On holographic structures

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Holography in physics is a special technique of producing three-dimensional images of objects.

One of the most remarkable properties of holograms:
any, even a very small part of it contains the whole information about the object.

Can we find a similar property for algebraic structures?

The talk is devoted to two concrete methods of defining algebraic structures via their finite parts. Structures that can be defined by these methods are called respectively **holographic** and **weakly holographic** structures.

We deal with predicate signatures only.

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For a signature σ, define its height as

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Definition

A structure $\mathfrak{M}$ of signature σ is called holographic if there exists a finite set $S \subseteq \mathfrak{M}$ such that for any $A \subseteq \mathfrak{M}$ such that $|A| \leqslant \|\sigma\|$ there exists a $\varphi \in \mathrm{Aut}\,\mathfrak{M}$ with the property $\varphi(A) \subseteq S$. Any such an S is called the set of prototypes for $\mathfrak{M}$.

It follows that for holographic structures it must be that $\|\sigma\| < \omega$.

Do we need infinite signatures?

Proposition

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Proof. Let P be an arbitrary signature predicate and S be the set of prototypes for $\mathfrak{M}$.Then for all $\bar{x} \in \mathfrak{M}$ holds

$$P(\bar{x}) \Leftrightarrow \exists \varphi \in \mathtt{Aut}\,\mathfrak{M}\Big(\varphi(\bar{x}) \subseteq S \wedge \langle \mathfrak{M} \cap S; P \rangle \models P(\varphi(\bar{x}))\Big),$$

Since the arities of such P are bounded by a natural number, there exists only a finite number of non–isomorphic structures $\langle \mathfrak{M} \cap S; P \rangle$. It follows that the number of possible P's is also finite.

Theorem

A structure of signature σ is holographic if and only if its automorphism group acts almost $\|\sigma\|$–transitive on it (i.e., the number of orbits of $\|\sigma\|$–tuples is finite).

Corollary

Any countable ω–categorical structure of finite signature is holographic. Any finite structure of finite signature is holographic.

Theorem

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Let F be a field of characteristic 0. Let $P_F = F \times F$ (2–dimensional plane). L_F: the set of all straight lines in P_F. The structure

$$\langle P_F \cup L_F; P_F, L_F, \in_F \rangle,$$

where $\in_F \subseteq P_F \times L_F$ is the membership relation, has the desired properties:
the number of orbits of 2-tuples is finite but the number of orbits of 3-tuples is infinite, since the relation $z - x = m(y - x)$ on P_F is definable, for each m.

On the other hand, all pairs $\langle x, y\rangle$ in $(P_F \cup L_F)^2$ form exactly 9 orbits that are described by the following conditions:

1. $x, y \in P_F$, $x = y$
2. $x, y \in L_F$, $x \neq y$
3. $x, y \in P_F$, $x \neq y$
4. $x, y \in L_F$, $x \cap y = \varnothing$
5. $x, y \in L_F$, $|x \cap y| = 1$
6. $x \in P_F$, $y \in L_F$, $x \in y$
7. $x \in P_F$, $y \in L_F$, $x \notin y$
8. $x \in L_F$, $y \in P_F$, $y \in x$
9. $x \in L_F$, $y \in P_F$, $y \notin x$

It would be interesting to find further examples of countable not–ω–categorical holographic structures.

Theorem

The Cartesian product of any two holographic structures is also holographic.

The set of prototypes for this product is the product of sets of prototypes of the components.

Theorem

Assume that $\mathfrak{A}$ *is a holographic structure of nonempty signature* σ *and let* M *be its set of prototypes. Then for any tuple* $\bar{a} \in \mathfrak{A}^{<\omega}$ *such that* $|\bar{a}| < \|\sigma\|$ *the Boolean algebra of sets definable by first order formulas with parameters* $\bar{a}$ *is finite and contains less or equal than* $|M|^{\|\sigma\|}$ *atoms.*

Theorem

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Corollary

Let $\mathfrak{M}$ be a holographic structure of signature σ. Then there exists an $m < \omega$ such that algebraic closure of any set containing less than $\|\sigma\|$ elements contains at most m elements.

Theorem

For any at most countable Boolean algebra $\mathfrak{B}$, the following conditions are equivalent:

1. $\mathfrak{B}$ *is holographic*
2. $\mathfrak{B}$ *contains a finite number of atoms*
3. $\mathfrak{B}$ *is ω–categorical*

In general case:

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In general case:

Theorem

For a Boolean algebra $\mathfrak{B}$, the following conditions are equivalent:

1. $\mathfrak{B}$ *is holographic*
2. *the number of isomorphism types of* $\mathfrak{B} \upharpoonright a$, $a \in \mathfrak{B}$ *is finite.*
3. *the number of isomorphism types of* $\langle \mathfrak{B}; a \rangle$, $a \in \mathfrak{B}$ *is finite.*

Theorem

A linear ordering is holographic if and only if it is almost 2–transitive, i.e. it has a finite number of orbits of 2–tuples.

Characterization of such countable orderings (J.G. Rosenstein):

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Characterization of such countable orderings (J.G. Rosenstein):
Operation τ.
Let $(L_i)_{i<\omega}$ be a family of linear orders and let $\mathbb{Q}_i$, $i < \omega$ be a partition of Q into its dense subsets. If $q \in Q_i$, we let $f(q) = i$.
The result of τ is $\sum_{q \in Q} L_{f(q)}$.

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The result of τ is $\sum_{q \in Q} L_{f(q)}$.

The class of all at most countable 2–transitive linear orders is the smallest class of orders containing 1 and closed under τ and finite sums.

Theorem

Let L be a countable linear ordering. Then the following conditions are equivalent:

1. *L is holographic*
2. *L is ω–categorical*
3. *L is almost 2–transitive*

Theorem

An Abelian group is holographic if and only if it is a group of bounded exponent, i.e., it satisfies some identity of kind $x^m = 1$, *for some* $m < \omega$.

It follows from the description of countable ω–categorical Abelian groups given by J.G. Rosenstein that for at most countable Abelian groups the properties of holographicity and ω–categoricity are equivalent.

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Any field is holographic if and only if it is finite.

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Assume that F is an infinite holographic field.
If $\mathrm{char}(F) = 0$ then the number of orbits of 1–tuples is infinite.

If $\mathrm{char}(F) > 0$ and there exists a transcendent $z \in F$ then the orbits of 2–tuples $\langle z, 1\rangle$, $\left\langle z, z^2\right\rangle$, $\left\langle z, z^3\right\rangle$ ... are distinct.

Theorem

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If $\mathrm{char}(F) > 0$ and F is infinite and algebraic over its prime subfield F then the set of elements algebraic over $\{1\}$ is infinite.
All these situations lead to contradictions.

For fields, holographicity = ω–categoricity.

Theorem

An equivalence is holographic if and only if the set of cardinalities of its classes is finite.

For countable case: holographicity = ω–categoricity.

Weak holographicity

Replace automorphisms with isomorphic embeddings:

Definition

A structure $\mathfrak{A}$ of signature σ is called weakly holographic *if there exists a finite $M \subseteq \mathfrak{A}$ such that for any $S \subseteq \mathfrak{A}$ of cardinality at most $\|\sigma\|$ there is an isomorphic embedding $\varphi : \mathfrak{A} \hookrightarrow \mathfrak{A}$ with the property $\varphi(S) \subseteq M$.*

The set M is called the set of prototypes.

All finite structures of finite signature are weakly holographic.

Holographicity $\Rightarrow$ weak holographicity.

Proposition

The Cartesian product of a finite family of weakly holographic structures is also weakly holographic.

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Proposition

Assume that $\mathfrak{A}$ is weakly holographic. Then the set of its signature predicates contains a finite subset such that each its signature predicate coincides with some predicate in this subset.

Theorem

Let $\mathfrak{A}$ be a weakly holographic structure of nonempty signature σ and let M be its set of prototypes. Then any its substructure generated by less than $\|\sigma\|$ elements contains at most $|M|^{\|\sigma\|}$ elements.

Theorem

Assume that an infinite Boolean algebra $\mathfrak{B}$ is weakly holographic. Then $\mathfrak{B}$ is not superatomic.

We do not assume $\mathfrak{B}$ to be countable.
For countable case this yields a criterion:

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We do not assume $\mathfrak{B}$ to be countable.
For countable case this yields a criterion:

Theorem

A <u>countable</u> infinite Boolean algebra is weakly holographic iff it is not superatomic.

The question for general case is open.

A linear ordering is called scattered if it does not contain suborderings of type η (ordering of the rational numbers).

Theorem

Any infinite scattered linear order is not weakly holographic.

For countable orderings we obtain a criterion:

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Theorem

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For countable orderings we obtain a criterion:

Theorem

For a countable linear ordering L the following conditions are equivalent:

1. *L is weakly holographic*
2. *L is not scattered*

The question for the general case is open.

Theorem

Let G be an Abelian group. Then the following conditions are equivalent:

1. *G is weakly holographic*
2. *G is a group of bounded exponent*
3. *G is holographic*

Theorem

Let P be a field. Then the following conditions are equivalent:

1. *P is holographic*
2. *P is weakly holographic*
3. *P is finite*

The only nontrivial implication is (2) $\Rightarrow$ (3).
Assume that P is a weakly holographic field. It cannot have characteristic 0 since in this case its unit generates an infinite set. It cannot either contain transcendent elements, otherwise any transcendent element t generates infinite set $t, t^2, t^3, \ldots$
Thus, P is algebraic over P_0, its prime subfield of non–zero characteristic. Each $\alpha \in P$ is a root of some polynomial f_α which is irreducible over P_0.

If α is a root of an irreducible polynomial of degree n then all

$$a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n,\ a_i \in P_0,\ i \leqslant n \tag{1}$$

are pairwise distinct, there are $|P_0|^{n+1}$ such elements, and all of them are generated by a single element α.
Next, the degrees of such polynomials f_α, $\alpha \in P$ are bounded by a finite natural number. If not, elements $\alpha \in P$ could generate subsets of arbitrarily big cardinality, a contradiction.
It follows that the degrees of polynomials f_α are bounded, and, since P_0 is finite, there can be only a finite number of f_α's. Each of them has a finite number of roots and P consists of such roots. It follows that P is finite.

Theorem

Let E be an equivalence on a countable set. Then the following conditions are equivalent:

1. *E is not weakly holographic*
2. *The number of infinite classes of E is finite and the cardinalities of its finite classes are not bounded.*

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- Boolean algebras
- Linear orderings
- Equivalences

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- Boolean algebras
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The properties of holographicity and weak holographicity are equivalent on the classes of

- Abelian groups
- Fields

Further studies: weakly co–holographic structures

Definition

A structure $\mathfrak{A}$ *of signature* σ *is called* *weakly co–holographic* *if there exists a finite* $M \subseteq \mathfrak{A}$ *such that for any* $S \subseteq \mathfrak{A}$ *of cardinality at most* $\|\sigma\|$ *there is an isomorphic embedding* $\varphi : \mathfrak{A} \hookrightarrow \mathfrak{A}$ *with the property* $\varphi(M) \supseteq S$.

Question: What are properties of such structures?

Publications:

- Kasymkhanuly B., Morozov A.S. On holographic structures Siberian Mathematical Journal. 2019. V.60. N2. P.312-318
- Kasymkhanuly B., Morozov A.S. On weakly holographic structures Siberian Mathematical Journal. 2022. V.63. N6 (in print)

Thank you!