

Some applications of recursion theory to geometric measure theory

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Let AD be the axiom of determinacy.

Definition

Strong Turing determinacy (sTD) says that for every set A of *reals* ranging Turing degrees cofinally, A has a pointed subset.

Theorem (Martin)

Over ZF, $AD \rightarrow sTD$.

Geometric measure theory (1)

Given a non-empty $U \subseteq \mathbb{R}$, the *diameter* of U is

$$\text{diam}(U) = |U| = \sup\{|x - y| : x, y \in U\}.$$

Given any set $E \subseteq \mathbb{R}$ and $d \geq 0$, let

$$\mathcal{H}^d(E) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_{i < \omega} |U_i|^d : \{U_i\} \text{ is an open cover of } E \wedge \forall i |U_i| < \delta \right\},$$

$\mathcal{P}_0^d(E) = \lim_{\delta \rightarrow 0} \sup \{ \sum_{i < \omega} |B_i|^d : \{B_i\} \text{ is a collection of disjoint balls of radii at most } \delta \text{ with centres in } E \}$
and

$$\mathcal{P}^d(E) = \inf \left\{ \sum_{i < \omega} \mathcal{P}_0^d(E_i) \mid E \subseteq \bigcup_{i < \omega} E_i \right\}.$$

Geometric measure theory (2)

Definition

Given any set E ,

- the *Hausdorff dimension* of E , or $\text{Dim}_H(E)$, is

$$\inf\{d \mid \mathcal{H}^d(E) = 0\};$$

- the *Packing dimension* of E , or $\text{Dim}_P(E)$, is

$$\inf\{d \mid \mathcal{P}^d(E) = 0\}.$$

Besicovitch and Davis theorem

Theorem (Besicovitch and Davis)

For any analytic set A , $\dim_H(A) = \sup_{F \subseteq A \wedge F \text{ is closed}} \dim_H(F)$.

Point-to-set principle and its applications

Let $\dim_H^x(y) = \underline{\lim}_{n \rightarrow \infty} \frac{K^x(y \upharpoonright n)}{n}$.

Theorem (Lutz and Lutz)

For any set A of reals, $\text{Dim}_H(A) = \min_x \max_{r \in A} \dim_H^x(r)$.

Theorem (Slaman)

Assume that $V = L$, then BD -theorem fails for a Π_1^1 -set.

One may slightly weaken the assumption to be " $(\mathbb{R})^L$ is not null".

Low for Hausdorff dimension

Theorem (Lempp, Miller, Ng, Turetsky and Weber)

For any real x , there is a real y low for Hausdorff dimension but $y' \geq_T x$.

BD-theorem under AD

Theorem (Peng, Wu and Y; Crone, Fishman and Jackson proves the consequence under $ZF + DC + AD$.)

Assume that $ZF + sTD$, BD-theorem holds for every set of reals.

Proof.

Fix any nonempty set A . For the simplicity, we may assume that $\text{Dim}_H(A) = 1$.

By the results above, there is some e so that $B = \{x \mid \Phi_e^x \in A \text{ has effective Hausdorff dimension 1 relative to } x\}$ ranges Turing degrees cofinally. By sTD , B has a pointed subset P . Then $C = \{r \mid \exists x \in P \Phi_e^x = r\}$ is an analytic subset of A with Hausdorff dimension 1. □

More results

Theorem (Joyce and Preiss)

For any analytic set A , $\text{Dim}_P(A) = \sup_{F \subseteq A \wedge F \text{ is closed}} \text{Dim}_P(F)$.

By a similar method, one may show that Joyce-Preiss theorem holds for arbitrary set under $ZF + sTD$. Note that Slaman's result holds for the packing dimension.

Some questions

Question

- ① *What is the consistency strength of BD- and JP-theorems for arbitrary sets?*
- ② *What is the consistency strength that every set of Turing degrees is measurable?*

Lutz-Lutz's theorem revisited (1)

Theorem

$\mathcal{H}^d(A) = 0$ if and only if there is a real x and a constant c so that $\forall z \in A \exists^\infty n (K^x(z \upharpoonright n) \leq dn + c)$.

Proof.

If direction, by Chaitin's counting theorem
($|\{\sigma \in 2^n \mid K(\sigma) \leq n + K(n) - d\}| \leq 2^{n-d+c_0}$).
Only if direction, by Chaitin-Kraft theorem. □

Two functions

Let

$$L_0(k) = \min\{n \mid \forall m \geq n, K(m) \geq k\}$$

$$L_1(k) = \min\{n \mid \sum_{m \geq n} 2^{-K(m)} \leq 2^{-k}\}.$$

Lemma

$$\exists c \forall k (L_1(k) \leq L_0(k + c)).$$

Proof.

Clearly $L_1(k) \geq L_0(k)$. Suppose that $\forall c \exists k L_1(k) - L_0(k) > c$. By the recursion theorem, there is a prefix-pre machine M_e : for any k , if $K(m) < k + e + 1$, then $\sum_{m' > m} 2^{-K(m')} > 2^{-k}$. Define $K_M(m) = k$. So $L_0(k + e) \geq L_0^M(k) \geq m \geq L_0(k + e + 1)$, a contradiction. □

Lutz-Lutz's theorem revisited (2)

Theorem

$\mathcal{H}^d(A) < \infty$ if and only if there is some c so that
 $\exists x \exists^\infty k \forall z \in A \exists n > L_0^x(k) (K^x(z \upharpoonright n) < dn + k + c).$

Proof.

If direction, by Chaitin's counting theorem and the Lemma.
Only if direction, by building a machine. □

Another Besicovitch-Davis theorem

A set A is σ -finite for \mathcal{H}^d if $A = \bigcup_n A_n$ and $\forall n (\mathcal{H}^d(A_n) < \infty)$.

Theorem (Besicovitch-Davis)

If a Σ_1^1 set A is not σ -finite for \mathcal{H}^d , then A has a compact subset that is not σ -finite.

On Π_1^1 -sets

For a real x , let ω_1^x be the least non- x -cursive ordinal.

Theorem (Spector, Gandy)

A set A is Π_1^1 if and only if there is a Σ_1 formula φ so that

$$x \in A \Leftrightarrow L_{\omega_1^x}[x] \models \varphi(x).$$

So a Π_1^1 -set can be viewed as a recursively enumerable set.

On thin sets

A set A of reals is *thin* if A has no perfect subset.
For a real x , let ω_1^x be the least non- x -cursive ordinal.

Theorem (Mansfield, Solovay)

The set $\mathcal{C} = \{x \mid x \in L_{\omega_1^x}\}$ is the largest Π_1^1 -thin set.

So every “master code” belongs to \mathcal{C} .

Slaman's result

Theorem (Slaman)

If $V = L$, then there is a thin Π_1^1 set A that is not σ -finite for \mathcal{H}^d .

Proof.

Slaman's original proof is a combination of two sophisticated methods from geometric measure theory and set theory.

Let $a_n = 2^{\sum_{m < n} a_m}$. Use the sequence partition ω . For any master code real x , there is some $y \equiv_T x'$ so that for any n ,

- If $i \in [a_n, 2a_n - \log a_n)$, then $y(i) = 0$;
- If $i = a_n - \log a_n$, then $y(i) = x'(n)$;
- If $i \in [2a_n - \log a_n, a_{n+1})$, then $y(i) = \Omega^x(i)$.

Then the collection of such y 's is the required set. □

谢谢