Generic Muchnik Reducibility

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Muchnik reducibility between structures

Definition

If $\mathcal{A}$ and $\mathcal{B}$ are countable structures, then $\mathcal{A}$ is *Muchnik reducible* to $\mathcal{B}$ (written $\mathcal{A} \leqslant_w \mathcal{B}$) if every ω-copy of $\mathcal{B}$ computes an ω-copy of $\mathcal{A}$.

- $\mathcal{A} \leqslant_w \mathcal{B}$ can be interpreted as saying that $\mathcal{B}$ is intrinsically at least as complicated as $\mathcal{A}$.
- This is a special case of Muchnik reducibility; it might be more precise to say that the problem of presenting the structure $\mathcal{A}$ is Muchnik reducible to the problem of presenting $\mathcal{B}$.
- Muchnik reducibility doesn't apply to uncountable structures.

Various approaches have been used to extend computable structure theory beyond the countable:

- Computability on admissible ordinals (aka α-recursion theory),
- Computability on separable structures, as in computable analysis,
- ...

Generic Muchnik reducibility

Noah Schweber extended Muchnik reducibility to arbitrary structures (see Knight, Montalbán, and Schweber [2016]):

Definition (Schweber)

If $\mathcal{A}$ and $\mathcal{B}$ are (possibly uncountable) structures, then $\mathcal{A}$ is *generically Muchnik reducible* to $\mathcal{B}$ (written $\mathcal{A} \leqslant_w^* \mathcal{B}$) if $\mathcal{A} \leqslant_w \mathcal{B}$ in some forcing extension of the universe in which $\mathcal{A}$ and $\mathcal{B}$ are countable.

It follows from Shoenfield absoluteness that generic Muchnik reducibility is robust.

Lemma (Schweber)

If $\mathcal{A} \leqslant_w^* \mathcal{B}$, then $\mathcal{A} \leqslant_w \mathcal{B}$ in *every* forcing extension that makes $\mathcal{A}$ and $\mathcal{B}$ countable.

In particular, for countable structures, $\mathcal{A} \leqslant_w^* \mathcal{B} \iff \mathcal{A} \leqslant_w \mathcal{B}$.

Collapsing the continuum

Goal. Understand the generic Muchnik degrees of (expansions of) Cantor space $\mathcal{C}$, Baire space $\mathcal{B}$, and the field of real numbers $(\mathbb{R}, +, \cdot)$.

Consider a forcing extension that makes these structures countable. Let I be the the ground model's copy of $2^\omega = \mathcal{P}(\omega)$.

By absoluteness, I is closed under

- Turing reduction,
- join,
- the Turing jump,
- ... and much more.

So I is (at least) a countable jump ideal in the Turing degrees.

Notation. We say that a function $f \in \omega^\omega$ is *in I* if it is computable from an element of I. We do the same for other countable objects, like trees $T \subseteq \omega^{<\omega}$ and real numbers.

Enumerations of ideals

Definition

An *enumeration* of a countable family of sets $S \subseteq 2^\omega$ is a sequence $\{X_n\}_{n\in\omega}$ of sets such that

$$S = \{X_n : n \in \omega\}.$$

The enumeration is *injective* if all of the X_n are distinct.

Lemma (folklore)

Let I be a countable ideal. Every enumeration of I computes an injective enumeration of I.

- This is proved by a simple finite injury argument.
- We can define an enumeration of a countable family of *functions* in the same way. The lemma also holds for the family of functions in a countable ideal I.

Initial example

Definition (Cantor space)

Let $\mathcal{C}$ be the structure with universe 2^ω and predicates $P_n(X)$ that hold if and only if $X(n) = 1$.

Observation (Knight, Montalbán, Schweber [2016])

$$\mathcal{C} \leqslant_w^* (\mathbb{R}, +, \cdot).$$

To understand this, take a forcing extension that collapses the continuum and let I be the ground model's version of 2^ω.

Let $\mathbb{R}_I$ be the real numbers in I and let $\mathcal{C}_I$ denote the restriction of $\mathcal{C}$ to sets in I.

In other words, $\mathbb{R}_I$ is the ground model's version of $\mathbb{R}$ and $\mathcal{C}_I$ is the ground model's version of $\mathcal{C}$.

Initial example

Facts

- From a copy of $(\mathbb{R}_I, +, <)$, we can compute an (injective) enumeration of I.
- A degree $\mathbf{d}$ computes a copy of $\mathcal{C}_I$ iff it computes an (injective) enumeration of I.

This shows that $\mathcal{C}_I \leqslant_w (\mathbb{R}_I, +, <)$. It is even easier to see that $(\mathbb{R}_I, +, <) \leqslant_w (\mathbb{R}_I, +, \cdot)$.

Therefore, $\mathcal{C} \leqslant_w^* (\mathbb{R}, +, <) \leqslant_w^* (\mathbb{R}, +, \cdot)$.

Question (KMS [2016]). Is $(\mathbb{R}, +, \cdot) \leqslant_w^* \mathcal{C}$?

This was answered by Igusa and Knight [2017], and independently (though later) by Downey, Greenberg, and M [2016].

First question

$$\text{Is } (\mathbb{R}, +, \cdot) \leqslant^*_w \mathcal{C}?$$

Downey, Greenberg, and M.'s solution

Definition (Baire space)

Let $\mathcal{B}$ be the structure with universe ω^ω and, for each finite string $\sigma \in \omega^{<\omega}$, a predicate $P_\sigma(f)$ that holds if and only if $\sigma \prec f$.

- From a copy of $(\mathbb{R}_I, +, \cdot)$, or even $(\mathbb{R}_I, +, <)$, we can compute an (injective) enumeration of the functions in I.
- A degree $\mathbf{d}$ computes a copy of $\mathcal{B}_I$ iff it computes an (injective) enumeration of the functions in I.

As before, we have $\mathcal{B} \leqslant_w^* (\mathbb{R}, +, <) \leqslant_w^* (\mathbb{R}, +, \cdot)$.

Theorem (DGM [2016])

Let I be a countable Scott ideal. There is an enumeration of I that does not compute an enumeration of the functions in I.

This implies that $\mathcal{B}_I \nleqslant_w \mathcal{C}_I$, so $\mathcal{B} \nleqslant_w^* \mathcal{C}$.

Theorem. $(\mathbb{R}, +, \cdot) \nleqslant_w^* \mathcal{C}$.

Many structures are equivalent (to $\mathcal{B}$)

Perhaps surprisingly, what makes $(\mathbb{R}, +, \cdot)$ more complicated than $\mathcal{C}$ has little to do with the field structure.

Theorem (DGM [2016]). $\mathcal{B} \equiv^*_w (\mathbb{R}, +, <) \equiv^*_w (\mathbb{R}, +, \cdot)$.

From an enumeration of the functions in a countable ideal I, we build a copy of $(\mathbb{R}_I, +, \cdot)$. We use quantifier elimination and decidability for real closed fields.

Around the same time (and still independently):

Theorem (Igusa, Knight, Schweber [2017])
$(\mathbb{R}, +, <) \equiv^*_w (\mathbb{R}, +, \cdot) \equiv^*_w (\mathbb{R}, +, \cdot, e^x)$.

They use the o-minimality of $(\mathbb{R}, +, \cdot, e^x)$ and the fact that its theory is in the ground model.

In both cases, tameness is used to recover from injury in the construction. Is it necessary?

Is o-minimality essential?

Summary

$$\mathcal{C} \qquad <^*_w \qquad \mathcal{B} \equiv^*_w (\mathbb{R}, +, <) \equiv^*_w (\mathbb{R}, +, \cdot) \equiv^*_w (\mathbb{R}, +, \cdot, e^x).$$

Going further, by the same method that they used for e^x:

Theorem (Igusa, Knight, Schweber [2017])

- If $(\mathbb{R}, +, \cdot, f)$ is o-minimal, then $(\mathbb{R}, +, \cdot) \equiv^*_w (\mathbb{R}, +, \cdot, f)$.
- $(\mathbb{R}, +, \cdot) \equiv^*_w (\mathbb{R}, +, \cdot, \sin)$.

Although $(\mathbb{R}, +, \cdot, \sin)$ is not o-minimal, $(\mathbb{R}, +, \cdot, \sin \restriction [0, \pi/2])$ is, and these structures are $\equiv^*_w$.

Question (Igusa, Knight, Schweber [2017])

Is there a continuous function $f \colon \mathbb{R} \to \mathbb{R}$ such that

$$(\mathbb{R}, +, \cdot) <^*_w (\mathbb{R}, +, \cdot, f)?$$

We will see that the answer is *no*.

Second question

Is there a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ s.t.

$$(\mathbb{R}, +, \cdot) <_w^* (\mathbb{R}, +, \cdot, f)?$$

(Joint work with Andrews, Knight, Kuyper, Lempp, and M. Soskova)

Enumeration with the running jump

When building a structure over I, it would be very helpful to have access to the jump. Our main lemma gives us that.

Definition. Let $\{X_n\}_{n\in\omega}$ be an enumeration of sets. The corresponding *running jump* is the sequence

$$\left\{\left(\bigoplus_{i\leqslant n} X_i\right)'\right\}_{n\in\omega}.$$

Note that computing the running jump is equivalent to uniformly being able to compute the jump of any join of members of the enumeration.

Lemma (AKKMS)

Let I be a countable jump ideal. Every enumeration of the functions in I computes an enumeration of I along with the running jump.

The proof is a delicate finite injury construction.

Enumeration with the running jump

Lemma (AKKMS). Let I be a countable jump ideal. Every enumeration of the functions in I computes an enumeration of the sets in I along with the running jump.

Main ideas

- To compute the next set in the running jump, we guess a function in I that majorizes the corresponding settling-time function. If we are wrong, there is an injury (and a new guess).
- When an injury occurs, we use the low basis theorem to "patch up" the enumeration consistently and keep control of the jumps.

Warnings

- We need to start with an enumeration of the functions in I so that we can search for settling-time functions.
- We can only hope to produce an enumeration of the sets in I. (We can't use the low basis theorem in Baire space.)

Continuous expansions of the reals

We can now expand the reals by continuous functions.

Theorem (AKKMS). Let $f_1, f_2, \ldots$ be continuous functions (of any arities) on $\mathbb{R}$. Then $(\mathbb{R}, +, \cdot, \{f_i\}_{i\in\omega}) \equiv^*_w (\mathbb{R}, +, \cdot) \equiv^*_w \mathcal{B}$.

Proof sketch.
Let $P \in 2^\omega$ be a parameter coding $\{f_i\}_{i\in\omega}$. Let I be a countable jump ideal including P. From any copy of $(\mathbb{R}_I, +, \cdot)$, we can enumerate I along with the running jump.

For $X \in 2^\omega$, let $0.X$ denote the real number in $[0, 1]$ with binary expansion X. For $z \in \mathbb{Z}$, let $z.X$ denote $z + 0.X$. Using $(X_0 \oplus X_1)'$, we can check if $z_0.X_0 = z_1.X_1$. Using $(P \oplus X_0 \oplus \cdots \oplus X_n)'$, we can check if $f_i(z_0.X_0, \ldots, z_{n-1}.X_{n-1}) = z_n.X_n$. Similarly, we can check $+$ and $\cdot$.

Therefore, we can build a copy of $(\mathbb{R}_I, +, \cdot, \{f_i\}_{i\in\omega})$. □

Note that the construction has no injury. We have moved all of the injury into building the enumeration of sets with the running jump.

Continuous expansions of Cantor space

The running jump lemma can also be used to build continuous expansions of $\mathcal{C}$.

Theorem (AKKMS)

Any expansion of $\mathcal{C}$ by countably many continuous functions is $\leqslant_w^* \mathcal{B}$.

Some natural expansions of $\mathcal{C}$ turn out to be equivalent to $\mathcal{B}$.

Let $\sigma\colon \omega^\omega \to \omega^\omega$ denote the *shift*: i.e., $\sigma(n_0 n_1 n_2 n_3 \cdots) = n_1 n_2 n_3 \cdots$. Let $\oplus\colon \omega^\omega \times \omega^\omega \to \omega^\omega$ denote the *join*. Both are continuous and both restrict to functions on 2^ω.

Proposition (AKKMS). $(\mathcal{C}, \sigma) \equiv_w^* (\mathcal{C}, \oplus) \equiv_w^* \mathcal{B}$.

In both cases, we can recognize the finite sets in a c.e. way, allowing a copy of $(\mathcal{C}_I, \sigma)$ or $(\mathcal{C}_I, \oplus)$ to enumerate the infinite sets (hence functions) in I.

Continuous expansions of Baire space

It turns out that continuous expansions of Baire space can be more complex than Baire space.

Note that

$$Z = \{(f \oplus g) \oplus h \colon h \text{ is the settling-time function for } f' \text{ and } g = f'\}$$

is a closed subset of ω^ω (in fact, a Π^0_1 class). Let F be a continuous function on ω^ω such that $Z = F^{-1}(0^\omega)$.

Proposition (AKKMS). Let I be a countable jump ideal. Any copy of $(\mathcal{B}_I, \oplus, F)$ computes an enumeration of the functions in I along with join and jump as functions on indices of the enumeration.

Proof idea.
A copy $\mathcal{A}$ of $(\mathcal{B}_I, \oplus, F)$ gives us a natural enumeration $\{f_n\}_{n\in\omega}$ of I such that $\oplus^{\mathcal{A}}$ is exactly a function that takes two indices to the index of the join. To find the jump of f_n, search for $m, j \in \omega$ such that $F^{\mathcal{A}}((n \oplus^{\mathcal{A}} m) \oplus^{\mathcal{A}} j)$ is the index of 0^ω. Then $f_m = f_n'$. □

Hyper-Scott ideals

Corollary. $(\mathcal{C}, \oplus, ') \leqslant^*_w (\mathcal{B}, \oplus, ') \leqslant^*_w (\mathcal{B}, \oplus, F)$.

We want to prove that $(\mathcal{C}, \oplus, ') \nleqslant^*_w \mathcal{B}$. (Note that although $\oplus$ is continuous, $'$ is not; it is Baire class 1.)

Definition

An ideal I is a *hyper-Scott* ideal if whenever a tree $T \subseteq \omega^{<\omega}$ in I has an infinite path, it has an infinite path in I. (Mention β-models.)

Fact. If I is the ground model's version of 2^ω, then it is a hyper-Scott ideal.

Proof.

(This is Shoenfield absoluteness in its simplest form.) If $T \subseteq \omega^{<\omega}$ is a tree in the ground model with no path, then in the ground model there is a rank function $\rho\colon T \to \omega_1$ witnessing that T is well-founded. But ρ also witnesses that T is well-founded in the extension. $\square$

Beyond the degree of Baire space

Theorem (AKKMS). Assume that I is a countable hyper-Scott ideal. There is an enumeration of the functions in I that does not compute an enumeration of the functions in I along with join and jump as functions on indices.

Corollary. $(\mathcal{C}, \oplus, ')$, $(\mathcal{B}, \oplus, ') \not\leqslant_w^* \mathcal{B}$.

Corollary. There is an expansion of $\mathcal{B}$ by continuous functions that is strictly above $\mathcal{B}$ in the generic Muchnik degrees.

In particular, $(\mathcal{B}, \oplus, F) \not\leqslant_w^* \mathcal{B}$.

It turns out that $(\mathcal{B}, \oplus, F) \equiv_w^* (\mathcal{B}, \oplus, ') \equiv_w^* (\mathcal{C}, \oplus, ')$. We have seen one direction; the other follows from the fact that $(\mathcal{C}, \oplus, ')$ is above all *Borel structures*.

Borel structures

Definition

A *Borel structure* has a presentation of the form $(D, E, f_1, f_2, \dots)$ where $D \subseteq \omega^\omega$ is Borel, E is a Borel equivalence relation on D, and $f_1, f_2, \dots$ are Borel functions (of any arities) on D that are compatible with E. (The domain of the structure is D/E.)

Examples

- Every structure we've talked about today,
- The Turing degrees with join and jump,
- The automorphism group of any countable structure,
- All Büchi automatic structures (Hjorth, Khoussainov, Montalbán, and Nies [2008]).

Theorem (AKKMS). Every Borel structure is $\leqslant^*_w (\mathcal{C}, \oplus, ')$.

Borel structures

Theorem (AKKMS). Every Borel structure is $\leqslant_w^* (\mathcal{C}, \oplus, ')$.

Proof idea.
Let I be a countable hyper-Scott ideal. From $(\mathcal{C}_I, \oplus, ')$ we can enumerate the functions in I along with join and jump as functions on indices.

For simplicity, we restrict our attention to a single Borel relation $R \subseteq \omega^\omega$. We may assume that R has a code $c \in I$. Since R is $\Delta_1^1[c]$, there are trees $T, S \subseteq \omega^{<\omega}$, both in I, such that

$$f \in R \iff (\exists h)\, f \oplus h \in [T] \iff (\forall h)\, f \oplus h \notin [S].$$

Using the enumeration of I, and the fact that $f \oplus h \in [T]$ can be checked using $(f \oplus h \oplus T)'$, we can computably determine if $R(f)$ holds for any function $f \in I$. $\square$

The story so far

$$\left.\begin{array}{c}(\mathcal{C}, \oplus, {}') \\ \\ \mathcal{B}\end{array}\right\} \text{Borel expansions of } \mathcal{B}$$

$$\left.\begin{array}{c}\mathcal{B} \\ \\ \mathcal{C}\end{array}\right\} \text{Continuous/closed expansions of } \mathcal{C}$$

- $\mathcal{B} \equiv_w^* $ any continuous/closed expansion of $(\mathbb{R}, +, \cdot)$.
- In terms of the *jumps* of these structures:
  - $\mathcal{C}' \equiv_w^* \mathcal{B}$, and
  - $\mathcal{B}' \equiv_w^* (\mathcal{C}, \oplus, {}')$.

Question

Is there a generic Muchnik degree strictly between $\mathcal{C}$ and $\mathcal{B}$? (Yes!)
Can it be the degree of a continuous expansion of $\mathcal{C}$? (No!)

Third question

Is there a generic Muchnik degree strictly between $\mathcal{C}$ and $\mathcal{B}$?

(Joint work with Andrews, Schweber, and M. Soskova)

Definability and post-extension complexity

It is going to be important to understand the complexity of definable sets both before and after the forcing extension.

Definition

We say that a relation R on a structure $\mathcal{M}$ is $\Sigma^c_n(\mathcal{M})$ if it is definable by a computable Σ_n $\mathcal{L}_{\omega_1\omega}$ formula with finitely many parameters.

Theorem (Ash, Knight, Manasse, Slaman; Chisholm)

If $\mathcal{M}$ is countable, then R is $\Sigma^c_n(\mathcal{M})$ if and only if it is relatively intrinsically Σ^0_n, i.e., its image in any ω-copy of $\mathcal{M}$ is Σ^0_n relative to that copy.

Computable objects and satisfaction on a structure are absolute, so:

Corollary. A relation R is $\Sigma^c_n(\mathcal{M})$ if and only if it is relatively intrinsically Σ^0_n in any/every forcing extension making $\mathcal{M}$ countable.

Definability and pre-extension complexity

In structures like $\mathcal{C}$ and $\mathcal{B}$, we can also measure the complexity of $\Sigma^c_n(\mathcal{M})$ relations in the projective hierarchy.

The "complexity profile" depends on the structure:

	Σ^c_2	Σ^c_3	Σ^c_4	Σ^c_5	Σ^c_6	$\dots$
$\mathcal{B}$	$\mathbf{\Sigma^1_1}$	$\mathbf{\Sigma^1_2}$	$\mathbf{\Sigma^1_3}$	$\mathbf{\Sigma^1_4}$	$\mathbf{\Sigma^1_5}$	$\dots$
$\mathcal{C}$	$\mathbf{\Sigma^0_2}$	$\mathbf{\Sigma^1_1}$	$\mathbf{\Sigma^1_2}$	$\mathbf{\Sigma^1_3}$	$\mathbf{\Sigma^1_4}$	$\dots$

- These bounds are sharp, e.g., every $\mathbf{\Sigma^1_1}$ relation on $\mathcal{B}$ is $\Sigma^c_2(\mathcal{B})$.
- The "lost quantifiers" correspond to the first order quantifiers needed in the normal form for Σ^1_n relations with function/set quantifiers.
- This gives us an easy (and essentially different) separation between the generic Muchnik degrees of $\mathcal{C}$ and $\mathcal{B}$.

Differentiating $\mathcal{C}$ and $\mathcal{B}$ with a linear order

Lemma (AMSS)

There is a linear order $\mathcal{L}$ such that $\mathcal{L} \leqslant_w^* \mathcal{B}$ but $\mathcal{L} \nleqslant_w^* \mathcal{C}$.

Proof Idea

For $X \subseteq \mathcal{C}$, we define a linear order $\mathcal{L}_X$ that codes X. It is essentially a shuffle sum of delimited ζ-representations of *all* elements of Cantor space along with markers for the sequences not in X.

It is designed so that:

- If X is $\Pi_3^c(\mathcal{B})$, then $\mathcal{L}_X \leqslant_w^* \mathcal{B}$,
- If $\mathcal{L}_X \leqslant_w^* \mathcal{C}$, then X is $\Sigma_4^c(\mathcal{C})$.

Now take $X \subseteq \mathcal{C}$ to be $\mathbf{\Pi^1_2}$ but not $\mathbf{\Sigma^1_2}$. By the analysis on the previous slide:

- X is $\Pi_3^c(\mathcal{B})$, so $\mathcal{L}_X \leqslant_w^* \mathcal{B}$,
- X is not $\Sigma_4^c(\mathcal{C})$, so $\mathcal{L}_X \nleqslant_w^* \mathcal{C}$. $\square$

A degree strictly between $\mathcal{C}$ and $\mathcal{B}$

Lemma (AMSS)

There is a linear order $\mathcal{L}$ such that $\mathcal{L} \leqslant_w^* \mathcal{B}$ but $\mathcal{L} \nleqslant_w^* \mathcal{C}$.

But linear orders are bad at coding:

Lemma (AMSS). If $\mathcal{L}$ is a linear order, then $\mathcal{B} \nleqslant_w^* \mathcal{C} \sqcup \mathcal{L}$.

This can be proved by showing that $\mathcal{C}$ and $\mathcal{C} \sqcup \mathcal{L}$ have the same Δ_2^c definable subsets of $\mathcal{C}$. The key fact used about linear orders is that their $\sim_2$-equivalence classes are tame (Knight 1986).

Now let $\mathcal{M} = \mathcal{C} \sqcup \mathcal{L}$, where $\mathcal{L}$ is the linear order from the first lemma.

Corollary (AMSS). There is an $\mathcal{M}$ such that $\mathcal{C} <_w^* \mathcal{M} <_w^* \mathcal{B}$.

Great! But…not the most satisfying example.

What kind of example would we like?

The initial attempts to find an intermediate degree involved natural expansions of $\mathcal{C}$, but without success. For example:

- $(\mathcal{C}, \oplus) \equiv^*_w (\mathcal{C}, \sigma) \equiv^*_w \mathcal{B}$, where σ is the shift operator on 2^ω.
- $(\mathcal{C}, \subseteq) \equiv^*_w (\mathcal{C}, \triangle) \equiv^*_w \mathcal{C}$.

Another approach would be to expand $\mathcal{C}$ with sufficiently generic relations. Greenberg, Igusa, Turetsky, and Westrick tried a version of this that involved adding infinitely many unary relations.

In both cases, we considered *expansions* of $\mathcal{C}$.

Open Question

Is there an expansion of $\mathcal{C}$ that is strictly between $\mathcal{C}$ and $\mathcal{B}$?

Fourth question

Is there an expansion of $\mathcal{C}$ that is strictly between $\mathcal{C}$ and $\mathcal{B}$?

(More joint work with Andrews, Schweber, and M. Soskova)

Expansions of $\mathcal{C}$ above $\mathcal{B}$

Let $\mathcal{M} = (\mathcal{C}, \text{Stuff})$ be an expansion of $\mathcal{C}$. First, we want a criterion that guarantees that $\mathcal{M} \geqslant^*_w \mathcal{B}$.

- If the set $\mathcal{F} \subset 2^\omega$ of sequences with finitely many ones is $\Delta^c_1(\mathcal{M})$, i.e., computable in every ω-copy of $\mathcal{M}$, then $\mathcal{M} \geqslant^*_w \mathcal{B}$.
  - Why? There is a natural bijection between $\mathcal{B}$ and $\mathcal{C} \setminus \mathcal{F}$.
- If $\mathcal{F}$ is $\Delta^c_2(\mathcal{M})$, then $\mathcal{M} \geqslant^*_w \mathcal{B}$.
  - Add a little injury.
  - This is how we show, for example, that $(\mathcal{C}, \oplus) \geqslant^*_w \mathcal{B}$.
- If any countable dense set is $\Delta^c_2(\mathcal{M})$, then $\mathcal{M} \geqslant^*_w \mathcal{B}$.
- If there is a perfect set $\mathcal{P} \subseteq \mathcal{C}$ with a countable dense $\mathcal{Q} \subset \mathcal{P}$ that is $\Delta^c_2(\mathcal{M})$, then $\mathcal{M} \geqslant^*_w \mathcal{B}$.

Expansions of $\mathcal{C}$ above $\mathcal{B}$

- If there is a perfect set $\mathcal{P} \subseteq \mathcal{C}$ with a countable dense $\mathcal{Q} \subset \mathcal{P}$ that is $\Delta_2^c(\mathcal{M})$, then $\mathcal{M} \geqslant_w^* \mathcal{B}$.

Lemma (AMSS)

If $\mathcal{M} \leqslant_w^* \mathcal{B}$ and $R \subseteq \mathcal{C}$ is $\Delta_2^c(\mathcal{M})$, then it is $\Delta_2^c(\mathcal{B})$, i.e., Borel.

Lemma (Hurewicz)

If $R \subseteq \mathcal{C}$ is Borel but not $\mathbf{\Delta_2^0}$, then there is a perfect set $\mathcal{P} \subseteq \mathcal{C}$ such that either $\mathcal{P} \cap R$ or $\mathcal{P} \setminus R$ is countable and dense in $\mathcal{P}$.

Putting it all together (and noting that arity doesn't matter here):

Lemma (AMSS). If $\mathcal{M} \leqslant_w^* \mathcal{B}$ is an expansion of $\mathcal{C}$ and $R \subseteq \mathcal{C}^n$ is $\Delta_2^c(\mathcal{M})$ but not $\mathbf{\Delta_2^0}$, then $\mathcal{M} \geqslant_w^* \mathcal{B}$.

Tameness and dichotomy

In the contrapositive (and using the fact that $\boldsymbol{\Delta^0_2} = \Delta^c_2(\mathcal{C})$):

Tameness Lemma (AMSS)

If $\mathcal{M} <^*_w \mathcal{B}$ is an expansion of $\mathcal{C}$, then $\Delta^c_2(\mathcal{M}) = \Delta^c_2(\mathcal{C})$.

Dichotomy Theorem for Closed Expansions (AMSS)

If $\mathcal{M} \leqslant^*_w \mathcal{B}$ is an expansion of $\mathcal{C}$ by closed relations (and/or continuous functions), then either $\mathcal{M} \equiv^*_w \mathcal{C}$ or $\mathcal{M} \equiv^*_w \mathcal{B}$.

Proof Idea

For a tuple $\overline{X} \subset \mathcal{C}$, let $p(\overline{X})$ be the (code for the) complete positive $\Sigma_1(\mathcal{M})$ type of $\overline{X}$. The relation that holds only on tuples of the form $(\overline{X}, p(\overline{X}))$ is $\Delta^c_2(\mathcal{M})$.

If it is not $\Delta^c_2(\mathcal{C})$, then $\mathcal{M} \geqslant^*_w \mathcal{B}$.

If it is $\Delta^c_2(\mathcal{C})$, then a delicate injury argument can be used to prove that $\mathcal{M} \leqslant^*_w \mathcal{C}$. □

Another dichotomy result

Combined with work of Greenberg, Igusa, Turetsky, and Westrick:

Dichotomy Theorem for Unary Expansions

If $\mathcal{M} \leqslant_w^* \mathcal{B}$ is an expansion of $\mathcal{C}$ by countably many unary relations, then either $\mathcal{M} \equiv_w^* \mathcal{C}$ or $\mathcal{M} \equiv_w^* \mathcal{B}$.

- If $\mathcal{M}$ is an expansion of $\mathcal{C}$ by finitely many $\mathbf{\Delta_2^0}$ unary relations, then $\mathcal{M} \leqslant_w^* \mathcal{C}$. This is a fairly simple finite injury argument.
- Expansions by infinitely many closed unary relations need not be below $\mathcal{C}$: For $\sigma \in 2^{<\omega}$, let U_σ hold only on $\sigma 0^\omega$. Then the set of sequences with finitely many ones is $\Sigma_1^c(\mathcal{C}, \{U_\sigma\}_{\sigma \in 2^{<\omega}})$.
- Greenberg, et al. supplied the right condition distinguishing the cases, and one direction of the proof.

The dichotomy results kill off a lot of possible natural (and many unnatural) examples of expansions.

Final comments

1. We still don't know if an expansion of $\mathcal{C}$ can be strictly between $\mathcal{C}$ and $\mathcal{B}$. (In particular, the non-unary $\boldsymbol{\Delta}^0_2$ case is open.)

2. (Gura) There is a chain $\mathcal{C} <^*_w \cdots <^*_w \mathcal{M}_4 <^*_w \mathcal{M}_3 <^*_w \mathcal{M}_2 <^*_w \mathcal{B}$.
   (AMSS) There is also an $\mathcal{M}_\infty$ with the same complexity profile as $\mathcal{C}$ and such that $\mathcal{C} <^*_w \mathcal{M}_\infty <^*_w \mathcal{B}$.

3. Are there incomparable degrees between $\mathcal{C}$ and $\mathcal{B}$?

4. This talk has focused on the interval between $\mathcal{C}$ and $\mathcal{B}$. For the interval between $\mathcal{B}$ and $(\mathcal{C}, \oplus, ')$, we have proved all of the analogous results (assuming $\boldsymbol{\Delta}^1_2$ Wadge determinacy)
   ...and the analogous questions are open.

Thank You!