Well orderings of graphs

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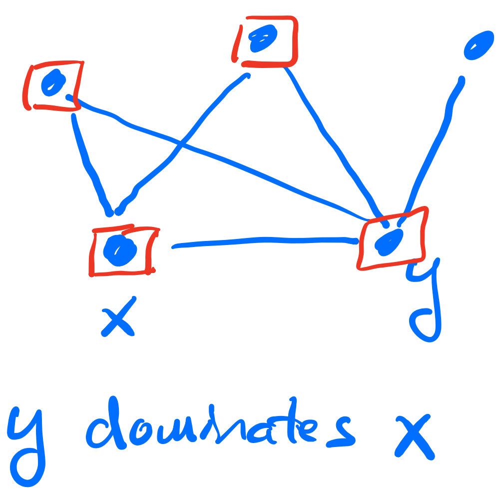
Neighbor sets and dominating nodes

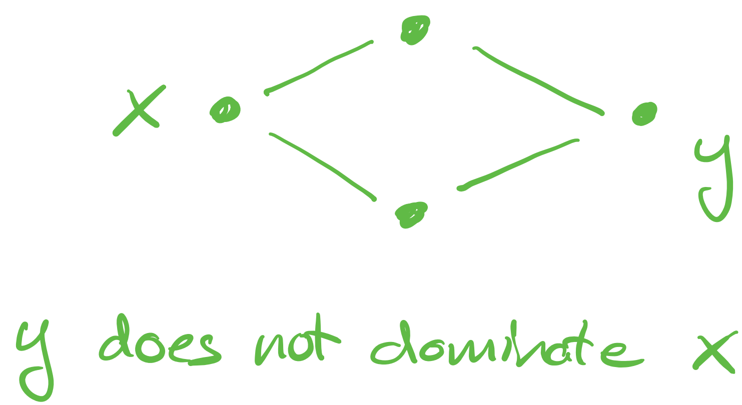
Let G be a (undirected, reflexive, connected and countable) graph. The **neighbors** of $x \in G$ are the nodes connected to x:

$$N_G[x] = \{v \in G \mid E(x, v)\}$$

We say y **dominates** x if $y \neq x$ and $N_G[x] \subseteq N_G[y]$.

Because E is reflexive, $x \in N_G[x]$. So, if y dominates x, then there is an edge between x and y.





Let $\leq$ be an ordering of the vertices of G. For $x \in G$,

$$G_{\leq x} = \{v \in G \mid v \leq x\} \quad \text{and} \quad G_{\geq x} = \{v \in G \mid v \geq x\}$$

as induced subgraphs.

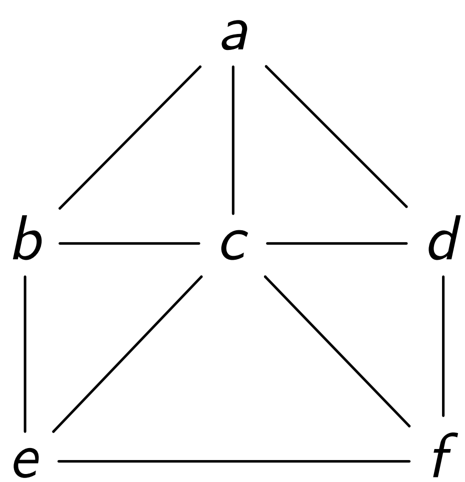
There are two natural well orders to put on a graph G. Both are used in graph theory in numerous settings.

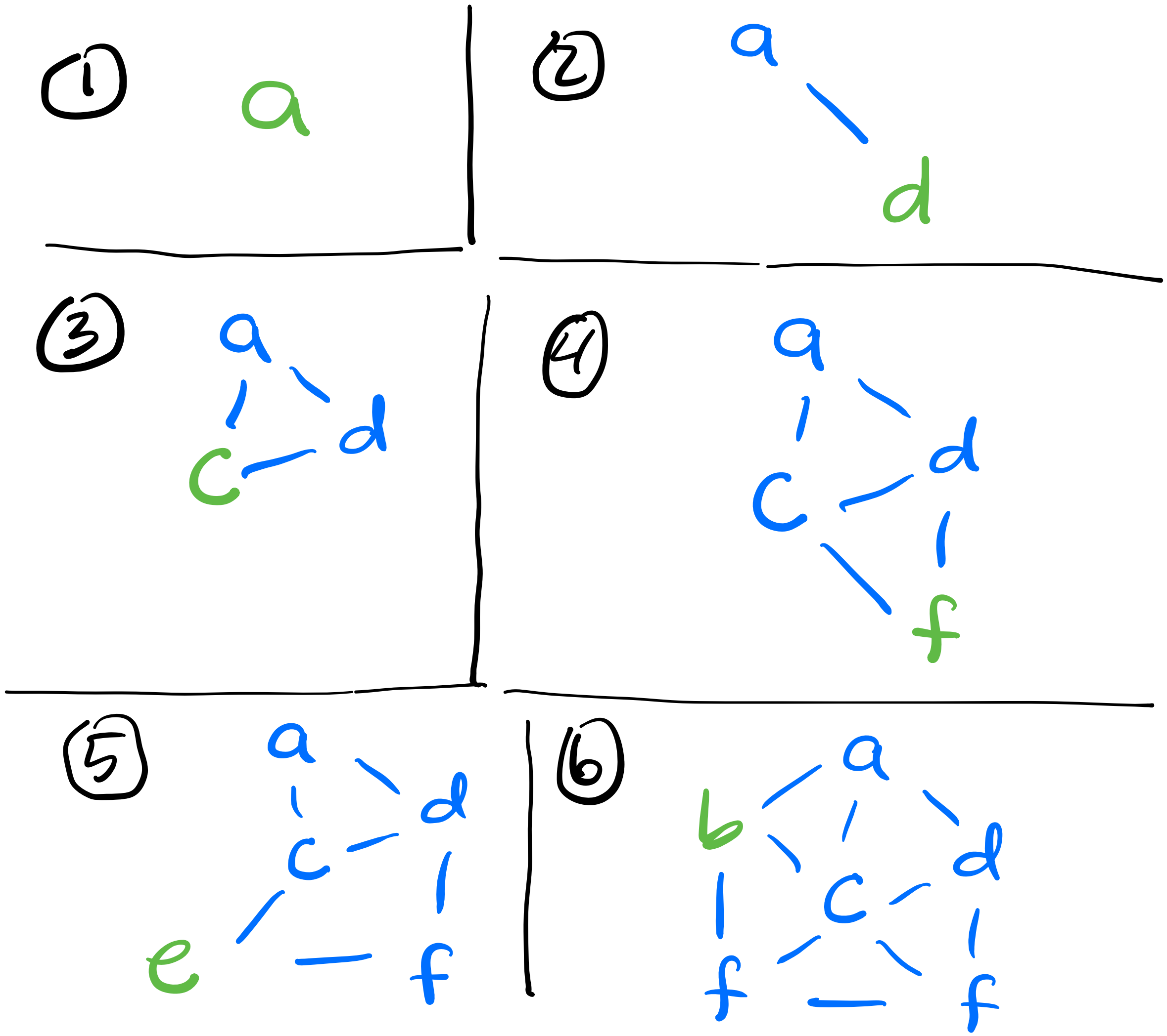
- A **constructible order** of G is a well order $\leq$ of G such that every vertex x is dominated in $G_{\leq x}$ (except the least vertex).
  - used in Helley graphs, bridged graphs, vertex pursuit games.
- A **dismantling order** of G is a well order $\leq$ of G such that every vertex x is dominated in $G_{\geq x}$ (except the greatest vertex).
  - used in constraint satisfactions problems, invariant subgraph and convexity properties.

A **constructible order** of G is a well order $\leq$ of G such that every vertex x is dominated in $G_{\leq x}$.

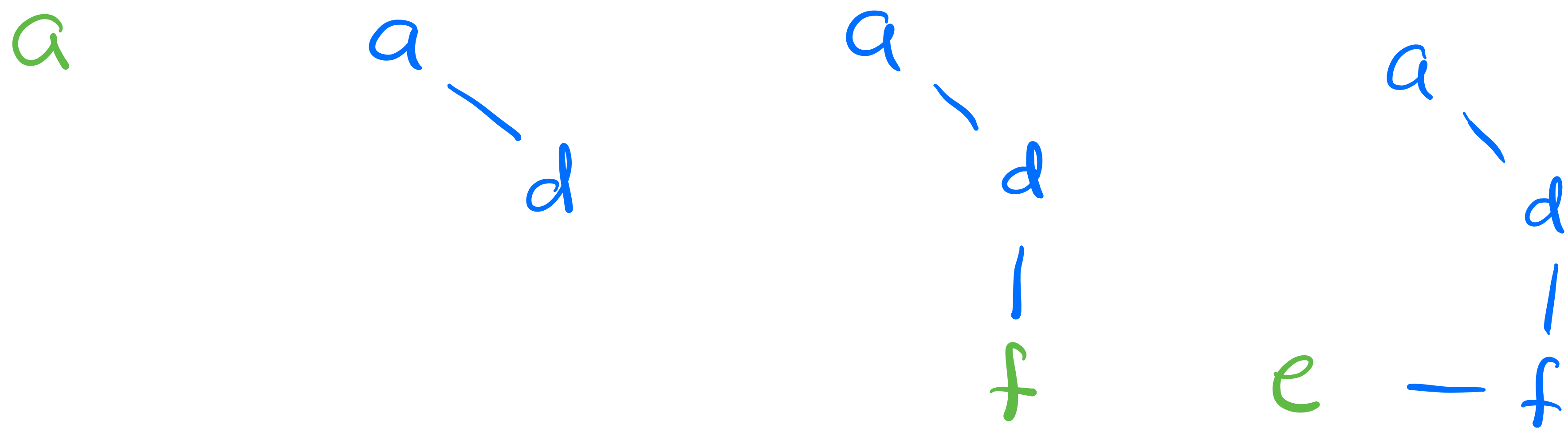
Idea: construct G one node at a time such that each node is dominated when it is added.

Example 1.

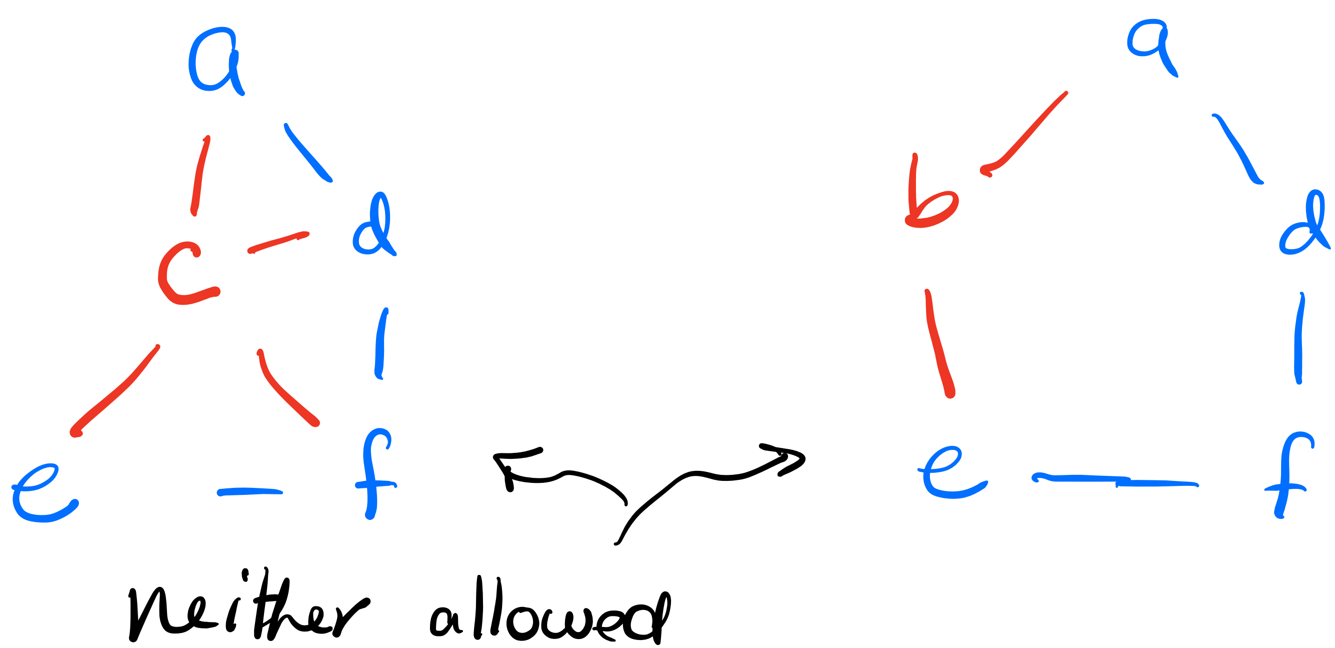




$a < d < c < f < e < b$



Stuck:



Example 2.

$$v_0 \text{ —— } v_1 \text{ —— } v_2 \text{ —— } v_3 \text{ —— } v_4 \text{ —— } v_5 \text{ —— } v_6 \text{ —— } \cdots$$

has a constructible order $v_0 < v_1 < v_2 < \cdots$

$$v_0 \qquad \text{(no need to dominate } v_0\text{)}$$

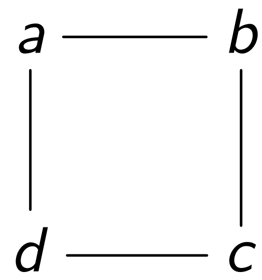
$$v_0 \text{ —— } v_1 \qquad (v_1 \text{ is dominated by } v_0)$$

$$v_0 \text{ —— } v_1 \text{ —— } v_2 \qquad (v_2 \text{ is dominated by } v_1)$$

and so on.

Note: There are other constructible orders on G.

Example 3. For finite G, the last element in a constructible order must be dominated in G.



has no constructible order because no vertex is dominated in G, e.g.

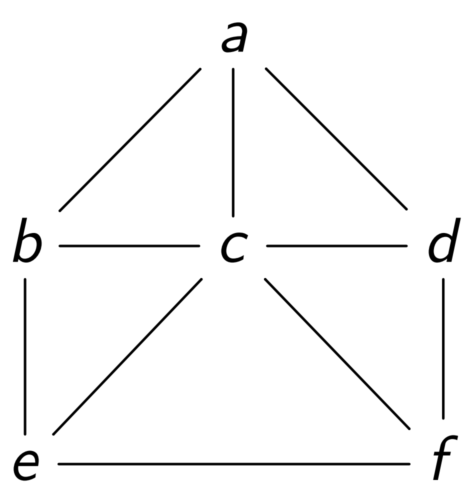
$$N_G[d] = \{a, c, d\}$$

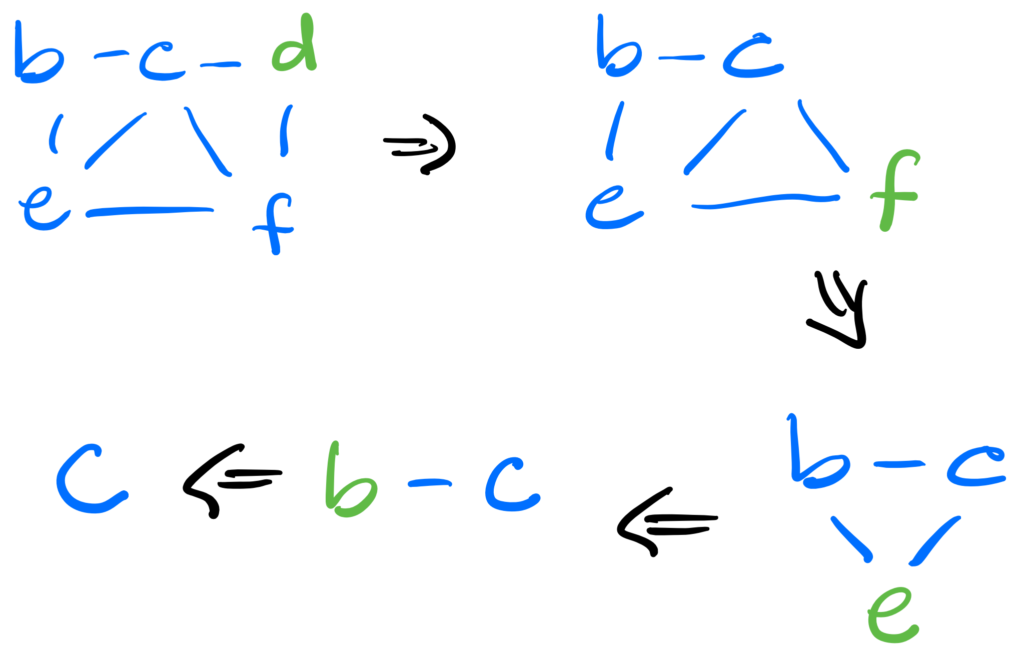
A **dismantling order** of G is a well order $\leq$ of G such that every vertex x is dominated in $G_{\geq x}$.

Idea: take G apart one node at a time such that each node is dominated when it is removed.

$a < d < f < e < b < c$

Example 1.





Note: It can matter which order dominated nodes are removed in.

Example 2.

$$v_0 \text{ --- } v_1 \text{ --- } v_2 \text{ --- } v_3 \text{ --- } v_4 \text{ --- } v_5 \text{ --- } v_6 \text{ --- } \cdots$$

has a dismantling order $v_0 < v_1 < v_2 < \cdots$

Remove v_0 first as it is the only dominated node in whole graph.

$$v_1 \text{ --- } v_2 \text{ --- } v_3 \text{ --- } v_4 \text{ --- } v_5 \text{ --- } v_6 \text{ --- } \cdots$$

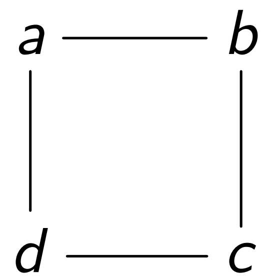
Remove v_1 next as it is the only dominated node in remaining graph.

$$v_2 \text{ --- } v_3 \text{ --- } v_4 \text{ --- } v_5 \text{ --- } v_6 \text{ --- } \cdots$$

Remove v_2 next and so on.

Note: This is the only dismantling order on G.

Example 3. For any G, the first element in a dismantling order must be dominated in G.



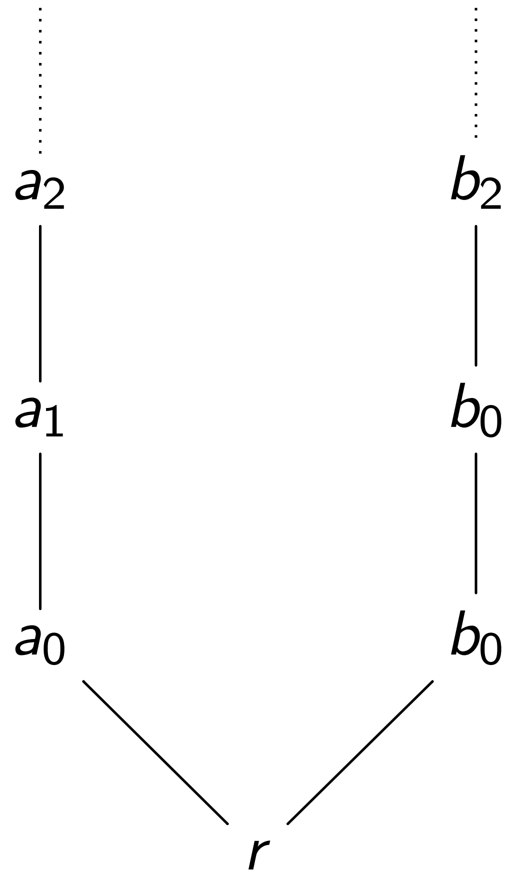
has no dismantlable order because no vertex is dominated in G.

Fact. If G is finite, then

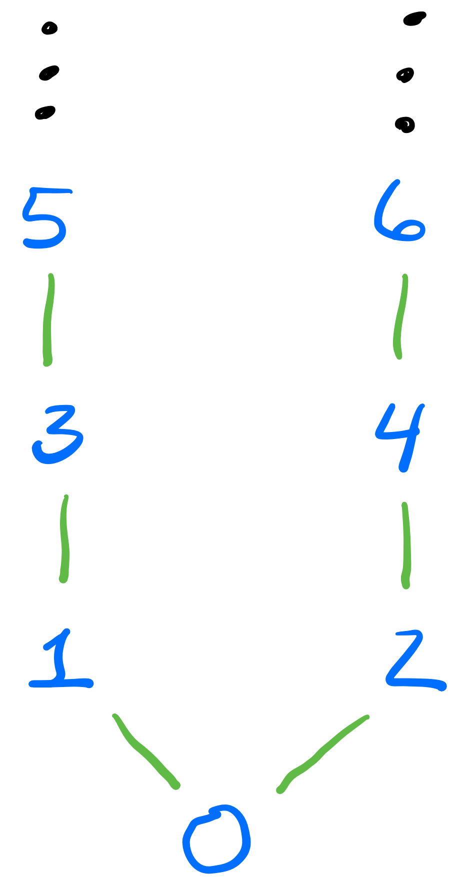
$$\leq \text{ is a constructible order} \quad \Leftrightarrow \quad \geq \text{ is a dismantling order}$$

So a finite graph is constructible if and only if it is dismantlable. However, neither direction is true for infinite graphs.

A tree consisting of two infinite paths (or equivalently, a $\mathbb{Z}$-chain graph) is constructible but not dismantlable.

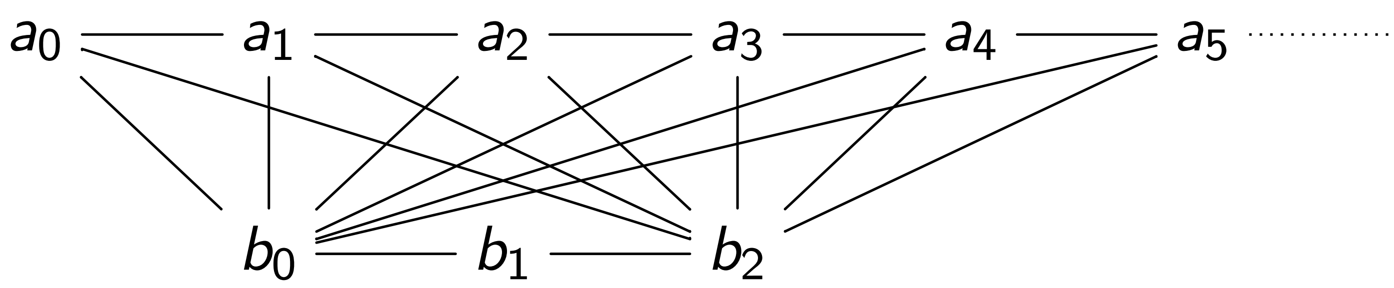


Constructible order



G is not dismantlable because there are no dominated vertices.

The following graph is dismantlable but not constructible.



(b_0 and b_2 connected to all the a_i, while b_1 connects to none of the a_i)

The dismantling order is

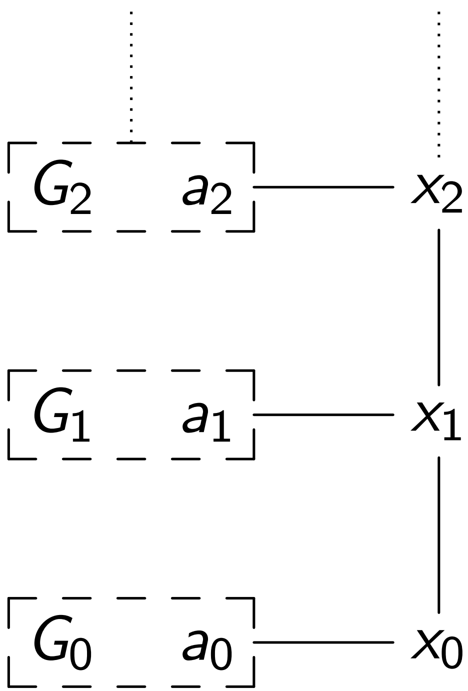
$$a_0 < a_1 < a_2 < \cdots < b_0 < b_1 < b_2$$

There are many natural computability questions to ask about these orders.

- What information can be coded into a dismantling or constructible order?
- How complicated are the index sets of dismantlible and constructible graphs?
- Do these results change for locally finite graphs?
- What order types are possible for constructible and dismantling orders?

Coding 0′ information using comb graphs

Let G_e be computable sequence of finite graphs with marked nodes a_e. Suppose each G_e has a constructible order with least element a_e.



- G has a constructible order and a dismantling order
- If $\leq$ is constructible order, then for all e (except possibly one) $\leq \upharpoonright G_e$ is constructible order with least element a_e.
- If $\leq$ is dismantling order, then for all e, $\leq \upharpoonright G_e$ is dismantling order with greatest element a_e.

Suppose each G_e has a constructible order with least element a_e.

- G has both a constructible order and a dismantling order.
- Let $\leq$ be a constructible order on G. For all e (except maybe one), the restriction of $\leq$ to G_e is a constructible order on G_e with least element a_e.
- Let $\preceq$ be a dismantling order on G. For all e, the restriction of $\preceq$ to G_e is a dismantling order on G_e with greatest element a_e.

Theorem (Evron, Markkanen, Solomon, Stahl)

There is a computable locally finite G such that every constructible order and every dismantling order computes $0'$.

G_e starts as

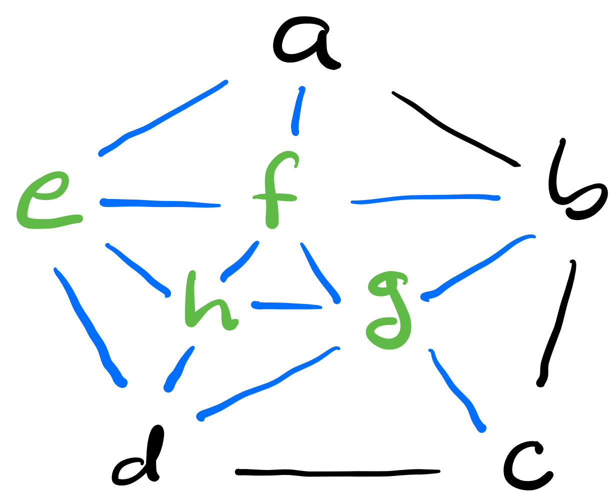
$a - b - c - d$

while $e \notin 0'_s$

if constructible $\leq$ has least element a, then

$c < d$

if e enters $0'$



if constructible $\leq$ has least element a, then $d < c$

(b/c only a and c are dominated)

More coding

Using more sophisticated graph constructions, we get:

Theorem (Evron, Solomon, Stahl)

There is a computable locally finite G such that every constructible order computes $0''$.

Theorem (Markkanen, Solomon, Stahl)

There is a computable G such that every dismantling order computes $0'''$.

Theorem (Markkanen, Solomon, Stahl)

For every $\alpha < \omega_1^{CK}$, there is a computable G such that for every dismantling order $\leq$ of G, the jump of $\leq$ computes 0^α.

Index set complexity

In general, "G is constructible (or dismantlable)" is a Σ^1_2 property:

$$\exists \leq \ (\leq \text{ is a well order of } G \wedge \leq \text{ has the dominating property})$$

Theorem (Evron, Solomon and Stahl)

The index set for constructible computable graphs is Π^1_1-hard.

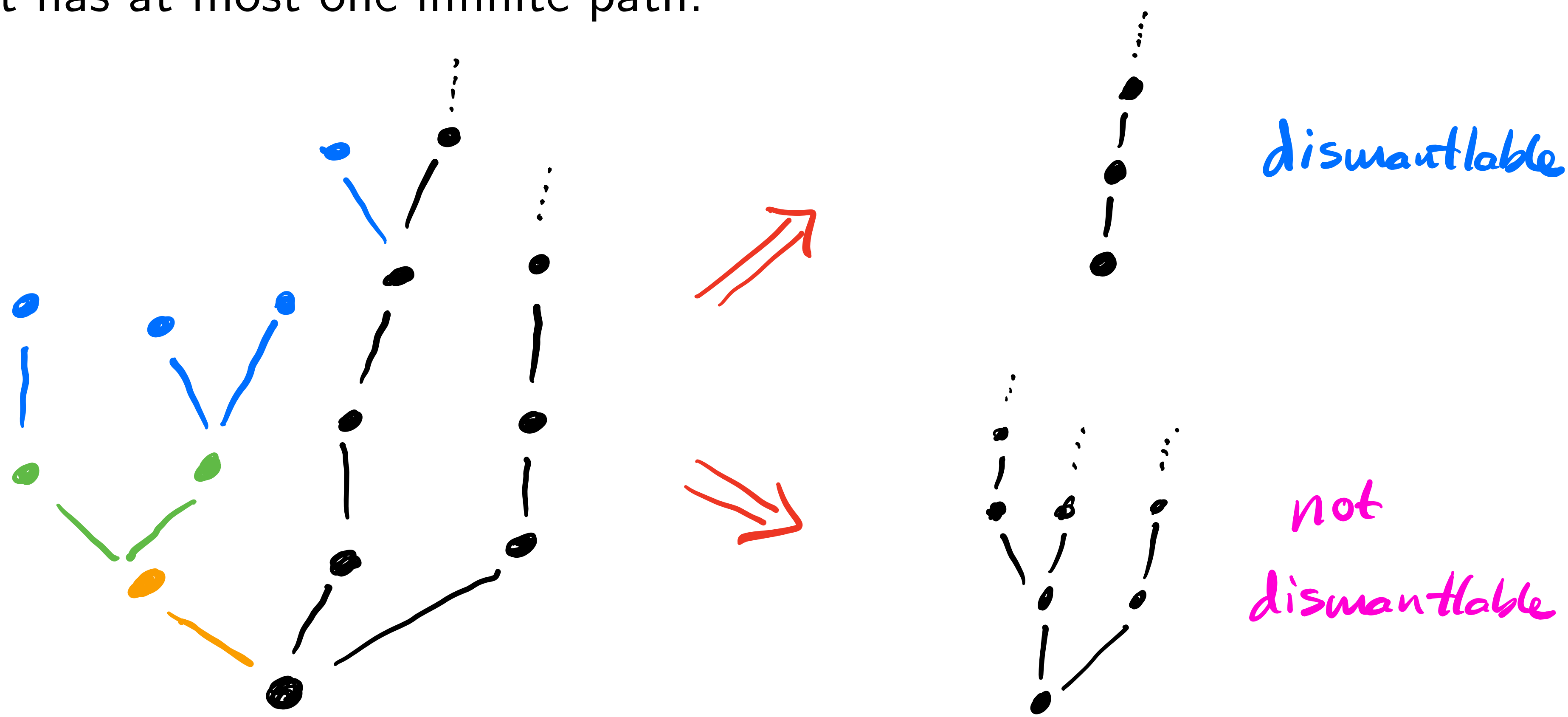
Theorem (Markkanen, Solomon and Stahl)

The index set for dismantlable computable graphs is Π^1_1-hard.

Theorem (Markkanen, Solomon and Stahl)

The index set for dismantlable computable graphs is Π^1_1-hard.

Lemma. A tree $T \subseteq \omega^{<\omega}$ (viewed as a graph) is dismantlable if and only if it has at most one infinite path.



Theorem (Markkanen, Solomon and Stahl)

The index set for dismantlable computable graphs is Π^1_1-hard.

Lemma. A tree $T \subseteq \omega^{<\omega}$ (viewed as a graph) is dismantlable if and only if it has at most one infinite path.

Let T_e be a list of all computable trees. Let G_{T_e} be the tree T_e with an additional infinite path attached (viewed as a graph).

$$G_{T_e} \text{ dismantlable } \Leftrightarrow T_e \text{ has no infinite path}$$

Theorem (Lehner)

A locally finite graph is constructible (or dismantlable) if and only if it has a constructible (or dismantable) order of order type ω.

For locally finite graphs, "G is constructible (or dismantlable)" is Σ^1_1.

$$\exists \leq \ ((G, \leq) \text{ has order type } \omega \wedge \ \leq \ \text{ has the dominating property})$$

Theorem (Evron, Solomon and Stahl)

The index set for locally finite constructible computable graphs is Π^0_4-hard.

Let the **constructible rank** of a constructible graph G is the least ordinal α such that G has a constructible order of type α.

Question. What are the ordinal values of the constructible rank of G?

The game of cops and robbers

The game of cops and robbers on a graph G is played as follows:

- Player C picks a vertex to start on, after which Player R picks a vertex to start on.
- In each round of the game, Player C moves to any vertex adjacent to her current position. If she lands on the vertex with Player R, then the game ends and Player C wins.
- Otherwise, Player R moves to any vertex adjacent to his current position. If he avoids Player C forever, then Player R wins.
- Because G is reflexive, the players can stay on their current vertex.

G is **C-win** if Player C has a winning strategy, and otherwise G is **R-win**.

Theorem (Nowakowski and Winkler; Quilliot)

A finite graph is C-win if and only if it is constructible (or equivalently dismantlable).

This equivalence fails for infinite graphs, mainly because it is too easy for Player R to run away. In general, infinite graphs are poorly behaved with respect to the basic results about the game for finite graphs.

Theorem (Stahl)

Every infinite locally finite graph is R-win.

Lehner considered the same game except that Player C wins if she lands on a vertex occupied by Player R **or** *if Player R occupies each vertex only finitely often during the infinite play of the game*.

Definition (Lehner)

G is **weakly C-win** if Player C has a winning strategy in this variant of the game.

Theorem (Lehner)

If G is constructible, then G is weakly C-win. (He also recovered other desirable properties from the finite case.)

Question (Lehner): Is every weakly C-win graph constructible?

Theorem (Evron, Solomon and Stahl)

There is a computable transformation from trees $T \subseteq \omega^{<\omega}$ to graphs G_T such that

(1) *G_T is constructible if and only if T is well founded,*

(2) *for every T, G_T is weakly C-win, and*

(3) *for well founded T, the constructible rank of G_T is greater than the tree rank of T.*

There are a number of immediate corollaries:

- Let T be a non-well founded tree. By (1), G_T is not constructible; by (2), G_T is weakly C-win. The answer to Lehner's question is no.
- For every $\alpha < \omega_1$, there is a graph G_T such that the constructible rank of G_T is greater than α.
- The index set of computable constructible graphs is Π^1_1 hard.

After posting our paper, Ivan, Leader and Walters answered two questions.

- For every $0 < \alpha < \omega_1$, there is a constructible graph G for which the constructible ordinal of G is exactly α.
- There is a graph G that is C-win (in the original game) but not constructible.

Lehner suggested dropping the well ordering condition for constructible orders to explore the connection to weakly C-win graphs.

Definition

A *weakly constructible order* of G is a linear order such that each vertex x (except the least) is dominated in $G_{\leq x}$.

Question (Lehner). For locally finite graphs, is there a connection between weakly C-win graphs and the order types of their weakly constructible orderings?

He was interested in locally finite graphs that are not constructible but have weakly constructible orders of type ω^* or $\omega^* + \omega$.

Recall: An infinite locally finite graph is constructible if and only if it has a constructible ordering of type ω.

Theorem (Solomon and Stahl)

There are computable locally finite graphs G_0, G_1 and G_2 that are weakly C-win, not constructible and

- *G_0 has a weakly constructible order of type ω^*,*
- *G_1 has a weakly constructible order of type $\omega^* + \omega$, but not one of type ω^*.*
- *G_2 has a weakling constructible order of type $\omega + \omega^*$, but not one of type ω^* or $\omega^* + \omega$.*

Open problems:

- Close the gap between Π^1_1 and Σ^1_2 for the index sets of computable constructible graphs and computable dismantlable graphs.
- Close the gap between Π^0_4 and Σ^1_1 for the index set of locally finite computable constructible graphs.
- Can we code more jumps into constructible or dismantling orders?
- Does every computable constructible (or dismantlable) graph have a hyperarithmetic such order?

Using an alternate characterization by Nowakowski and Winkler, Shelley settled the corresponding index set question for C-win graphs in the original game.

Theorem (Stahl)

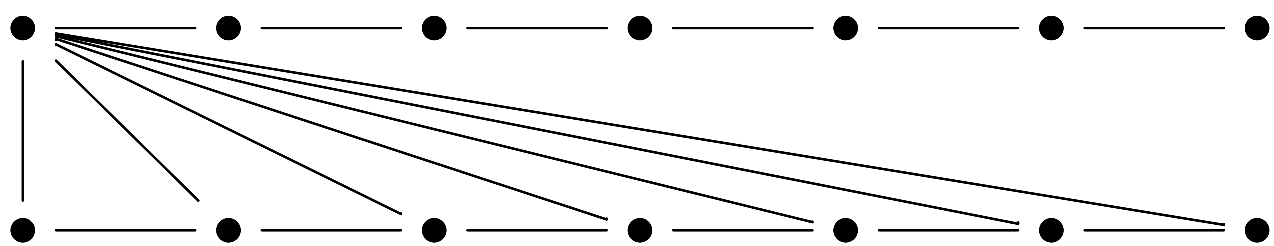
The index set of computable C-win graphs is Π^1_1 complete.

What does the tree transformation look like?

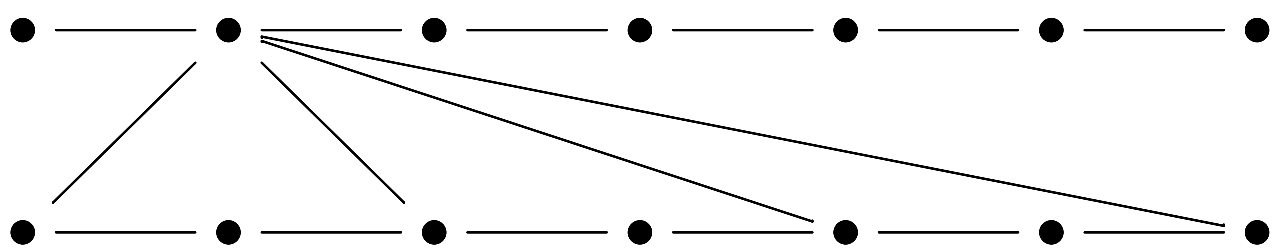
Consider a graph consisting of (stacks of) rows of the form

$$v_0 \text{ --- } v_1 \text{ --- } v_2 \text{ --- } v_3 \text{ --- } v_4 \text{ --- } v_5 \text{ --- } v_6$$

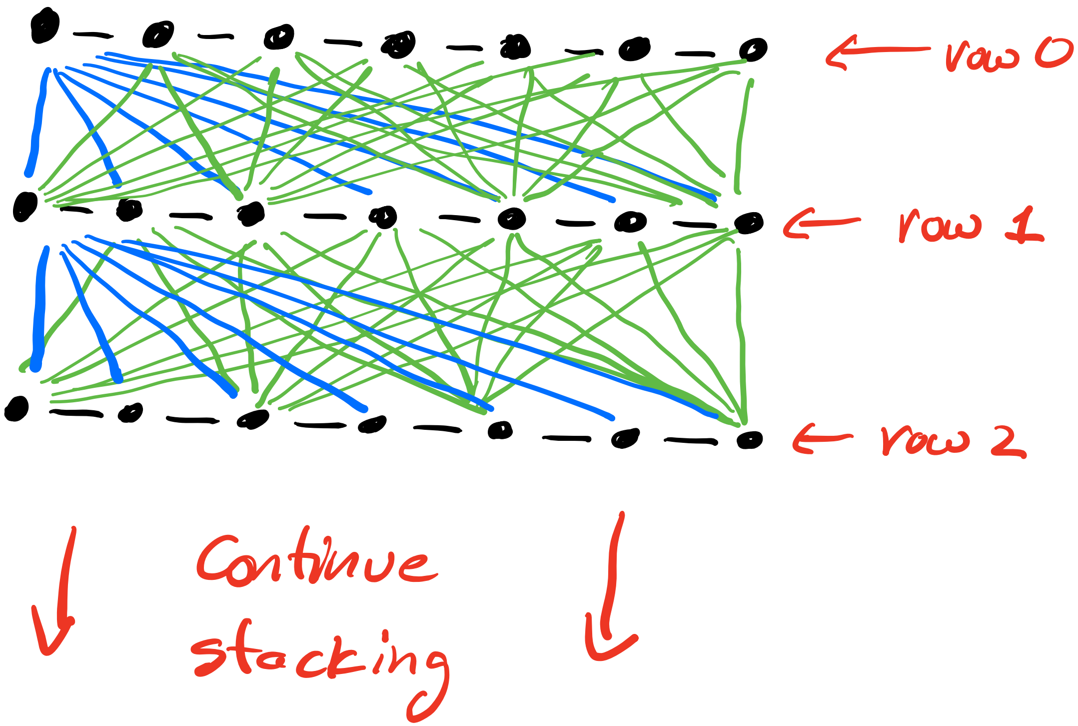
We connect the first node in each row to all nodes in the next row down



and connect the remaining nodes to the even index nodes in the next row



Stacking the rows

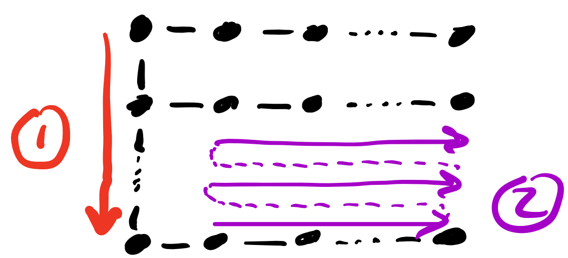


Fact: If $\leq$ is constructible and m_i = greatest even index node in row i, then

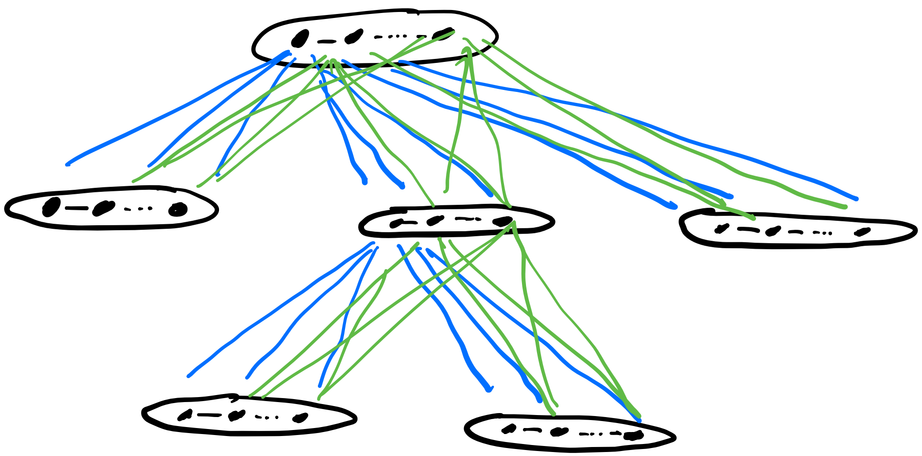
$$m_{i+1} < m_i$$

$\therefore$ An infinite stack has no constructible order

But a finite stack does have a constructible order.



Transforming trees

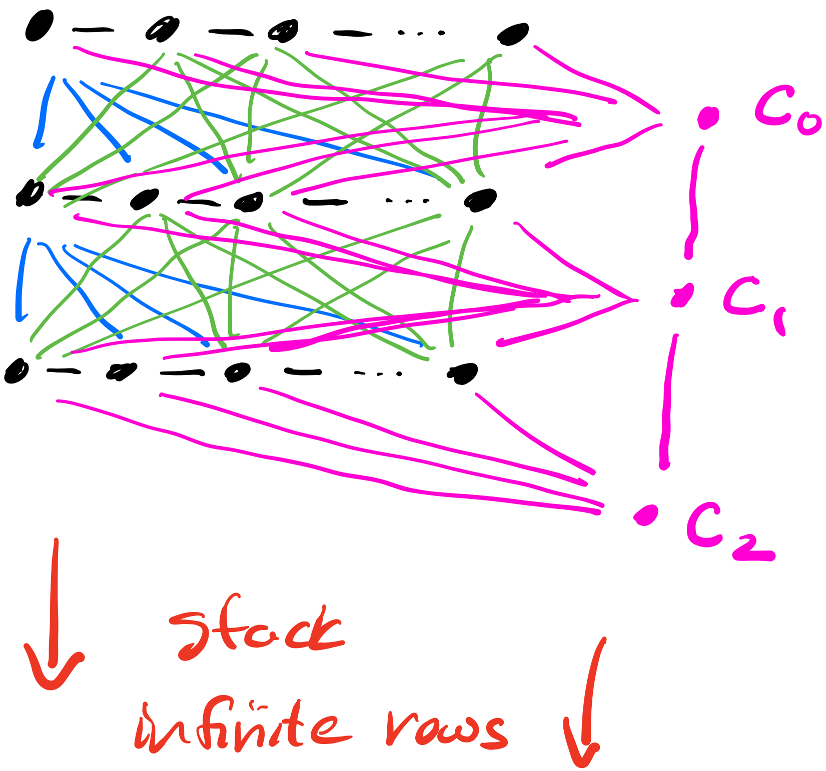


↓ Continue as tree grows downward ↓

If T has an infinite path, it acts like an infinite stack and has no constructible order.

If T is well founded, it has a constructible order. First order leftmost points in each group of 7, then order remainder.

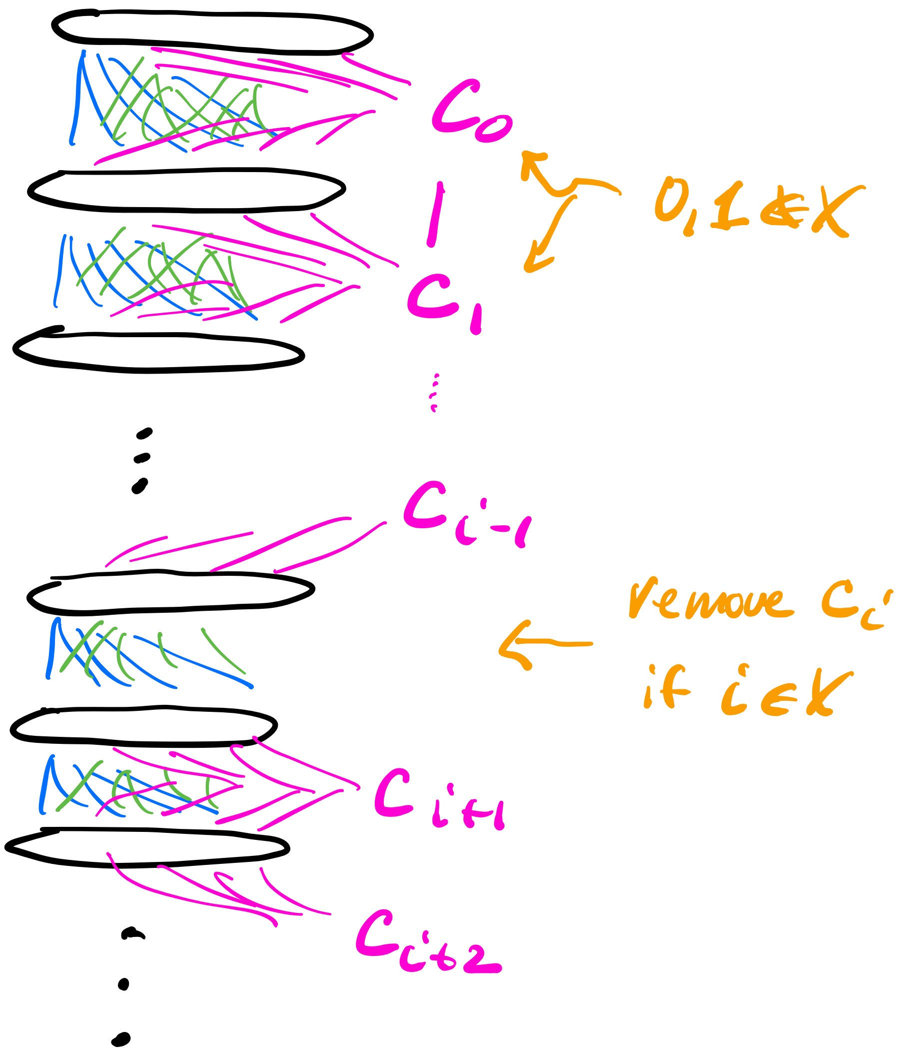
The graph K for coding in constructible orders



Start with infinitely many stacked rows, but add a chain of auxiliary nodes c_i.

With these auxiliary nodes, the graph has a constructible order.

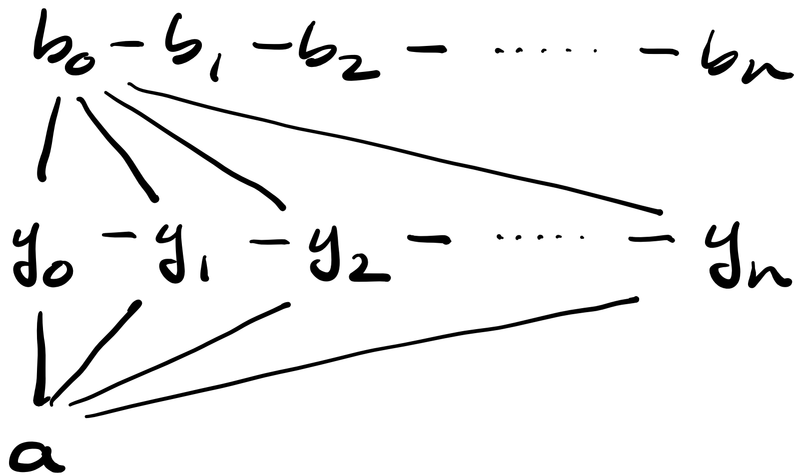
The graph $K(X)$



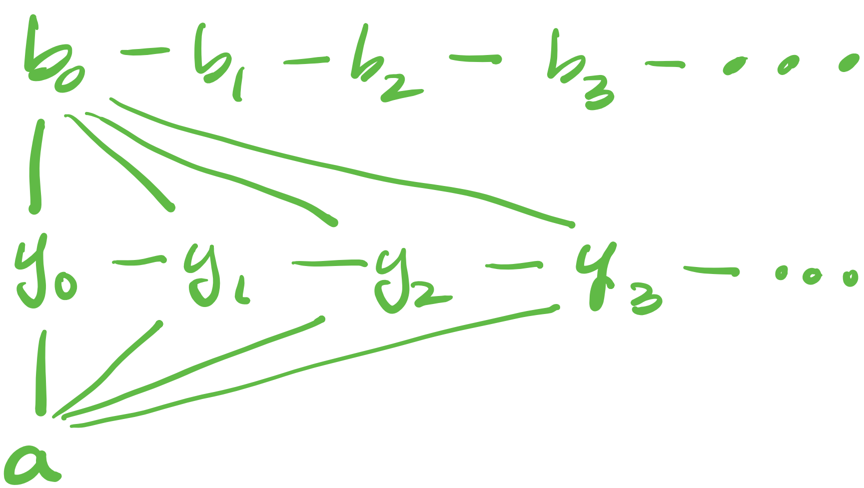
For $X \subseteq \omega$, start with graph K, but remove C_i for all $i \in X$.

Fact: $K(X)$ has a construction order $\iff$ X is finite

Gadget for coding into dismantling orders



If $\leq$ is dismantling order in which a is greatest, then $b_1 < b_0$



For any dismantling order, $b_0 < b_1$.

Thank you!