Computability-Theoretic Reduction Games in Reverse Mathematics

Denis R. Hirschfeldt

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Stronger systems include ACA_0, which is RCA_0 plus arithmetical comprehension.

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We write $P \leqslant_\omega Q$ if every ω-model of $\mathsf{RCA}_0 + Q$ is a model of P.

If $\mathsf{RCA}_0 \vdash Q \to P$ then $P \leqslant_\omega Q$, but not always vice-versa.

Ramsey's Theorem

$[X]^n$ is the set of n-element subsets of X.

A k-coloring of $[X]^n$ is a map $c : [X]^n \to k$.

$H \subseteq X$ is homogeneous for c if $|c([H]^n)| = 1$.

RT^n_k: Every k-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.

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Thm (Hirst). $\mathsf{RT}^1 \leqslant_\omega \mathsf{RT}^1_2$, but $\mathsf{RCA}_0 \nvdash \mathsf{RT}^1_2 \to \mathsf{RT}^1$.

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We assume that our problems always have Δ^0_1 instances.

Reduction Games

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Let P and Q be Π^1_2 problems, and let $\mathfrak{M}$ be a family of models in the language of second-order arithmetic with countable first-order parts.

We describe the two-player reduction game $G^{\mathfrak{M}}(Q \to P)$.

Player 1 challenges Player 2 to show that P is reducible to Q over the models in $\mathfrak{M}$, in a computable (Δ^0_1-definable) way.

Player 1 will play a P-instance X_0 in some model in $\mathfrak{M}$.

Player 2 will try to obtain a solution to X_0 by asking Player 1 to solve various Q-instances.

If Player 2 ever plays such a solution, it wins, and the game ends.

If the game never ends then Player 1 wins.

If a player cannot make a move, the opponent wins.

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For a model $\mathcal{N}$ of first-order arithmetic and $X_0, \ldots, X_n \subseteq |\mathcal{N}|$, let $\mathcal{N}[X_0, \ldots, X_n] = (\mathcal{N}, S)$ where S consists of all subsets of $|\mathcal{N}|$ that are Δ^0_1-definable from parameters in $|\mathcal{N}| \cup \{X_0, \ldots, X_n\}$.

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Second Move:

Player 1: A solution X_1 to Y_1 in S.

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Third Move:

Player 1: A solution X_2 to Y_2 in S.

Player 2: Either a solution to X_0 in $\mathcal{N}[X_0, X_1, X_2]$, or a Q-instance $Y_3 \in \mathcal{N}[X_0, X_1, X_2]$.

$\vdots$

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P is computably reducible to Q (written $P \leqslant_c Q$) if Player 2 has a winning strategy for $G(Q \to P)$ ensuring victory by its second move.

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P is generalized Weihrauch reducible to Q (written $P \leqslant_{\mathrm{gW}} Q$) if Player 2 has a computable winning strategy for $G(Q \to P)$.

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$$n + (j-1)(n-2) < m \leqslant n + j(n-2).$$

Then

$$\mathsf{RT}^m_k \leqslant^{j+1}_{\mathrm{gW}} \mathsf{RT}^n_k \quad \text{but} \quad \mathsf{RT}^m_k \not\leqslant^{j}_{\omega} \mathsf{RT}^n_k.$$

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Patey determined the least m s.t. $\mathsf{RT}^n_k \leqslant_\omega^m \mathsf{RT}^n_j$ for $n \geqslant 2$ and $j < k$.

For $n \geqslant 3$, this m is always 2. For $n = 2$ it is more complicated and goes to infinity as k increases.

We can also fix a weak base theory Γ, e.g. RCA_0, and consider the game $G^{\mathfrak{M}}(Q \to P)$ where $\mathfrak{M}$ is the family of models of Γ with countable first-order parts.

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But we can also define $P \leqslant_{\mathrm{c}}^{\Gamma} Q$, $P \leqslant_{\mathrm{W}}^{\Gamma} Q$, and $P \leqslant_{\mathrm{gW}}^{\Gamma} Q$, as well as the instance-counting versions $\Gamma \vdash^{n} Q \to P$ and $P \leqslant_{\mathrm{gW}}^{\Gamma,n} Q$.

It is possible to have $P \leqslant_\omega Q$ but not have $P \leqslant_\omega^n Q$ for any n, and similarly for gW.

For example, $\mathsf{RT} \leqslant_{\mathrm{gW}} \mathsf{RT}_2^3$ but $\mathsf{RT} \nleqslant_\omega^n \mathsf{RT}_2^3$ for all n.

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It also follows that $\mathsf{ACA}_0 \nvdash \mathsf{RT}$, though $\mathsf{ACA}_0 \vdash \mathsf{RT}^n$ for each n.

Indeed, as noted by Wang, if

$$\mathsf{ACA}_0 \vdash \forall X\, [\varphi(X) \rightarrow \exists Y\, \psi(X,Y)]$$

where φ and ψ are arithmetic, then there is an $n \in \omega$ s.t.

$$\mathsf{ACA}_0 \vdash \forall X\, [\varphi(X) \rightarrow \exists Y \in \Sigma^{0,X}_n\, \psi(X,Y)].$$

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Thm (Cholak, Jockusch, and Slaman). $\mathrm{RCA}_0 + \mathrm{RT}^2_2 \nvdash \mathrm{RT}^2$.

Using results of Patey and the first theorem above, we have:

Thm (Dzhafarov, Hirschfeldt, and Reitzes / Slaman and Yokoyama). Let Γ be RCA_0 together with all Π^1_1 formulas true in the natural numbers. Then $\Gamma + \mathrm{RT}^2_2 \nvdash \mathrm{RT}^2$.

$\mathsf{I}\Sigma^0_n$ is the axiom scheme

$$(\varphi(0) \land \forall n\,[\varphi(n) \rightarrow \varphi(n+1)]) \rightarrow \forall n\,\varphi(n)$$

for Σ^0_n formulas φ, and similarly for $\mathsf{I}\Pi^0_n$.

$\mathsf{I}\Delta^0_n$ is the axiom scheme

$$\forall n\,[\varphi(n) \leftrightarrow \psi(n)] \rightarrow (\varphi(0) \land \forall n\,[\varphi(n) \rightarrow \varphi(n+1)]) \rightarrow \forall n\,\varphi(n)$$

for Σ^0_n formulas φ and Π^0_n formulas ψ.

$\mathsf{B}\Sigma^0_n$ is the axiom scheme

$$\forall k\,[\forall n < k\,\exists i\,\varphi(n,i) \rightarrow \exists b\,\forall n < k\,\exists i < b\,\varphi(n,i)]$$

for Σ^0_n formulas φ, and similarly for $\mathsf{B}\Pi^0_n$.

Thm (Paris and Kirby / Slaman). Over RCA_0 (or RCA_0^*), we have the following strict implications and equivalences:

$$\begin{array}{c} \vdots \\ \downarrow \\ \mathsf{B}\Sigma^0_3 \equiv \mathsf{B}\Pi^0_2 \equiv \mathsf{I}\Delta^0_3 \\ \downarrow \\ \mathsf{I}\Sigma^0_2 \equiv \mathsf{I}\Pi^0_2 \\ \downarrow \\ \mathsf{B}\Sigma^0_2 \equiv \mathsf{B}\Pi^0_1 \equiv \mathsf{I}\Delta^0_2 \\ \downarrow \\ \mathsf{I}\Sigma^0_1 \equiv \mathsf{I}\Pi^0_1 \end{array}$$

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A Π^1_2 problem is first-order if its solutions are natural numbers.

These levels of induction have several first-order equivalents.

Bound*: For any simultaneous enumeration of bounded sets $F_0, \ldots, F_n$, there is a common bound for the F_i's.

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Thm (Dzhafarov, Hirschfeldt, and Reitzes). Bound$^* <_{\mathrm{gW}} \mathsf{RT}^1$, and in fact Bound$^* <_{\mathrm{gW}}^{\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2} \mathsf{RT}^1$, but Bound$^* \mid_{\mathrm{gW}}^{\mathsf{RCA}_0} \mathsf{RT}^1$.

In the following theorem, all instances are in the standard model.

Thm (Reitzes). Let P and Q be first-order and s.t.:

1. There is a computable way to determine, given a P-instance X, a number k such that X has a solution bounded by k.
2. There is an infinite tree $S \subseteq 2^{<\omega}$ s.t. for each $\sigma \in S$ and each k, there is a path on S extending σ that is a Q-instance with no solution bounded by k.

Then $Q \nleqslant^{n}_{\mathrm{gW}} P$ for all n, so for any weak base theory Γ, we have $Q \nleqslant^{\Gamma,n}_{\mathrm{gW}} P$ for all n, and hence $Q \nleqslant^{\Gamma}_{\mathrm{gW}} P$.

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Then $Q \not\leq^n_{\mathrm{gW}} P$ for all n, so for any weak base theory Γ, we have $Q \not\leq^{\Gamma,n}_{\mathrm{gW}} P$ for all n, and hence $Q \not\leq^{\Gamma}_{\mathrm{gW}} P$.

For example, we can take $P \equiv \mathsf{RT}^1$ and $Q \equiv \mathsf{Bound}^*$.

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$\mathsf{F}\Pi^0_1$: An instance is an enumeration of the complement of $A \subsetneq \mathbb{N}$. A solution is an $n \notin A$ s.t. either $n = 0$ or $n - 1 \in A$.

Both of these are equivalent to $\mathsf{I}\Sigma^0_1$ over RCA^*_0.

Thm (Reitzes). $\mathsf{F}\Pi^0_1 \leqslant^{\mathsf{RCA}^*_0}_{\mathsf{W}} \mathsf{F}\Sigma^0_1$.

Levels of induction have several equivalents, e.g. least number principles and bounded comprehension.

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Thm (Reitzes). $\mathsf{F}\Pi^0_1 \leqslant^{\mathsf{RCA}^*_0}_{\mathsf{W}} \mathsf{F}\Sigma^0_1$.

However, $\mathsf{F}\Pi^0_1$ is weak in the sense of the previous theorem, while $\mathsf{F}\Sigma^0_1$ is strong, so we have:

Thm (Reitzes). $\mathsf{F}\Sigma^0_1 \nleqslant^{\Gamma}_{\mathsf{gW}} \mathsf{F}\Pi^0_1$ for any weak base theory Γ.

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