le B\|f\|^2, \quad f \in L^2(\mathbb{R})$$

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Dual representations and Main Question

Proposition

If $\mathcal{G}(g, \alpha, \beta)$ is a frame, then there exists a dual window γ such that

$$f = \sum_{k,m} (f, T_{\alpha k} M_{\beta m} g) T_{\alpha k} M_{\beta m} \gamma = \sum_{k,m} (f, T_{\alpha k} M_{\beta m} \gamma) T_{\alpha k} M_{\beta m} g.$$

In particular, $\mathcal{G}(\gamma, \alpha, \beta)$ also is a frame.

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General Theory

Theorem

If system $\mathcal{G}(g,\alpha,\beta)$ is a frame, then $\alpha\beta \le 1$.

Theorem

If $g \in W$, i.e. $\operatorname{esssup}_{x\in[0,1]} \sum_n |g(x+n)| < \infty$, then $\mathcal{G}(g,\alpha,\beta)$ is a frame for sufficiently small $\alpha,\beta > 0$.

If $\alpha\beta = 1$, then frame operator S

$$Sf = \sum_{k,m} (f, T_{\alpha k} M_{\beta m} g) T_{\alpha k} M_{\beta m} g$$

is unitary equivalent (by Zak transform) to multiplication operator in space $L^2([0,\alpha]\cdot[0,\alpha^{-1}])$. In particular, if g is smooth and decreasing ($g, \hat{g} \in W$, $g \in C(\mathbb{R})$), then $\mathcal{G}(g,\alpha,\alpha^{-1})$ is not a frame.

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Known Results

- Gaussian $g(t) = e^{-\pi t^2}$, $\mathcal{G}(g, \alpha, \beta)$ is a frame iff $\alpha\beta < 1$ [Lyubarskii, Seip, 1992];
- Hyperbolic secant $g(t) = (e^t + e^{-t})^{-1}$, $\mathcal{G}(g, \alpha, \beta)$ is a frame iff $\alpha\beta < 1$ [Janssen - Strohmer, 2002];
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Since,

$$\widehat{\mathcal{G}(g, \alpha, \beta)} = \mathcal{G}(\hat{g}, \beta, \alpha),$$

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Totally positive functions

Definition

We will say that function g is totally-positive if for any finite sequences $x_1 < x_2 < ... < x_N$, $y_1 < y_2 < .... < y_N$ we have

$$\det[g(x_j - y_k)]_{1 \le j,k \le N} \ge 0.$$

Theorem (Shoenberg, 1951)

Function g is totally positive iff

$$\hat{g}(\xi) = ce^{-a\xi^2} e^{ib\xi} \prod_{k=1}^{M \text{ or } \infty} (1 + i\delta_k \xi)^{-1} e^{i\delta_k \xi},$$

$a \ge 0$, $\delta_k \in \mathbb{R}$, $\sum_k \delta_k^2 < \infty$.

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Totally positive functions meet Gabor analysis

Theorem (Grochenig, Stockler, 2011)

If function g is totally positive of finite type $M \geq 2$, i.e.

$$\hat{g}(\xi) = \prod_{k=1}^{M} (1 + i\delta_k \xi)^{-1},$$

then the system $\mathcal{G}(g, \alpha, \beta)$ is a frame iff $\alpha\beta < 1$ (for $M = 1$, $\alpha\beta \leq 1$).

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Non-frame systems

First Hermite polynomial.

Proposition (Lyubarskii, Nes, 2006)

If $g \in W$ and $g(-x) = -\overline{g(x)}$, then the system $\mathcal{G}(g, \alpha, \beta)$ is not a frame for $\alpha\beta = \frac{n-1}{n}$, $n = 2, 3, \ldots$.

Conjecture

Let $g(x) = xe^{-\pi x^2}$. Then the system $\mathcal{G}(g, \alpha, \beta)$ is a frame iff $\alpha\beta < 1$ and

$$\alpha\beta \neq \frac{n-1}{n}, n = 2, 3, \ldots.$$

Calculations supports that many systems $\mathcal{G}(g, \alpha, \beta)$ with $\alpha\beta < 1$ are not frame systems. For second Hermite polynomials there are other exceptional values.

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Main results

Theorem (Belov, Kulikov, Lyubarskii)

Let

$$g(x) = \sum_{k=1}^{N} \frac{a_k}{x + iw_k}.$$

If $a_k > 0$, $w_k > 0$, then the system $\mathcal{G}(g, \alpha, \beta)$ is a frame iff $\alpha\beta \le 1$.

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If $N = 2$, $a_k, w_k \in \mathbb{R}$, then the system $\mathcal{G}(g, \alpha, \beta)$ is a frame iff $\alpha\beta \le 1$ (< 1 if $a_1 = -a_2$).

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If $\operatorname{Re} w_k > 0$, $\operatorname{Re} w_1 > \operatorname{Re} w_l$, $\hat{g}(\xi) \neq 0$, $\xi > 0$, $\alpha\beta < 1$ *and* $\alpha\beta$ *is irrational, then the system* $\mathcal{G}(g, \alpha, \beta)$ *is a frame.*

There exist many rational functions g and α, β with irrational $\alpha\beta < 1$ such that $\mathcal{G}(g, \alpha, \beta)$ is not a frame.

Moreover, for any $\delta > 0$ there exists $q_0 = q_0(\delta, g, \beta)$ such that if $\alpha\beta = \frac{p}{q} < 1 - \delta$ and $q > q_0$, then the system $\mathcal{G}(g, \alpha, \beta)$ is a frame.

Corollary

For any fixed $\beta > 0$ *for any* $\alpha \leq \beta^{-1}$ *except at most discrete subset (finite or* $\to \beta^{-1}$*) the system* $\mathcal{G}(g, \alpha, \beta)$ *is a frame.*

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Previous tools: Bargmann transform, Theta-functions, Totally positive functions, Zeevi-Zibulski matrices for rational $\alpha\beta$.

New tools: Paley-Wiener spaces, Dynamical systems, N-diagonal matrices.

We can assume that $\beta = 1$. So, $\alpha \le 1$. Let

$$Z(z,\xi) := \sum_{k=1}^{N} \frac{a_k e^{2\pi\xi w_k}}{1 - ze^{2\pi w_k/\alpha}} = \frac{m_0(\xi) + zm_1(\xi) + \ldots + z^{N-1}m_{N-1}(\xi)}{\prod_{k=1}^{N}(1 - ze^{2\pi w_k/\alpha})}.$$

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The system $\mathcal{G}(g,\alpha,\beta)$ is a frame iff

$$B\|f\|_2^2 \ge \int_0^{\frac{1}{\alpha}} \sum_n \Big|\sum_{s=0}^{N-1} f\Big(\xi + n + \frac{s}{\alpha}\Big) m_s(\xi)\Big|^2 d\xi \ge \varepsilon\|f\|_2^2.$$

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Proposition

The system $\mathcal{G}(g,\alpha,\beta)$ is a frame iff

$$B\|f\|_2^2 \ge \int_0^{\frac{1}{\alpha}} \sum_n \Big| \sum_{s=0}^{N-1} f\left(\xi + n + \frac{s}{\alpha}\right) m_s(\xi)\Big|^2 d\xi \ge \varepsilon \|f\|_2^2.$$

Thank you for your attention!