The Scott Rank of Countable Structures and the Isomorphism Relation

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The Scott Analysis of Countable Structures

Throughout this talk we use the following notation:

- L: a recursive first-order signature
- X_L or $\mathrm{Mod}(L)$: the space of all countable L-structures with universe ω
- $L_{\omega_1\omega}$: the infinitary logic consisting of formulas of the following form:
  1. atomic formulas;
  2. if φ is a formula then $\neg\varphi$, $\exists x\varphi$ and $\forall x\varphi$ are formulas;
  3. if Φ is a countable set of formulas, then $\bigwedge\Phi$ and $\bigvee\Phi$ are formulas.

An $L_{\omega_1\omega}$ sentence is a formula without free variables.

All formulas we consider will have only finitely many free variable (they are subformulas of $L_{\omega_1\omega}$ sentences).

Inductively define a hierarchy of formulas Σ_α and Π_α:

- $\Sigma_0 = \Pi_0$ consists of all finite Boolean combinations of atomic formulas;
- if $\alpha > 0$, then $\varphi \in \Sigma_\alpha$ if φ is a countable disjunction of formulas of the form $\exists x \psi$, where ψ is Π_β for some $\beta < \alpha$.
- if $\alpha > 0$, then $\varphi \in \Pi_\alpha$ if φ is a countable conjunction of formulas of the form $\forall x \psi$, where ψ is Σ_β for some $\beta < \alpha$.

Let $M \in X_L$ be a countable L-structure and $\overline{a}$ be a tuple in M. Inductively define the canonical Scott formula of (quantifier) rank α as follows:

$$\varphi_0^{M,\overline{a}}(\overline{v}) = \bigwedge\{\theta(\overline{v}) : \theta \text{ is atomic or negated atomic and } M \models \theta[\overline{a}]\}$$

$$\varphi_{\alpha+1}^{M,\overline{a}}(\overline{v}) = \varphi_\alpha^{M,\overline{a}}(\overline{v}) \wedge \bigwedge_{b\in\omega}(\exists u)\varphi_\alpha^{M,\overline{a}^\frown b}(\overline{v}^\frown u) \wedge (\forall u) \bigvee_{b\in\omega} \varphi_\alpha^{M,\overline{a}^\frown b}(\overline{v}^\frown u)$$

$$\varphi_\lambda^{M,\overline{a}}(\overline{v}) = \bigwedge_{\alpha<\lambda} \varphi_\alpha^{M,\overline{a}}(\overline{v})$$

Note that for each limit $\lambda > 0$, φ_λ is Π_λ and $\varphi_{\lambda+n}$ is $\Pi_{\lambda+2n}$.

For $M, N \in X_L$, $\overline{a} \in M$, $\overline{b} \in N$, and $\alpha < \omega_1$, we define α-equivalence by

$$(M, \overline{a}) \equiv_\alpha (N, \overline{b})$$

if

$$\varphi_\alpha^{M,\overline{a}} = \varphi_\alpha^{N,\overline{b}},$$

which is equivalent to

$$N \models \varphi_\alpha^{M,\overline{a}}[\overline{b}]$$

and/or

$$M \models \varphi_\alpha^{N,\overline{b}}[\overline{a}].$$

Theorem (Karp)

The following are equivalent for $M, N \in X_L$ and $\bar{a} \in M$, $\bar{b} \in N$:

- $(M, \bar{a}) \equiv_\infty (N, \bar{b})$, i.e., for all $L_{\omega_1\omega}$ formulas φ, $M \models \varphi[\bar{a}]$ iff $N \models \varphi[\bar{b}]$
- $(M, \bar{a}) \cong (N, \bar{b})$, i.e., there is an isomorphism $j : M \to N$ such that $j(\bar{a}) = \bar{b}$.

For $M \in X_L$ and $\overline{a} \in M$, define $\rho(\overline{a})$ to be the least β such that for all $\overline{b} \in M$, if $(M, \overline{a}) \equiv_\beta (M, \overline{b})$ then $(M, \overline{a}) \equiv_\infty (M, \overline{b})$.

The Scott rank of M, $\mathrm{SR}(M)$, is the least α such that $\alpha > \rho(\overline{a})$ for all $\overline{a} \in M$.

There is a least ordinal ρ such that for all $\overline{a}, \overline{b} \in M$, if $(M, \overline{a}) \equiv_\rho (M, \overline{b})$, then $(M, \overline{a}) \equiv_{\rho+1} (M, \overline{b})$.

In fact, for this ordinal ρ, for all $\overline{a}, \overline{b} \in M$, if $(M, \overline{a}) \equiv_\rho (M, \overline{b})$, then $(M, \overline{a}) \equiv_\infty (M, \overline{b})$. In other words, $\rho \geq \rho(\overline{a})$ for all $\overline{a} \in M$.

The canonical Scott sentence of M is

$$\varphi_M = \varphi_\rho^{M,\varnothing} \wedge \bigwedge_{\overline{a}} \forall \overline{v} \left(\varphi_\rho^{M,\overline{a}}(\overline{v}) \rightarrow \varphi_{\rho+1}^{M,\overline{a}}(\overline{v}) \right)$$

Note that if ρ is a limit ordinal, then φ_M is equivalent to a $\Pi_{\rho+2}$ formula.

Theorem (Scott)
For $M, N \in X_L$ the following are equivalent:

- $M \cong N$
- $M \models \varphi_N$
- $N \models \varphi_M$
- $M \equiv_\infty N$

Theorem (essentially Nadel)
For any $M \in X_L$, $\mathrm{SR}(M) \leq \omega_1^M + 1$, where ω_1^M is the least ordinal not computable in M.

Corollary
For $M \in X_L$, $[M]_\cong$ is $\mathbf{\Pi}^0_{\omega_1^M+2}$.

The Scott Rank of Computable Structures

Theorem (Calvert–Goncharov–Knight)
Let L be a computable relational vocabulary, $\mathcal{F}$ be the fragment of $L_{\omega_1\omega}$ consisting of computable formulas, and $M \in X_L$ a computable structure. Then

1. $\mathrm{SR}(M) \leq \omega_1^{\mathrm{CK}} + 1$.
2. $\mathrm{SR}(M) = \omega_1^{\mathrm{CK}}$ iff M is $\mathcal{F}$-atomic.
3. $\mathrm{SR}(M) < \omega_1^{\mathrm{CK}}$ iff $\varphi_M \in \mathcal{F}$.

Example (Harrison)
There is a computable linear ordering with order type

$$\omega_1^{\mathrm{CK}}(1+\eta)$$

where η is the order type of $(\mathbb{Q}, <)$. The Scott rank of this structure is $\omega_1^{\mathrm{CK}} + 1$.

For each oracle $x \in 2^\omega$, there is a computable-in-x linear ordering with order type

$$\omega_1^x(1+\eta).$$

The Scott rank of this structure is $\omega_1^x + 1$.

It is much more difficult to construct computable structures with Scott rank ω_1^{CK}.

- (Makkai, 1981) There is a Δ_1^1 structure M with Scott rank $\omega_1^M = \omega_1^{\mathrm{CK}}$.
- (Knight–Millar, 2010) There is a computable structure with Scott rank ω_1^{CK}.
- More computable structures with Scott rank ω_1^{CK} were produced by Calvert–Knight–Millar, Calvert–Goncharov–Knight, Fokina–Knight–Melnikov–Quinn–Safranski, Harrison-Trainor–Igusa–Knight, etc.

Conjecture (Becker)

Assume AD. There is no function F from all Turing degrees to isomorphism types of countable structures such that for all Turing degree $\tilde{d}$, $F(\tilde{d})$ has Scott rank $\omega_1^{\tilde{d}}$.

Observation

There is a Δ^1_1 function $F : 2^\omega \to X_{\{<\}}$ such that for all $x \in 2^\omega$, $F(x)$ is an x-Harrion linear ordering, i.e.,

(i) for each $x \in 2^\omega$, $F(x)$ is computable in x;

(ii) if $x \equiv_T y$ then $F(x) \cong F(y)$; and

(iii) the Scott rank of $F(x)$ is $\omega_1^x + 1$.

Theorem (Becker)

Assume AD. Let Q be a well-orderable set of isomorphism types of countable structures. Then for a Turing cone of Turing degrees $\tilde{d}$, no members of Q has Scott rank $\omega_1^{\tilde{d}}$.

Conjecture

There is no Δ_1^1 function $F : 2^\omega \to X_L$ such that

(i) for each $x \in 2^\omega$, $F(x)$ is computable in x;

(ii) if $x \equiv_T y$ then $F(x) \cong F(y)$; and

(iii) the Scott rank of $F(x)$ is ω_1^x.

The Countable Admissible Ordinal Equivalence

The equivalence relation F_{ω_1} is

$$xF_{\omega_1}y \iff \omega_1^x = \omega_1^y$$

Observation

There is a Δ_1^1 function $F : 2^\omega \to X_L$ such that for all $x, y \in 2^\omega$,

(i) $\omega_1^x = \omega_1^y$ iff $F(x) \cong F(y)$ (i.e., F is a reduction from F_{ω_1} to $\cong$ on X_L); and

(ii) the Scott rank of $F(x)$ is $\omega_1^x + 1$.

F_{ω_1} has been considered to be related to a possible counterexample to the infinitary Vaught's conjecture.

Infinitary Vaught's Conjecture

Let φ be an $L_{\omega_1\omega}$ sentence. Then either φ has only countably many nonisomorphic countable models or there is a perfect subset of X_L consisting of pairwise nonisomorphic models of φ.

- (Marker) F_{ω_1} is not the orbit equivalence relation of a continuous action of a Polish group on 2^ω.
- (Becker) F_{ω_1} is not the orbit equivalence relation of a Δ^1_1 action of a Polish group on 2^ω.

Observation

There is a Δ^1_1 reduction of F_{ω_1} to an orbit equivalence relation of a Δ^1_1 action of a Polish group on 2^ω.

Theorem (Chan)

Suppose F is a Δ^1_1 reduction of F_{ω_1} to $\cong$ on X_L. Then for all $x \in 2^\omega$, the Scott rank of $F(x)$ is $\geq \omega_1^x$.

Question (Chan–Harrison-Trainor–Marks)

Is there a Δ^1_1 reduction F of F_{ω_1} to $\cong$ on X_L so that the Scott rank of $F(x)$ is ω_1^x for all $x \in 2^\omega$?

Theorem (Chan–Harrison-Trainor–Marks, Becker)
Suppose F is a Δ^1_1 reduction of F_{ω_1} to $\cong$ on X_L. Then there is $x \in 2^\omega$ such that the Scott rank of $F(x)$ is $\omega_1^x + 1$. In other words, there is no Δ^1_1 reduction from F_{ω_1} to $\cong$ on X_L such that for all $x \in 2^\omega$, the Scott rank of $F(x)$ is ω_1^x.

Question (Chan, Chan–Harrison-Trainor–Marks)
Suppose F is a Δ^1_1 reduction of F_{ω_1} to $\cong$ on X_L. Is the Scott rank of $F(x)$ necessarily $\omega_1^x + 1$ for all $x \in 2^\omega$?

Theorem
Suppose F is a Δ^1_1 reduction of F_{ω_1} to $\cong$ on X_L. Then the Scott rank of $F(x)$ is $\omega_1^x + 1$ for all $x \in 2^\omega$.

The Hyperarithmetic Equivalence

Consider hyperarithmetic equivalence $\equiv_h$ defined as

$$x \equiv_h y \iff \Delta^1_1(x) = \Delta^1_1(y)$$

Then

$$x \equiv_T y \Rightarrow x \equiv_h y \Rightarrow \omega_1^x = \omega_1^y$$

Theorem (Becker)
If E is a Σ^1_1 equivalence relation on 2^ω and for all $x, y \in 2^\omega$,

$$x \equiv_h y \Rightarrow xEy.$$

Then for all $x, y \in 2^\omega$,

$$\omega_1^x = \omega_1^y \Rightarrow xEy.$$

Theorem (Becker)
Let $F : 2^\omega \to X_L$ be a Δ^1_1 function such that for all $x, y \in 2^\omega$, $x \equiv_h y \Rightarrow F(x) \cong F(y)$. Then either $F(0)$ has Scott rank $\omega_1^{\mathrm{CK}} + 1$ or else for all $x, y \in 2^\omega$, $F(x) \cong F(y)$.

Theorem (Becker)
Let C be a hypercone, i.e. $C = \{x \in 2^\omega : u \in \Delta^1_1(x)\}$ for some $u \in 2^\omega$. Let $F : C \to X_L$ be a Δ^1_1 function such that for all $x, y \in C$, $x \equiv_h y \Rightarrow F(x) \cong F(y)$. Then exactly one of the following holds:

(i) there is a hypercone $D \subseteq C$ such that for all $x \in D$, $F(x)$ has Scott rank $\omega_1^x + 1$;

(ii) there is a hypercone $D \subseteq C$ such that for all $x, y \in D$, $F(x) \cong F(y)$.

Technical Theorem (Becker)

Let S be a subset of 2^ω such that

1. S is Σ^1_1;
2. S is $\Delta^1_1(\mathcal{O})$, where $\mathcal{O}$ is Kleene's O;
3. S is $\mathbf{\Sigma}^0_{\omega_1^{\mathrm{CK}}+2}$;
4. For all $x, y \in 2^\omega$, if $x \equiv_h y$ and $x \in S$ then $y \in S$;
5. $0 \in S$;

Then $S = 2^\omega$.

Relativized Technical Theorem

Let $\sigma \in 2^\omega$ and S be a subset of 2^ω such that

1. S is $\Sigma^1_1(\sigma)$;
2. S is $\Delta^1_1(\mathcal{O}^\sigma)$;
3. S is $\mathbf{\Sigma}^0_{\omega_1^\sigma+2}$;
4. For all $x, y \in 2^\omega$ with $\sigma \in \Delta^1_1(x) \cap \Delta^1_1(y)$, if $x \equiv_h y$ and $x \in S$ then $y \in S$;
5. $\sigma \in S$;

Then $\{x \in 2^\omega : \sigma \in \Delta^1_1(x)\} \subseteq S$.

The proof uses Steel forcing and other results of Steel, Harrington, Kechris, Thomason, etc.

Thanks!