PRIMITIVE RECURSIVE REVERSE MATHEMATICS

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- For any pair of computable dense linear orders without endpoints there exists a computable isomorphism between them

$$a_0 <_A \dots\dots\dots <_A a_n <_A \mathbf{a}$$

$$b_0 <_B \dots\dots\dots <_B b_n \qquad \exists b\,(b_n <_B b)$$

unbounded search

- For any computable poset there exists a computable linear extension of it

$$x_0 <_L \cdots <_L x_n \qquad \mathbf{x}$$

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unbounded search $\sim$ NON PRIM REC

- For any computable poset there exists a computable linear extension of it PRIM REC

$$x_0 <_L \cdots <_L x_n \qquad \mathbf{x}$$

Why primitive recursion?

Kalimullin, Melnikov, Ng 2017 : good balance between computable (unbounded delay) and polinomial

Punctual structures (domain N and operations/relations uniformely primitive recursive):

- which structures have a primitive recursive representations?
- which are p.r. categorical?

Our questions

- is a statement primitively recursively true?

- does a statement imply the existence of all computable functions?

Formal framework

- Symbols for any primitive recursive function -

PRA

$\mathsf{PA}^- + \mathrm{I}\Delta_0^0$

defining equations for any primitive recursive function

Formal framework

- Symbols for any primitive recursive function -
- Variables for numbers and for functions -

PRA^2

PRA

$\mathsf{PA}^- + \mathrm{I}\Delta_0^0$

defining equations for any primitive recursive function

defining equations for any primitive recursive functional

$$\forall f\ \forall n \quad \big(\mathsf{Add}(0, f, n) = f(n) \land \mathsf{Add}(s(m), f, n) = s\,(m + f(n))\big)$$

Ulrich Kohlenbach, Things that can and things that can’t be done in PRA, 2000

Models of PRA^2

$\mathcal{X} \subseteq \{f \mid f\colon M \to M\}$

closed under comp and prim recur

$M \vDash \mathsf{PRA}$

$(M, \mathcal{X}, 0, s, <, \{f_n\})$

prim recur functions

RCA_0

RCA_0 $\quad$ $\mathsf{PRA}^2 + \mathrm{I}\Sigma^0_1$ + closure under minimisation

$\mathcal{X} \supseteq \mathsf{Comp}$

$M \models \mathsf{PRA} + \mathrm{I}\Sigma^0_1$

$(M, \mathcal{X}, 0, s, <, \{f_n\})$

2-RCA_0

$2\text{-RCA}_0 \qquad \text{PRA}^2 + \mathrm{I}\Sigma^0_1 + \Delta^0_1\text{-CA}$

$\Delta^0_1\text{-CA} \;\Big|\; \exists f\, \forall n\, ((\varphi(n) \to f(n) = 1) \wedge (\neg\varphi(n) \to f(n) = 0), \;\; \varphi \text{ is } \Delta^0_1$

$\mathcal{X} \supseteq$ 2-valued Comp Functions

$M \vDash \text{PRA} + \mathrm{I}\Sigma^0_1$

$(M, \mathcal{X}, 0, s, <, \{f_n\})$

• 2-$\mathsf{RCA}_0 \nvdash \mathsf{RCA}_0$ •

Proof's sketch.

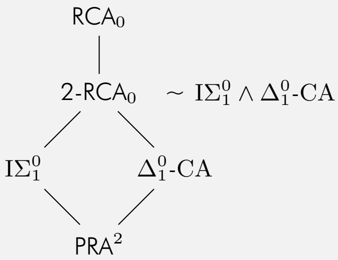
Let $(\omega, \mathcal{X})$ such that

- $\mathcal{X}$ contains all 0-1-valued computable functions and
- $\mathcal{X}$ is closed under composition and primitive recursion.

Then $\mathcal{X}$ does not contain all computable functions.

Use the fact that any 2-valued computable function is bounded by a primitive recursive function. □

2-RCA_0	$\Delta^0_1\text{-CA} \wedge \mathrm{I}\Sigma^0_1$
Δ^0_1-CA	$\exists f\ \forall n\ ((f(n) = 1 \to \varphi(n)) \wedge (f(n) = 0 \to \neg\varphi(n)))$, φ is Δ^0_1


RCA_0
$2\text{-}\mathsf{RCA}_0$
$\sim \mathrm{I}\Sigma^0_1 \wedge \Delta^0_1\text{-CA}$
$\mathrm{I}\Sigma^0_1$
$\Delta^0_1\text{-CA}$
PRA^2

Theorem

Over PRA^2, the following are equivalent:

- RCA_0
- categoricity of dense linear orders without endpoints
- categoricity of random graphs
- categoricity of atomless Boolean algebras

Theorem

PRA^2 proves the followings:

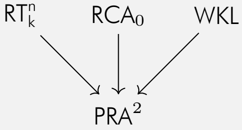
- every poset can be linearised
- every complete consistent theory has a model
- every field can be embedded into its algebraic closure
- every vector space over a finite field has a basis

RCA_0 **vs** RCA_0

RT^n_k	For each colouring $c\colon [\mathbb{N}]^n \to k$ there exists an infinite homogeneous set
WKL	For each infinite tree $T \subseteq 2^{<\mathbb{N}}$ there exists an infinite path

Theorem

Over PRA^2, $\mathsf{RT}^n_k \nrightarrow \mathsf{RCA}_0$ and $\mathsf{WKL} \nrightarrow \mathsf{RCA}_0$

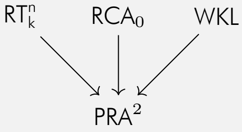


RCA$_0$ **vs** RCA$_0$

RT^n_k	For each colouring c: $[\mathbb{N}]^n \to k$ there exists an infinite homogeneous set
WKL	For each infinite tree $T \subseteq 2^{<\mathbb{N}}$ there exists an infinite path

Theorem

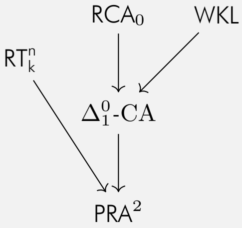
Over PRA^2,
RT^n_k and WKL do not imply
the existence of all computable functions



Theorem

Over PRA^2, $\mathsf{WKL} \to \Delta^0_1\text{-CA}$

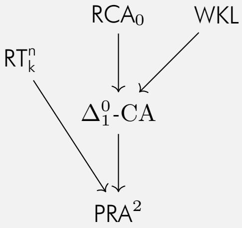
Over PRA^2, $\mathsf{RT}^n_k \nrightarrow \Delta^0_1\text{-CA}$



Theorem

Over PRA^2, WKL $\rightarrow \Delta^0_1$-CA

Over PRA^2, RT^n_k do not imply the existence of all computable 0-1-valued functions



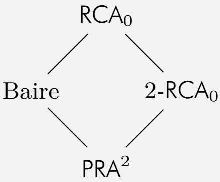
Baire categoricity theorem

Any sequence $\langle V_n \mid n \in \mathbb{N} \rangle$ of dense open subset of $\mathbb{N}^{\mathbb{N}}$ has not empty intersection.

Theorem

Over PRA^2, RCA_0 $\underset{\not\leftarrow}{\rightarrow}$ BaireCategory

Over PRA^2, $2\text{-}RCA_0 \perp$ BaireCategory



- $\mathsf{BaireCategory} \nvdash 2\text{-}\mathsf{RCA}_0$ •
- $\mathsf{RT}^n_k \nvdash 2\text{-}\mathsf{RCA}_0$ •

Proof's sketch.

Goal: define $(\omega, \mathcal{X}) \models \mathsf{PRA}^2 + \mathsf{BaireCategory}$ such that $g \neq P(f)$, for $f \in \mathcal{X}$ and g 2-valued computable non p.r. function.

Suppose $(V_n) \in \mathcal{X}_n$ and that σ is an initial segment of the solution.

Consider the p.r. functional P and $m \in \mathbb{N}$. Then there exists $k, j \in \mathbb{N}$ such that $g(j) \neq P^{\sigma \oplus m^k}(j)$. Extend $\sigma \oplus m^k$ to meet another bit of $(V_n) \in \mathcal{X}_n$. $\square$

N. Bazhenov, R. Downey, I. Kalimullin, A. Melnikov
Foundations of online structure theory, 2019

N. Bazhenov, M. Fiori Carones, L. Liu, A. Melnikov
Primitive recursive reverse mathematics (submitted)

R. Downey, A. G. Melnikov, K. M. Ng
Foundations of online structure theory II, 2021

I. Kalimullin, A. Melnikov, K. M. Ng
Algebraic structures computable without delay, 2017

U. Kohlenbach
Things that can and things that cannot be done in PRA, 2000