# "Jump inversion" for linear orders

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We will review results whose proofs use following general idea:

- To construct a computable linear order $\mathcal{L}$ with some given property, firstly, we construct a linear order $\mathcal{M}$ using a suitable $\mathbf{0}^{(n)}$-oracle.
- Then we reduce the complexity of the linear order stage by stage. Namely, we built a sequence of linear orders $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m$ such that:
  - $\mathcal{L}_1 = \mathcal{M}$;
  - $\mathcal{L}_i$ is $\mathbf{0}^{(n_i)}$-computable;
  - if $i < j$ then $n_i > n_j$;
  - $n_m = 0$, i. e. $\mathcal{L}_m$ is a computable linear order.

# Outline

1. Preliminaries
2. Low linear orders
3. “Jump inversion” theorems
4. Applications

# Preliminaries

# Linear orders

- A linear order $\mathcal{L}$ is an algebraic system $(L, <_{\mathcal{L}})$ with one binary relation having the properties of antireflexivity, antisymmetry, transitivity, and for any $x,\ y \in L$ either $x \leq_{\mathcal{L}} y$, or $y \leq_{\mathcal{L}} x$ (where $x \leq_{\mathcal{L}} y \Leftrightarrow x = y \vee x <_{\mathcal{L}} y$).
- If $\mathcal{L} = (L, <_{\mathcal{L}})$ is a linear order, then we define $\mathcal{L}^*$ as $\mathcal{L}^* = (L, <_{\mathcal{L}^*})$, where $(\forall x)(\forall y)[x <_{\mathcal{L}^*} y \longleftrightarrow y <_{\mathcal{L}} x]$.
- An interval of a linear order $\mathcal{L}$ is a suborder $\mathcal{M} = (M, <_{\mathcal{L}})$ such that $(\forall x, y \in M)(\forall z \in L)[x <_{\mathcal{M}} z <_{\mathcal{M}} y \longrightarrow z \in M]$. A closed interval of a linear order $\mathcal{L}$ with ends $x$ and $y$ is the set $[x,\ y]_{\mathcal{L}} = \{z \in L \mid x \leq_{\mathcal{L}} z \leq_{\mathcal{L}} y\}$. An open interval of a linear order $\mathcal{L}$ with ends $x$ and $y$ is the set $(x,\ y)_{\mathcal{L}} = \{z \in L \mid x <_{\mathcal{L}} z <_{\mathcal{L}} y\}$.

# Relations on linear orders

Let $\mathcal{L}$ be a linear order.

- A binary relation $S_{\mathcal{L}}(x,y) \rightleftharpoons (x <_{\mathcal{L}} y) \,\&\, ([x,y]_{\mathcal{L}} = \{x,y\})$ is called *the successor relation*.
- A binary relation $F_{\mathcal{L}}(x,y) \rightleftharpoons (x = y) \vee (x <_{\mathcal{L}} y) \,\&\, (|[x,y]_{\mathcal{L}}| < \infty) \vee (y <_{\mathcal{L}} x) \,\&\, (|[y,x]_{\mathcal{L}}| < \infty)$ is called *the block relation*.

If $\mathcal{L}$ is computable then $S_{\mathcal{L}}$ is $\mathbf{0}'$-computable, and $F_{\mathcal{L}}$ is $\mathbf{0}''$-computable.

## Condensations

The block relation is a congruence. An equivalence class is called *a block* and denoted by $[x]_{\mathcal{L}} = \{y \mid F_{\mathcal{L}}(x, y)\}$. The linear order on the set of blocks induced by $\mathcal{L}$ is called *condensation* and denoted by $\mathcal{L}/F_{\mathcal{L}}$.

We define the relation $F^{\alpha}_{\mathcal{L}}$ for every ordinal $\alpha$:

1. $F^0_{\mathcal{L}}(x, y) \Leftrightarrow x = y$, $[x]^0_{\mathcal{L}} \rightleftharpoons \{y \in L \mid F^0_{\mathcal{L}}(x, y)\}$;
2. if an ordinal $\alpha = \beta + 1$ is a successor then $F^{\beta+1}_{\mathcal{L}}(x, y) \Leftrightarrow F_{\mathcal{L}/F^{\beta}_{\mathcal{L}}}([x]^{\beta}_{\mathcal{L}}, [y]^{\beta}_{\mathcal{L}})$, $[x]^{\beta}_{\mathcal{L}} \rightleftharpoons \{y \in L \mid F^{\beta}_{\mathcal{L}}(x, y)\}$;
3. if an ordinal $\alpha$ is a limit ordinal then $F^{\alpha}_{\mathcal{L}}(x, y) \Leftrightarrow (\exists \beta < \alpha)[F_{\mathcal{L}/F^{\beta}_{\mathcal{L}}}([x]^{\beta}_{\mathcal{L}}, [y]^{\beta}_{\mathcal{L}})]$,
   $[x]^{\alpha}_{\mathcal{L}} \rightleftharpoons \{y \in L \mid (\exists \beta < \alpha)[F^{\beta}_{\mathcal{L}}(x, y)]\}$.

The least ordinal $\alpha$ such that $F^{\alpha}_{\mathcal{L}} = F^{\alpha+1}_{\mathcal{L}}$ is called the Hausdorff's rank of $\mathcal{L}$.

# Classes of linear orders

- A linear order is dense if for any two points there exists a point between of them. The type of countable dense linear order with no endpoints is denoted by $\eta$.
- A linear order is discrete if every element has both an immediate predecessor and an immediate successor except for the possible first and last elements.
- A linear order is scattered if it has no an infinite dense subset.
- A linear order is called $\eta$-like linear orders if it is infinite and it does not contain an infinite block. If block sizes bounded by a fixed number $k$ then the linear order is called strongly $\eta$-like.
- Let $\{a_0, a_1, a_2, \dots\}$ be an enumeration of a set $A \subseteq \omega$, perhaps with repetitions. Then a linear order $\mathcal{L}$ of the order type $\eta + a_0 + \eta + a_1 + \eta + a_2 + \eta + \dots$ is called an $\eta$-representation of the set $A$.

# Scattered linear orders and *VD*-rank.

1. $\mathbf{VD_0} = \{\mathbf{0},\ \mathbf{1}\}$;
2. $\mathbf{VD}_\alpha = \{\sum_{\mathbf{i}\in\tau} \mathcal{L}_\mathbf{i} \mid \mathcal{L}_\mathbf{i} \in \bigcup_{\beta<\alpha} \mathbf{VD}_\beta,\ \tau \in \{\omega,\ \omega^*,\ \zeta,\ \mathbf{0},\ \mathbf{1},\ \mathbf{2}, \ldots\}\}$.

$\mathbf{VD} = \bigcup_{\alpha} \mathbf{VD}_\alpha$.

**Theorem (Hausdorff)**

*A countable linear order $\mathcal{L}$ is scattered if and only if $\mathcal{L} \in \mathbf{VD}$.*

- The least ordinal $\alpha$ such that $\mathcal{L} \in \mathbf{VD}_\alpha$ is called the *VD*-rank of $\mathcal{L}$.
- The least ordinal $\alpha$ such that $\mathcal{L}$ is a finite sum of linear orders with *VD*-rank less or equal than $\alpha$ is called the *VD**-rank of $\mathcal{L}$.
- The *VD*-rank of a scattered linear order $\mathcal{L}$ is equal to the Hausdorff's rank of $\mathcal{L}$.

# Low linear orders

**Theorem (Frolov, 2010; Montalban, 2009)**

*A linear order has a low copy if and only if it has a $\mathbf{0}'$-computable copy with a $\mathbf{0}'$-computable successor relation.*

**Question (Knight)**

*Is it true that every low linear order has a computable copy?*

**Theorem (Jockusch, Soare, 1991)**

*For every $\Delta^0_2$-degree $\mathbf{x}$ there is an $\mathbf{x}$ computable linear order with no computable copies.*

# The main problem

**Theorem (Downey, Moses, 1989)**

*Every low discrete linear order has a computable copy.*

**Question (Downey, 1998)**

*Describe a property $P$ of classical order types that guarantee that if $\mathcal{L}$ is a low linear order and $P$ holds for the order type of $\mathcal{L}$ then $\mathcal{L}$ is isomorphic to a computable linear order.*

# $k$-quasidiscrete linear orders

**Definition**

If block sizes of a linear order $\mathcal{L}$ either is bounded by a fixed number $k$ or is infinite then $\mathcal{L}$ is called $k$-quasidiscrete.

**Theorem (Frolov, 2010)**

*Every low k-quasidiscrete linear order is $\mathbf{0}'''$-isomorphic to a computable linear order.*

# $\eta$-like linear orders

**Theorem (Frolov, 2006)**

*Every low strongly $\eta$-like linear order is $\mathbf{0}'$-isomorphic to a computable linear order.*

**Theorem (Frolov, 2010)**

*If a low linear order with the dense condensation has no strongly $\eta$-like subinterval then it is $\mathbf{0}''$-isomorphic to a computable linear order.*

# Limitwise monotonic functions

## Definition

A function $F$ is called **x**-limitwise monotonic if there is an **x**-computable function $f(x, s)$ such that
1) $(\forall x)(\forall s)[f(x, s) \leq f(x, s+1)]$;
2) $(\forall x)[F(x) = \lim_{s \to \infty} f(x, s)]$.

## Definition

If $\mathcal{L} \cong \sum_{q \in \mathbb{Q}} F(q)$ then we say that the order type of $\eta$-like linear order $\mathcal{L}$ is defined by **x**-limitwise monotonic function $F$.

# A criteria of existence of a LMF

## Definition

*A block $[x]_{\mathcal{L}}$ is called a left (right) local maximal block, if there are $[y]_{\mathcal{L}} <_{\mathcal{L}} [x]_{\mathcal{L}}$ ($[y]_{\mathcal{L}} >_{\mathcal{L}} [x]_{\mathcal{L}}$) such that for all $[z]_{\mathcal{L}}$ if $[y]_{\mathcal{L}} <_{\mathcal{L}} [z]_{\mathcal{L}} <_{\mathcal{L}} [x]_{\mathcal{L}}$ ($[y]_{\mathcal{L}} >_{\mathcal{L}} [z]_{\mathcal{L}} >_{\mathcal{L}} [x]_{\mathcal{L}}$), then $|[z]_{\mathcal{L}}| < |[x]_{\mathcal{L}}|$.*

## Theorem (Z., 2017)

*If sizes of the left and the right local maximal blocks of a low $\eta$-like linear order are bounded by a fixed number then the order type of this linear is defined by a $\mathbf{0}'$-limitwise monotonic function on rationals, and, consequently, this linear order has a computable copy.*

# $\Pi^0_2$-initial segments

**Theorem (Coles, Downey, Khoussainov, 1997)**

*For every $\Sigma^0_3$ set $A$ there is an $\eta$-representation $\mathcal{L}$ of $A$ such that $\mathcal{L} + \omega^*$ has a computable copy.*

**Theorem (Z., 2009)**

*For every $\Sigma^0_3$ set $A$ there is an $\eta$-representation $\mathcal{L}$ of $A$ such that $\mathcal{L} + \omega^*$ has a low copy.*

**Theorem (Z., 2009)**

*Suppose that $\mathcal{L}$ is an $\eta$-representation of $A$. If $\mathcal{L} + \omega^*$ has a low copy then there is an $\eta$-representation $\mathcal{L}'$ such that $\mathcal{L}' + \omega^*$ has a computable copy.*

# Connection with low linear orders

**Theorem (Z., ta)**

*Every low linear order of the form $\mathcal{L} + \omega^*$ (where $\mathcal{L}$ is an $\eta$-representation) is $\mathbf{0}''$-isomorphic to a computable linear order.*

**Theorem (Frolov, Z., 2022)**

*There is a low strongly $\eta$-representation which has no computable copy.*

# "Jump inversion" theorems

**Theorem (Downey, Knight, 1992)**

*A linear order $\mathcal{L}$ has a $\mathbf{0}'$-computable presentation if and only if $(\eta + 2 + \eta) \cdot \mathcal{L}$ has a computable presentation.*

**Proof sketch.**

If a linear order $\mathcal{L}$ has a $\mathbf{0}'$-computable presentation then $(\eta + 2 + \eta) \cdot \mathcal{L}$ has $\mathbf{0}'$-computable presentation with $\mathbf{0}'$-computable successor relation. The linear order $(\eta + 2 + \eta) \cdot \mathcal{L}$ is strongly $\eta$-like linear order, and, consequently, has a computable copy. □

**Theorem (Watnick, 1984)**

*A linear order $\mathcal{L}$ has a $\mathbf{0}''$-computable presentation if and only if $\zeta \cdot \mathcal{L}$ has a computable presentation.*

**Theorem (Ash, Knight, 2000)**

*A linear order $\mathcal{L}$ has a $\mathbf{0}''$-computable presentation if and only if $\omega \cdot \mathcal{L}$ has a computable presentation.*

# Watnick's theorem: the proof's idea

- If $\mathcal{L}$ has a $\mathbf{0}''$-computable presentation then $\zeta \cdot \mathcal{L}$ ($\omega \cdot \mathcal{L}$) has a $\mathbf{0}'$-computable presentation with $\mathbf{0}'$-computable successor relation.
- $\zeta \cdot \mathcal{L}$ is a low discrete (quasi-discret) linear order. By the result of Downey-Moses (Frolov), it has a computable presentation.

**Theorem (Frolov, 2006)**

*Suppose that $\tau$ is a computable linear order which has no the greatest and the least elements. If $\mathcal{L}$ is $\mathbf{0}'$-computable linear order then $\tau \cdot \mathcal{L}$ has a computable copy.*

**Theorem (Frolov, 2012)**

*Suppose that $\tau$ is a $\mathbf{0}'$-computable linear order with $\mathbf{0}'$-computable successor relation such that:*
$(\forall x)(\forall n)(\exists x', y')[(x <_\tau x' <_\tau y')\&|[x', y']_\tau| = n]$ *or*
$(\forall x)(\forall n)(\exists x', y')[(x >_\tau x' >_\tau y')\&|[x', y']_\tau| = n]$
*and for any linear order $\mathcal{L}$ if $\tau \cdot \mathcal{L}$ has a low copy then it has a computable copy. If $\mathcal{L}$ has a $\mathbf{0}''$-computable copy then $\tau \cdot \mathcal{L}$ has a computable copy.*

**Corollary (Frolov, 2012)**

*Suppose that* $\tau$ *has one of the following order types:* $\zeta + 1 + \zeta$, $\omega^* + \eta + \zeta$. *A linear order* $\mathcal{L}$ *has* $\mathbf{0}''$*-computable copy if and only if* $\tau \cdot \mathcal{L}$ *has a computable copy.*

# Applications

# Definitions

**Definition**

A computable algebraic structure is called $\Delta^0_\alpha$-categorical if for any two computable copy of it there exists a $\Delta^0_\alpha$-isomorphism between of them.

**Definition**

A computable algebraic structure is called relatively $\Delta^0_\alpha$-categorical if for any two computable copy $\mathcal{L}_1$ and $\mathcal{L}_2$ of it there exists a $\Delta^x_\alpha(\mathcal{L}_1 \oplus \mathcal{L}_2)$-isomorphism between of them.

# Computably and $\Delta_2^0$-categorical linear orders

**Theorem (Goncharov, Dzgoev, 1980; Remmel, 1981)**

*A computable linear order is computably categorical iff it has finitely many successors.*

**Theorem (McCoy, 2003)**

*A computable linear order $\mathcal{L}$ with end points is relatively $\Delta_2^0$-categorical iff $\mathcal{L}$ is a finite sum of finite linear orders, $\zeta$, $\omega$, $\omega^*$, $1 + k \cdot \eta + 1$.*

**Theorem (McCoy, 2003)**

*If a linear order $\mathcal{L}$ has a computable copy with a computable successor relation, and computable left and right limit points then $\mathcal{L}$ is $\Delta_2^0$ categorical iff it is relatively $\Delta_2^0$ categorical.*

# $\Delta_n^0$-categorical linear orders

**Theorem (Ash, 1986)**

*If an ordinal $\alpha$ such that $\omega^{\delta+n} \leq \alpha < \omega^{\delta+n+1}$ then $\alpha$ is $\Delta^0_{\delta+2n}$ categorical, and is not $\Delta^0_\beta$ categorical for any $\beta < \delta + 2n$.*

**Theorem (Bazenov, 2016)**

*If an ordinal $\alpha$ such that $\omega^n \leq \alpha < \omega^{n+1}$ then $\mathbf{0}^{2n-1}$ is the degree of categoricity of $\alpha$.*
*If an ordinal $\alpha$ such that $\omega^{\delta+n} \leq \alpha < \omega^{\delta+n+1}$ $(\delta \geq \omega)$ then $\mathbf{0}^{\delta+2n}$ is the degree of categoricity of $\alpha$.*

# Scattered linear orders

**Theorem (Frolov, Z.,ta)**

*If a scattered linear order $\mathcal{L}$ has VD$^*$-rank $\delta + n$, where $\delta$ is a constructive limit ordinal, n is finite ordinal, then $\mathcal{L}$ is relatively $\Delta^0_{\delta+2n}$ categorical.*

**Theorem (Frolov, Z., ta)**

*For any constructive ordinal $\delta + n \geq 2$, where $\delta$ is a constructive limit ordinal, n is finite ordinal and any $\beta$ such that $3 \leq \beta \leq \delta + 2n$, and $\beta$ is not a successor of a limit ordinal there exist a computable scattered linear order $\mathcal{L}$ with rank $\delta + n$ which is relatively $\Delta^0_\beta$ categorical and is not $\Delta^0_\gamma$ categorical for any $\gamma < \beta$.*

# Proof's sketch

- Suppose that $\nu : \mathbb{Z} \to \mathbb{Z}$ is computable strictly increasing function such that $|\nu| : \mathbb{Z} \to \mathbb{N} \setminus \{0\}$ is 1–1 function with computable range. If for $\alpha + n = 2$ and $\beta = 3$ then the following linear order $\mathcal{L}(\nu) = \sum_{i \in \mathbb{Z}} (\zeta + |\nu(i)|)$ is satisfying the theorem conditions. It is easy to see that $\mathcal{L}(\nu)$ has a "good" copy $\mathcal{L}_g$.
- We define $Inf^{\zeta}_{\mathcal{L}} = \{x \in L \mid |[x]_F| \cong \zeta\}$.
- Lemma. There is a $\mathbf{0}'$-computable structure $\mathcal{L}_2 = \langle L_2; <_{\mathcal{L}_2}, S_{\mathcal{L}_2}, F_{\mathcal{L}_2} \rangle$ such that $Inf^{\zeta}_{\mathcal{L}_2} \in \Delta^0_2$, $\mathcal{L}_g \cong \langle L_2; <_{\mathcal{L}_2} \rangle$ and there is no $\mathbf{0}'$-computable isomorphism between of them.
- We satisfy the following requirements:
  $R_e : \varphi^{\mathbf{0}'}_e(x)$ is not an isomorphism between $\widehat{\mathcal{L}}_g$ and $\widehat{\mathcal{L}}_2$.

# Low linear orders

**Theorem (Frolov, Z., ta)**

*If $\mathcal{L}$ is a low scattered linear order such that Inf$_{\mathcal{L}} \in \Delta_2^0$, then $\mathcal{L}$ has a computable copy via $\mathbf{0}'''$-isomorphism. Moreover,*

- *if each infinite block of the $\mathcal{L}$ has order type $\zeta$ then an isomorphism can be $\mathbf{0}''$-computable;*
- *if each infinite block of the $\mathcal{L}$ has order type $\zeta$ and there exists a finite block between any two different infinite blocks then an isomorphism can be $\mathbf{0}'$-computable.*

*The complexities of all isomorphisms are optimal.*

# Case of a finite rank $n = m + 2 \geq 2$ and $\beta = 2m + 3 \geq 3$

- The desired example would be $\zeta^m \mathcal{L}_g$.
- By relativization, we have that there exists a $\mathbf{0}^{2m}$-computable copy $\widetilde{\mathcal{L}}$ of the linear order $\mathcal{L}_g$ such that there is no $\mathbf{0}^{2m+1}$-computable isomorphism between them.
- Applying Watnick's theorem to the $\widetilde{\mathcal{L}}$, we can construct a sequence of linear orders $\zeta\widetilde{\mathcal{L}}$, $\zeta^2\widetilde{\mathcal{L}}$,..., $\zeta^m\widetilde{\mathcal{L}}$ such that $\zeta^i\widetilde{\mathcal{L}}$ are $\mathbf{0}^{2m-2i}$-computable, and there exists a $\mathbf{0}^{2m-2i}$-computable homomorphism from $\zeta^i\widetilde{\mathcal{L}}$ onto $\zeta^{i-1}\widetilde{\mathcal{L}}$.
- The composition of homomorphisms is a $\mathbf{0}^{2m}$-computable homomorphism from $\zeta^m\widetilde{\mathcal{L}}$ onto $\widetilde{\mathcal{L}}$. Consequently, there is no $\mathbf{0}^{2m}$-computable isomorphism between the "good" copy of $\zeta^m\mathcal{L}_g$ and the linear order $\zeta^m\widetilde{\mathcal{L}}$.

# Bi-embeddable categoricity

## Definition (Fokina, Rossegger, San Mauro, 2018)

Let $\mathbf{d}$ be a Turing degree. A computable structure $\mathcal{S}$ is $\mathbf{d}$-computably bi-embeddably categorical if for any computable structure $\mathcal{A} \approx \mathcal{S}$, there are $d$-computable isomorphic embeddings $f : \mathcal{A} \hookrightarrow S$ and $g : \mathcal{S} \hookrightarrow \mathcal{A}$. The bi-embeddable categoricity spectrum of $\mathcal{S}$ is the set
$CatSpec_{\approx}(\mathcal{S}) = \{\mathbf{d} \mid \mathcal{S} \text{ is } \mathbf{d}\text{-computably bi-embeddably categorical}\}$.
A degree $\mathbf{c}$ is the degree of bi-embeddable categoricity of $S$ if $\mathbf{c}$ is the least degree in the spectrum $CatSpec_{\approx}(\mathcal{S})$.

## Definition

A computable structure $\mathcal{S}$ is $\Delta^0_\alpha$-bi-embeddably categorical if for any computable structure $\mathcal{A} \approx \mathcal{S}$, there are $\Delta^0_\alpha$-isomorphic embeddings $f : \mathcal{A} \hookrightarrow \mathcal{S}$ and $g : \mathcal{S} \hookrightarrow \mathcal{A}$.

# Non scattered linear orders

**Theorem (Cantor)**

*Every countable linear order embeddable to a linear order of type $\eta$.*

If two linear order have suborders of type $\eta$ then they are bi-embeddable to each other.

**Theorem (Handy; Harrison)**

*The linear order $\omega_1^{CK}(1+\eta)$ with no infinite hyperarithmetic descending chains has a computable copy.*

# Levels of bi-embeddable categoricity

**Theorem (Bazhenov, Rossegger, Z., 2022)**

*Suppose that $\mathcal{L}$ is a computable scattered linear order of a finite $VD^*$-rank $n$. Then it is $\Delta^0_{2n}$-bi-embeddably categorical.*

**Theorem (Bazhenov, Rossegger, Z., 2022)**

*Suppose that $\mathcal{L}$ is a computable scattered linear order of a finite rank $n$. Then $\mathcal{L}$ is not $\Delta^0_{2n-1}$-bi-embeddably categorical.*

# Proof's sketch

- Suppose that $\mathcal{L}$ is indecomposable of rank $n+1$, there is an h-indecomposable linear order together with its signed tree $T$ of rank $n+1$. Given $T$ of rank $n+1$, let $\sigma \in T$ of length $n$ and let $P(\sigma)$ be the tree $\{\tau : \tau \preceq \sigma\}$ with sign function inherited from $T$. WLOG assume that $s_T(\emptyset) = +$. We define $\mathcal{G} = \sum_{i\in\omega}\left(\sum_{j\leq i} lin(T_{\langle j\rangle}) + lin(P(\sigma))\right)$. $\mathcal{L}$ and $\mathcal{G}$ are bi-embeddable.
- Let $\mathcal{G}^n$ be a standard copy of $\omega$ with the elements labelled by the nodes of the tree of height 1, i.e., we can write $\mathcal{G}^n$ as

$$\mathcal{G}^n = t_0 + g_0 + t_{0,1} + t_{1,1} + g_1 + t_{0,2} + t_{1,2} + t_{2,2} + g_2 + \dots .$$

Clearly we can take $\mathcal{G}$ such that there is a computable function $\psi : \mathcal{G} \to \mathcal{G}^n$ taking $x$ in the $i^{th}$ copy of $lin(T_{\langle j\rangle})$ in $\mathcal{G}$ to $t_{j,i}$ and $x$ in the $i^{th}$ copy of $lin(P(\sigma))$ in $\mathcal{G}$ to $g_i$.

# A construction of a $\Delta^0_{2n+1}$-computable linear order

- We want to build $\mathcal{B}$ such that no embedding $\mathcal{B} \hookrightarrow \mathcal{G}$ has degree $\Delta^0_{2n+1}$. We construct $\mathcal{B}^n$ in stages. At stage 0, $\mathcal{B}^n$ is $\mathcal{G}^n$ with the difference that every element $g_i$ is replaced by successive elements $b_{i,1}, b_{i,2}$, i.e., we can write $\mathcal{B}^n$ as

$$\mathcal{B}^n = t_0 + b_{0,1} + b_{0,2} + t_{0,1} + t_{1,1} + b_{1,1} + b_{1,2} + t_{0,2} + \dots.$$

- We satisfy the requirements

$$R_e : \quad \varphi_e : \mathcal{B}^n \not\hookrightarrow \mathcal{G}^n.$$

- At stage $s$, if for $e < s$, $s$ is the first stage greater than $e$ such that $\varphi_{e,s}(b_{e,2}){\downarrow} = x$ for some $x$ and $g_k$ is least such that $x \geq g_k$, then put elements into $(b_{e,1}, b_{e,2})$ such that $|(b_{e,1}, b_{e,2})| > |[t_0, x]|$.

# A construction of a computable linear order

- We use Ash-Knight theorem.
  - We replace every interval $[b_{i,1}, b_{i,2}]$ of $\mathcal{B}^n$ by a copy of $\omega \cdot [b_{i,1}, b_{i,2}]$ if $\sigma(0) = +$ and by a copy of $\omega^* \cdot [b_{i,1}, b_{i,2}]$ otherwise.
  - Elements labelled $t_{i,j}$ are replaced by computable disjoint copies of $T_{\langle i \rangle}$. We obtain a linear order $\mathcal{B}^{n-1}$ which is $\Delta^0_{2n-1}$.
  - We repeat this procedure, replacing $\mathcal{B}^j_i$ with $\omega \cdot \mathcal{B}^j_i$ or $\omega^* \cdot \mathcal{B}^j_i$ depending on whether $\sigma(j) = +$ or $\sigma(j) = -$.
  - By induction we end up with a linear order $\mathcal{B}^0 = \mathcal{B}$ which is computable, and it is easy to see that $\mathcal{B} \approx \mathcal{G}$.
- Our construction of $\mathcal{B}^i$ from $\mathcal{B}^{i+1}$ also provides us with $\Delta^0_i$ computable embeddings $\varphi_i : \mathcal{B}^{i+1} \to \mathcal{B}^i$. Assume that there is a $\Delta^0_{2n+1}$ embedding $\chi$ of $\mathcal{B}$ into $\mathcal{G}$, then the composition of the embeddings $\varphi_i$ and $\chi$ gives a $\Delta^0_{2n+1}$ embedding of $\mathcal{B}^n$ into $\mathcal{G}^n$, a contradiction.

# Definable sets

- Let $L$ be a computable signature, and let $\mathcal{S}$ be a computable $L$-structure. Consider a first-order $L$-formula $\psi(x)$ without parameters. Recall that a subset $A \subseteq \mathrm{dom}(\mathcal{S})$ is *definable* by the formula $\psi$ if

$$A = \{a \in \mathrm{dom}(\mathcal{S}) : \mathcal{S} \models \psi(a)\}.$$

  In this case, we write $A = \psi[\mathcal{S}]$.
- For a non-zero natural number $n$, by $\Sigma^0_n(\mathcal{S})$ we denote the family of all unary relations that are definable inside $\mathcal{S}$ by finitary $\Sigma^0_n$-formulas without parameters.

# Friedberg numberings

- Let $\mathcal{F}$ be a family of c.e. sets. A *computable numbering* of the family $\mathcal{F}$ is a map $\nu$ from the set of natural numbers $\omega$ onto $\mathcal{F}$ such that the set $\{(k, x) : x \in \nu(k)\}$ is computably enumerable. In other words, a computable numbering $\nu$ gives a uniform enumeration of the family $\mathcal{F}$.
- There is an injective computable numbering $\nu$ of the family of all c.e. sets (i.e., $\nu(k) \neq \nu(\ell)$ for all $k \neq \ell$).
- There is a computable list $\{\psi_i(x)\}_{i \in \omega}$ of $\Sigma^0_1$-formulas that lists all $\Sigma^0_1$-definable unary relations in $(\mathbb{N}; +, \times, \leq, 0, 1)$ without repetitions.

# Classifications

## Definition

Let $\mathbf{d}$ be a Turing degree. A computable structure $\mathcal{S}$ has a $\mathbf{d}$-computable $\Sigma^0_n$-classification if there exists a $\mathbf{d}$-computable list $\{\psi_i(x)\}_{i\in\omega}$ of $\Sigma^0_n$-formulas without parameters such that:

(a) For any subset $A \subseteq \mathrm{dom}(\mathcal{S})$ $\Sigma^0_n$-definable (without parameters), there is an index $i \in \omega$ such that the formula $\psi_i$ defines the set $A$ inside $\mathcal{S}$.

(b) If $i \neq j$, then the sets $\psi_i[\mathcal{S}]$ and $\psi_j[\mathcal{S}]$ are not equal.

Informally speaking, a $\mathbf{d}$-computable $\Sigma^0_n$-classification lists the class $\Sigma^0_n(\mathcal{S})$ without repetitions.

In case $\mathbf{d} = \mathbf{0}$, we often just say that $\mathcal{S}$ has a $\Sigma^0_n$-classification.

# $\Sigma^0_1$- and $\Sigma^0_2$-classifications

- [Goncharov and Kogabaev, 2008] There is a computable structure $\mathcal{S}$ in an infinite signature $L$ such that $\mathcal{S}$ has no computable $\Sigma^0_1$-classification.
- [Boyadzhiyska, Lange, Raz, Scanlon, Wallbaum, and Zhang, 2019] There is a computable equivalence structure $\mathcal{E}$ such that $\mathcal{E}$ admits a $\Sigma^0_2$-classification, but has no $\Sigma^0_1$-classification.
- [Boyadzhiyska, Lange, Raz, Scanlon, Wallbaum, and Zhang, 2019] On the other hand, every unbounded computable injection structure has both $\Sigma^0_1$-classification and $\Sigma^0_2$-classification.

# $\Sigma^0_n$-classifications

**Proposition**

*Let n be a non-zero natural number. Suppose that a computable structure $\mathcal{S}$ has infinitely many subsets that are definable by $\Sigma^0_n$-formulas without parameters. Then $\mathcal{S}$ admits a $\mathbf{0}^{(n)}$-computable $\Sigma^0_n$-classification.*

**Theorem (Aleksandrova, Bazhenov, Z., 2022)**

*For every non-zero n, there exists a computable structure $\mathcal{A}$ with the following properties:*

(i) *$\mathcal{A}$ has infinitely many subsets that are definable by $\Sigma^0_n$-formulas without parameters, and*

(ii) *$\mathcal{A}$ does not admit a $\mathbf{0}^{(n-1)}$-computable $\Sigma^0_n$-classification.*

# Proof sketch. Odd case

- We satisfy the following requirements:

  $\mathcal{R}_e$: The list $\{\xi_{e,i}(x)\}_{i\in\omega}$ is not a $\Sigma^0_{2m+1}$-classification for the structure $\mathcal{A}$.

- We build a $\mathbf{0}^{(2m)}$-computable partial order $\mathcal{B}$. The poset $\mathcal{B}$ contains only *finite* linearly ordered components.
- For each $e \in \omega$, we assign two components inside $\mathcal{B}$ to the number $e$. As before, they are called $\mathcal{C}_e$ and $\mathcal{D}_e$, and we have

$$\mathcal{C}_e \cong \mathbf{2e+1}, \quad \mathcal{D}_e \cong \mathbf{2e+2}.$$

  The component $\mathcal{D}_e$ never changes, and $\mathcal{C}_e$ can only grow once — by adding a fresh element to the right end of $\mathcal{C}_e$.

# Proof sketch. Odd case

- If we find two formulas defining $\{2e+1\}$ and $\{2e+2\}$ then then extend $\mathcal{C}_e$ by adding a fresh element. This procedure can be arranged effectively with the oracle $\mathbf{0}^{(2m)}$.
- [Ash-Knight] theorem (for linear orders with a least element) implies that the poset $\omega^m \cdot B$ has a computable copy. Let $A$ be such a computable copy.
- $A$ is the required structure.

# Proof sketch. Even case

- The construction of $B$ is the same but with the oracle $\mathbf{0}^{(2m+1)}$.
- [Downey-Knight] theorem (for linear orders with a least element) implies that the poset $(\eta + 2 + \eta) \cdot B$ has a $\mathbf{0}^{(2m)}$-computable copy.
- [Ash-Knight] theorem (for linear orders with a least element) implies that the poset $\omega^m \cdot (\eta + 2 + \eta) \cdot B$ has a computable copy. Let $A$ be such a computable copy.
- Lemma. Let $m$ be a natural number. The theory $Th(\omega^m \cdot (\eta + 2 + \eta) \cdot \omega)$ is decidable.

# Thanks!