On the complexity of the Quantified Constraint Satisfaction Problem

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Quantified Equality Constraints

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QCSP$(\mathbb{N}; x = y)$

Given a sentence $\forall x_1 \exists x_2 \ldots \forall x_{n-1} \exists x_n \; (x_{i_1} = x_{j_1} \wedge \cdots \wedge x_{i_s} = x_{j_s})$.
Decide whether it holds.

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Given a sentence $\forall x_1 \exists x_2 \ldots \forall x_{n-1} \exists x_n \, (R(\ldots) \wedge \cdots \wedge R(\ldots))$.
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What is the complexity of QCSP($\mathbb{N}; x = y \to y = z$)?

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Lemma [Zhuk, Martin, 2021]

QCSP($\mathbb{N}; x = y \to y = z$) is PSpace-hard.

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What is the complexity of $\mathrm{QCSP}(\mathbb{N}; x = y \rightarrow y = z)$?

- $\mathrm{QCSP}(\mathbb{N}; x = y \rightarrow y = z)$ is coNP-hard [Bodirsky, Chen, 2010].

Lemma [Zhuk, Martin, 2021]

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Theorem [Zhuk, Martin, Bodirsky, Chen, 2021]

Suppose relations $R_1, \ldots, R_s$ are definable by some Boolean combination of atoms of the form $(x = y)$. Then $\mathrm{QCSP}(\mathbb{N}; R_1, \ldots, R_s)$ is either in P, NP-complete, or PSpace-complete.

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What is the complexity of QCSP($\mathbb{Q}; x = y \to y \geq z$)?
Nobody knows!

Quantified Constraint Satisfaction Problem

Γ is a set of relations on a finite set A.

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QCSP(Γ)

Given: a sentence

$$\exists y_1 \forall x_1 \ldots \exists y_t \forall x_t (R_1(\ldots) \wedge \cdots \wedge R_s(\ldots)),$$

where $R_1, \ldots, R_s \in \Gamma$.
Decide: whether it holds.

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Question

What is the complexity of QCSP(Γ) for different Γ?

Σ	dual-Σ	Classification	Complexity Classes

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$\{\exists, \vee\}$	$\{\forall, \wedge\}$	Trivial	L

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Decide whether it holds.

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$\{\exists, \wedge\}$	$\{\forall, \vee\}$	CSP Dichotomy	P, NP-complete

Constraint Satisfaction Problem:

Given a sentence $\exists y_1 \ldots \exists y_t (R_1(\ldots) \wedge \cdots \wedge R_s(\ldots))$,
where $R_1, \ldots, R_s \in \Gamma$.
Decide whether it holds.

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$\{\exists, \wedge, \vee\}$	$\{\forall, \wedge, \vee\}$	Trivial iff the core has one element	L NP-complete

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$\{\exists, \forall, \wedge, \vee, \neg\}$		Trivial iff Γ is trivial	L PSPACE-complete

Given a sentence
$\exists y_1 \forall x_1 \ldots \exists y_t \forall x_t ((\neg R_1(\ldots) \vee R_2(\ldots)) \wedge \neg R_3(\ldots))$,
where $R_1, \ldots, R_3 \in \Gamma$.
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- If Γ contains all predicates then QCSP(Γ) is PSPACE-complete.

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- If Γ consists of linear equations in a finite field then QCSP(Γ) is in P.

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- If Γ contains all predicates then QCSP(Γ) is PSPACE-complete.
- If Γ consists of linear equations in a finite field then QCSP(Γ) is in P.

Theorem [Schaefer 1978 + Creignou et al. 2001/ Dalmau 1997.]

Suppose Γ is a constraint language on $\{0, 1\}$. Then

- QCSP(Γ) is in P if Γ is preserved by an idempotent WNU operation,
- QCSP(Γ) is PSPACE-complete otherwise.

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- Put $A' = A \cup \{*\}$, Γ' is Γ extended to A'. Then QCSP(Γ') is equivalent to CSP(Γ).

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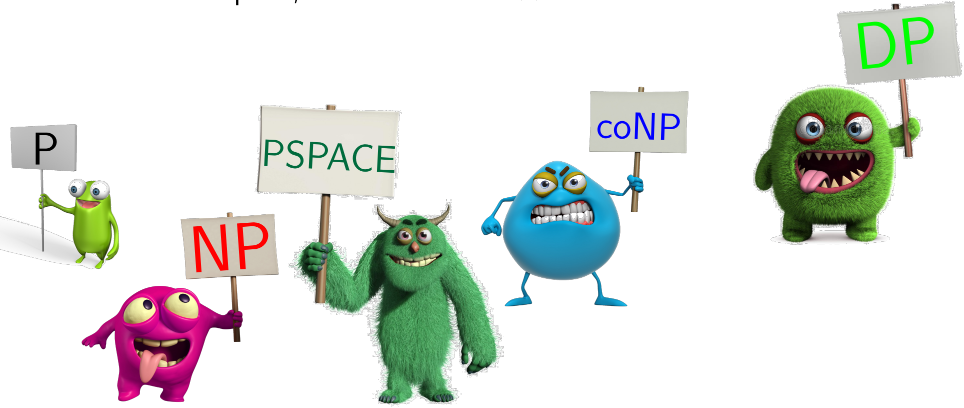

QCSP Complexity Classes

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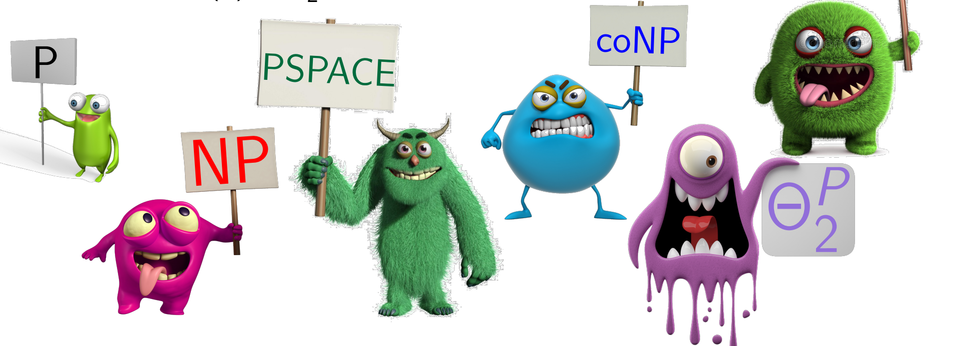
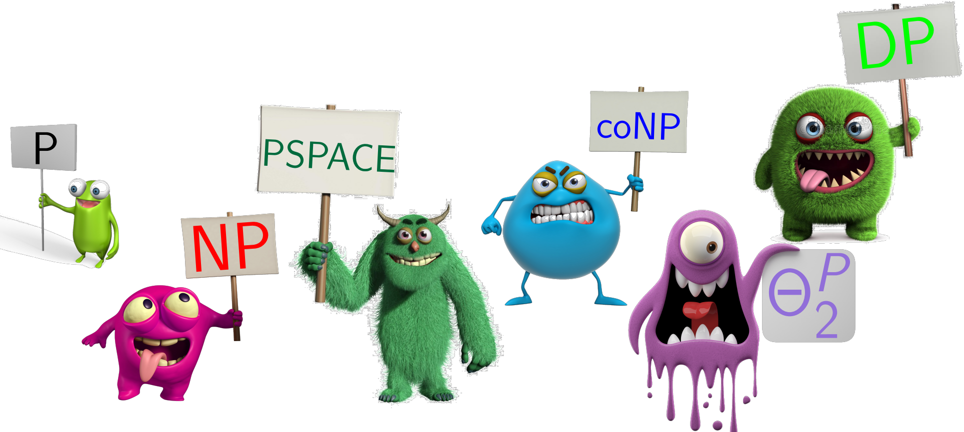


QCSP Complexity Classes



QCSP Complexity Classes

Theorem [Zhuk, Martin, 2019]

Suppose Γ is a constraint language on $\{0, 1, 2\}$ containing $\{x = a \mid a \in \{0, 1, 2\}\}$. Then QCSP($\Gamma$) is

- in P, or
- NP-complete, or
- coNP-complete, or
- PSPACE-complete.

Given a sentence $\exists y_1 \forall x_1 \exists y_2 \forall x_2 \ldots \exists y_n \forall x_n (R_1(\ldots) \wedge \cdots \wedge R_s(\ldots))$.

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DP $=$ **NP** $\wedge$ **coNP**: Each plays its own game. Yes-instance: EP wins and UP loses.

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$\mathbf{DP} = \mathbf{NP} \wedge \mathbf{coNP}$: Each plays its own game. Yes-instance: EP wins and UP loses.
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What is in the middle?

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CSP Dichotomy Theorem [Bulatov, Zhuk, 2017]

CSP(Γ)

- is either NP-complete,
- or in P.

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PSpace-hardness

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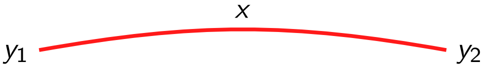

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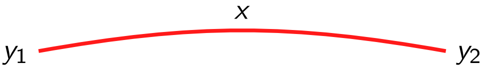

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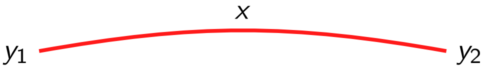

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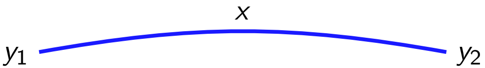

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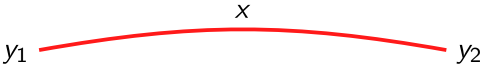

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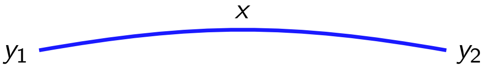
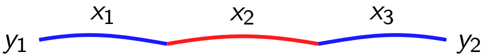


$$\exists u_1 \exists u_2 R_1(y_1, u_1, x_1) \wedge R_0(u_1, u_2, x_2) \wedge R_1(u_2, y_2, x_3)$$



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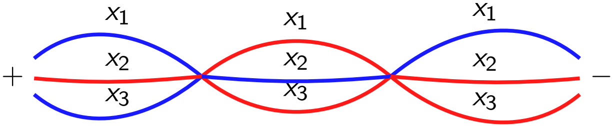

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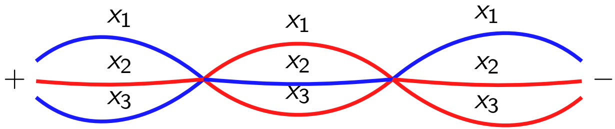

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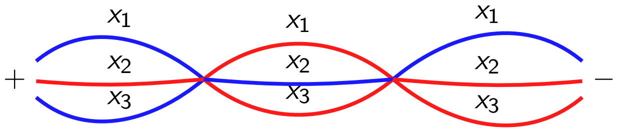

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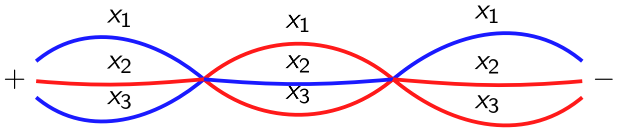

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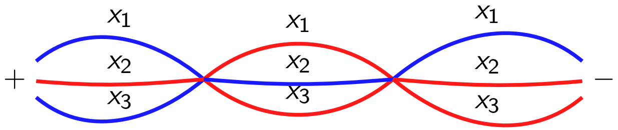

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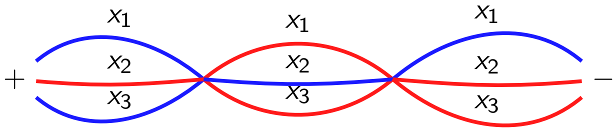

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Claim

QCSP(Γ) is coNP-hard.

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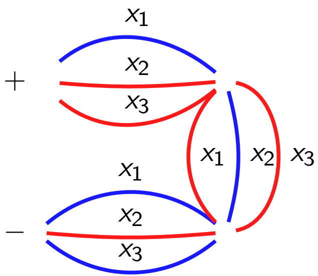

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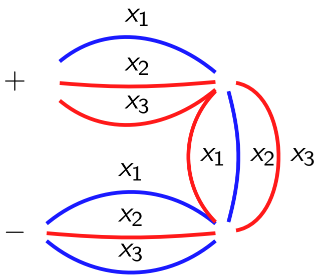

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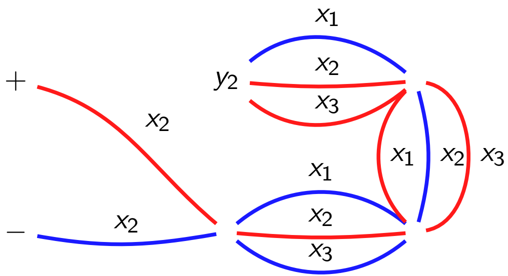

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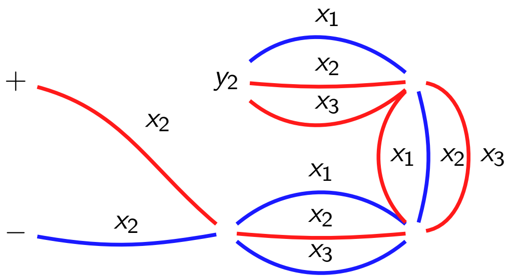

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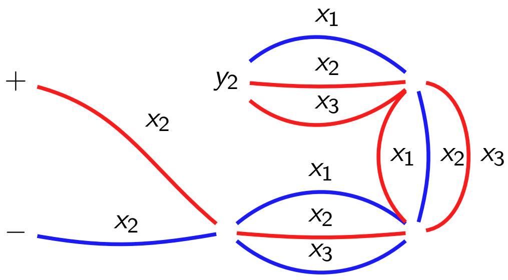

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Claim

QCSP(Γ) is PSpace-hard.

PSpace-hardness

PSpace-hardness

Let $A = \{+, -, \boxed{0}, \boxed{1}\}$

$R_0(y_1, y_2, x) = (y_1, y_2 \in \{+, -\}) \wedge (x = 0 \rightarrow y_1 = y_2)$

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Lemma

QCSP($R_0, R_1, \{+\}, \{-\}$) is PSpace-hard.

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Lemma

$\mathrm{QCSP}(R_0, R_1, \{+\}, \{-\})$ is PSpace-hard.

Theorem

Suppose

1. Γ contains $\{x = a \mid a \in A\}$
2. $\mathrm{QCSP}(\Gamma)$ is PSpace-hard.

Then there exist

- $D \subseteq A$
- a nontrivial equivalence relation σ on D
- $\varnothing \subsetneq B, C \subsetneq A$, $B \cap C = \varnothing$

s.t. $B(x) \to \sigma(y_1, y_2)$ and $C(x) \to \sigma(y_1, y_2)$ are definable over Γ.

QCSP Dichotomy

CSP Dichotomy Theorem [Bulatov, Zhuk, 2017]

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Reduction to CSP

QCSP Instance

$\Psi = \exists y_1 \forall x_1 \exists y_2 \forall x_2 \ldots \exists y_n \forall x_n \Phi.$

Reduction to CSP

QCSP Instance

$\Psi = \exists y_1 \forall x_1 \exists y_2 \forall x_2 \dots \exists y_n \forall x_n \Phi.$

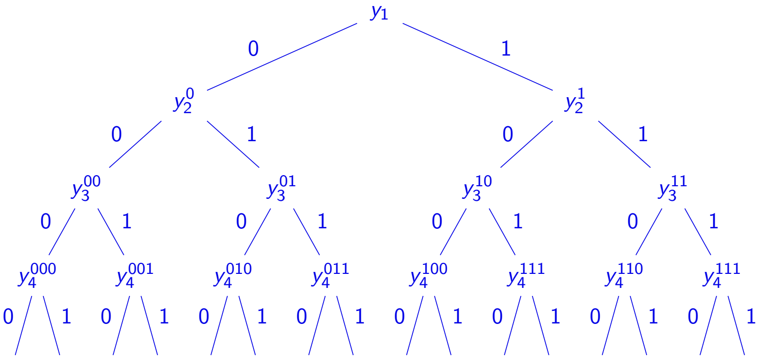
Put $R(y_1, \dots, y_n, x_1, \dots, x_n) = \Phi$.

Reduction to CSP

QCSP Instance

$\Psi = \exists y_1 \forall x_1 \exists y_2 \forall x_2 \dots \exists y_n \forall x_n \Phi.$

Put $R(y_1, \dots, y_n, x_1, \dots, x_n) = \Phi$.



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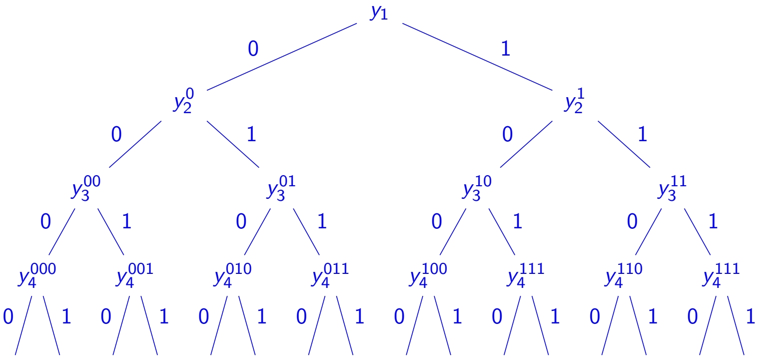

$R(y_1, y_2^0, y_3^{00}, y_4^{000}, 0, 0, 0, 0) \wedge$

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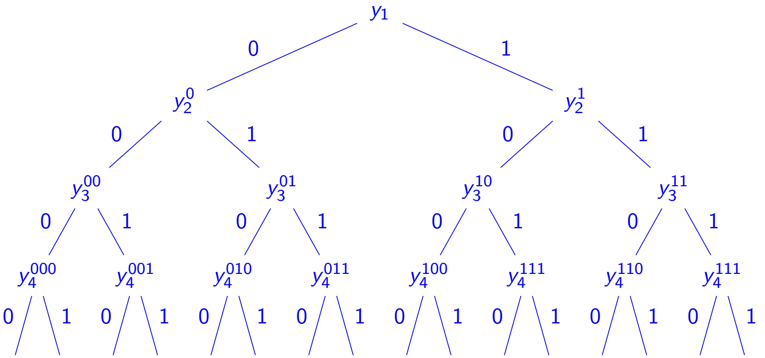

$R(y_1, y_2^0, y_3^{00}, y_4^{000}, 0, 0, 0, 0) \wedge R(y_1, y_2^0, y_3^{00}, y_4^{000}, 0, 0, 0, 1) \wedge$

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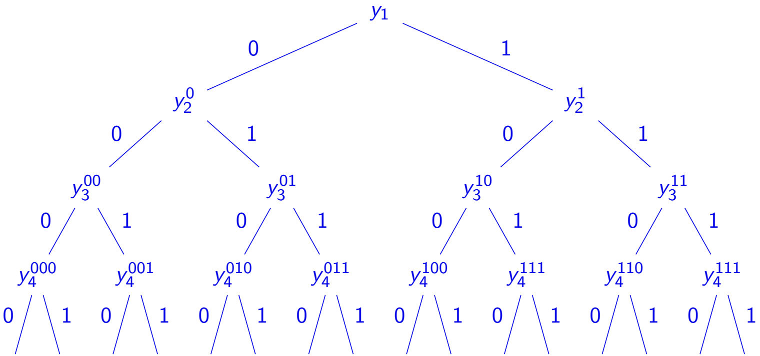

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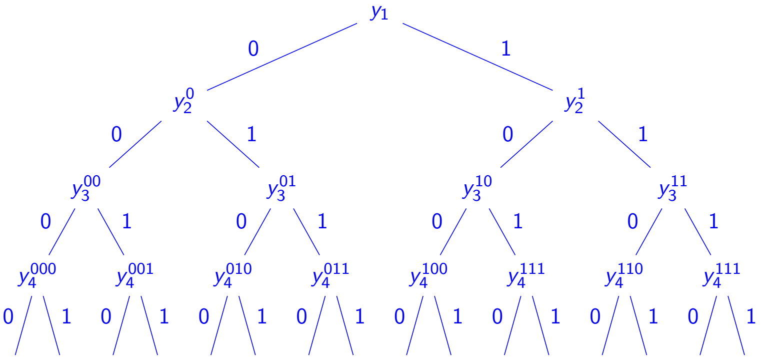

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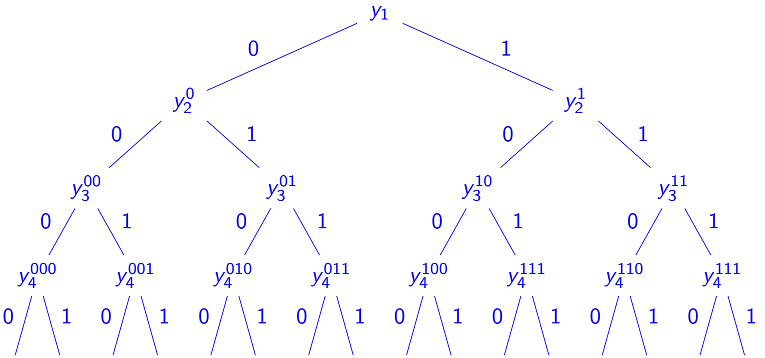

$$R(y_1, y_2^0, y_3^{00}, y_4^{000}, 0, 0, 0, 0) \wedge R(y_1, y_2^0, y_3^{00}, y_4^{000}, 0, 0, 0, 1) \wedge$$
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$$\wedge\, R(y_1, y_2^1, y_3^{11}, y_4^{111}, 1, 1, 1, 0) \wedge$$

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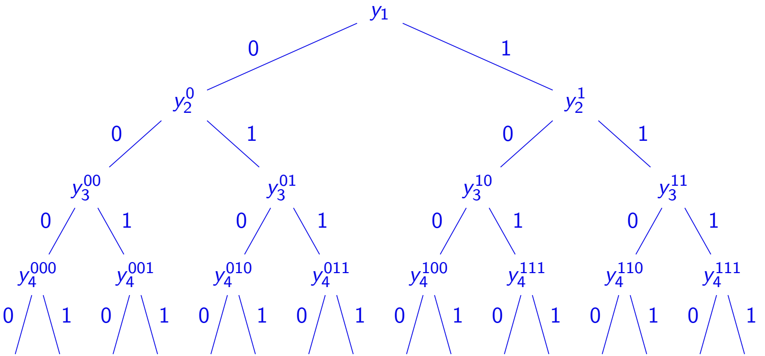

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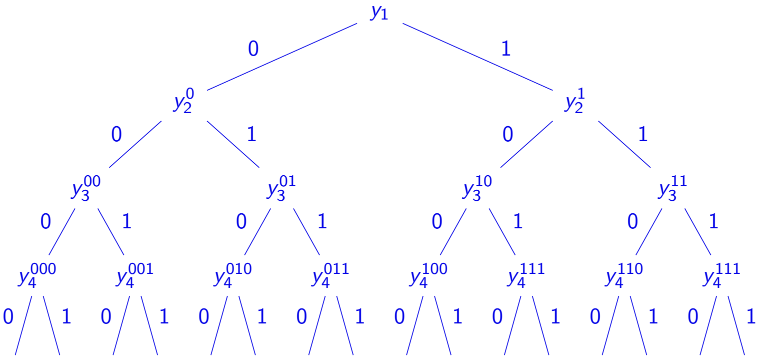

$$\mathrm{ExpCSP}_R^n = R(y_1, y_2^0, y_3^{00}, y_4^{000}, 0, 0, 0, 0) \wedge R(y_1, y_2^0, y_3^{00}, y_4^{000}, 0, 0, 0, 1) \wedge$$
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Complexity class Π_2^P

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Π_2^P is the class of problems $\mathcal{U}$

$$\mathcal{U}(Z) = \forall X^{|X|<p(|Z|)} \exists Y^{|Y|<q(|Z|)} \mathcal{V}(X, Y, Z),$$

where $\mathcal{V} \in \mathrm{P}$, and p and q are polynomials.

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$$\Psi \Leftrightarrow \forall \Omega \subseteq \mathrm{ExpCSP}_R^n \;\; ^{|\Omega|<p(|\Phi|)} \;\; (\exists (y_1, y_2^0, y_2^1, y_3^{00}, \dots)\ \Omega)$$

Theorem (Π_2^P vs PSpace)

QCSP(Γ)

- *is either PSpace-hard*
- *or in* Π_2^P.

* *if* Γ *contains* $\{x = a \mid a \in A\}$ *then* QCSP(Γ) *is PSpace-hard IFF there exist a nontrivial equivalence relation* σ *on* $D \subseteq A$, $\varnothing \subsetneq B, C \subsetneq A$, $B \cap C = \varnothing$, *s.t.* $B(x) \to \sigma(y_1, y_2)$ *and* $C(x) \to \sigma(y_1, y_2)$ *are definable over* Γ.

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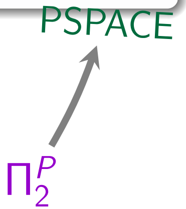

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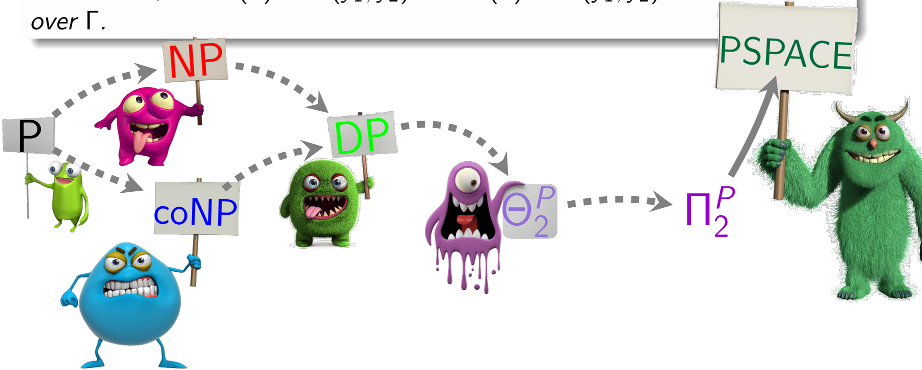

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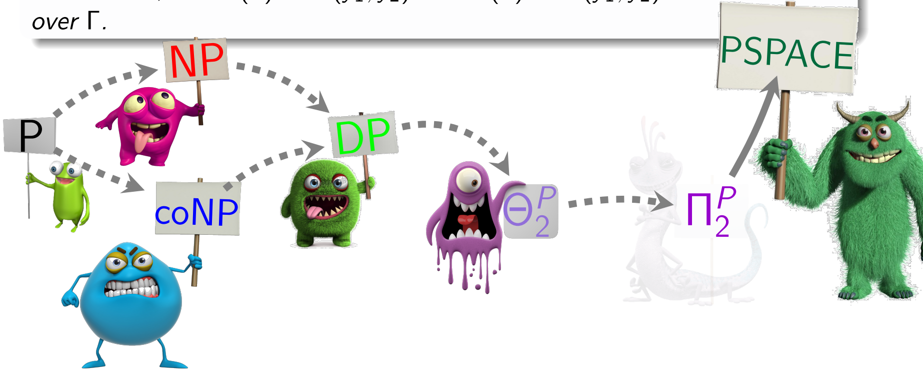

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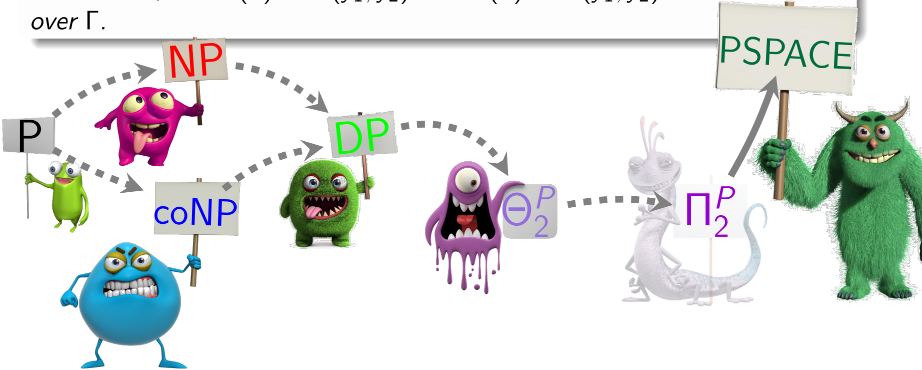

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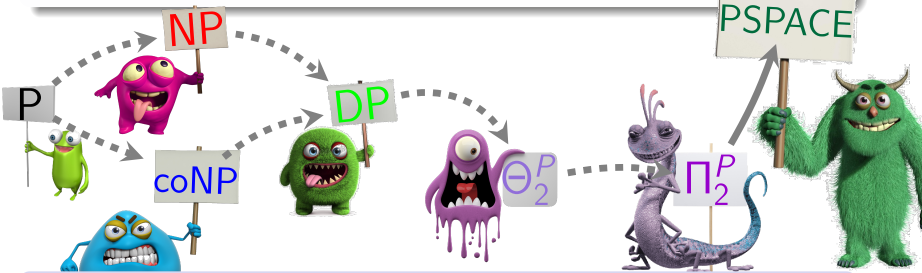

Lemma

There exists Γ on a 6-element set such that QCSP(Γ) is Π_2^P-complete.

Π_2^P-example

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Π_2^P-complete problem on $\{0,1\}$

$\forall x_1 \ldots \forall x_m \exists x_{m+1} \ldots \exists x_n \; 1\mathrm{IN}3(x_{i_1}, x_{i_2}, x_{i_3}) \wedge \cdots \wedge 1\mathrm{IN}3(x_{i_{3l-2}}, x_{3l-1}, x_{3l})$

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$\forall x_1 \ldots \forall x_m \exists x_{m+1} \ldots \exists x_n \; \mathrm{1IN3}(x_{i_1}, x_{i_2}, x_{i_3}) \wedge \cdots \wedge \mathrm{1IN3}(x_{i_{3l-2}}, x_{3l-1}, x_{3l})$

$A = \{0, 1, 2\}$, variables are of 2 sorts, EP and UP play on different sorts.

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$$\mathrm{1IN3} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

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$$\mathrm{AND} = \begin{pmatrix} 0 & 1 & 0 & 1 & 2 & \forall \\ 0 & 0 & 1 & 1 & \forall & 2 \\ 0 & 0 & 0 & 1 & \forall & \forall \end{pmatrix}$$

$$\mathrm{OR} = \begin{pmatrix} 0 & 1 & 0 & 1 & 2 & \forall \\ 0 & 0 & 1 & 1 & \forall & 2 \\ 0 & 1 & 1 & 1 & \forall & \forall \end{pmatrix}$$

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$\forall x_1^0 \forall x_1^1 \forall x_2^0 \forall x_2^1 \dots \forall x_m^0 \forall x_m^1 \exists y_1 \exists y_2 \dots \exists y_n$

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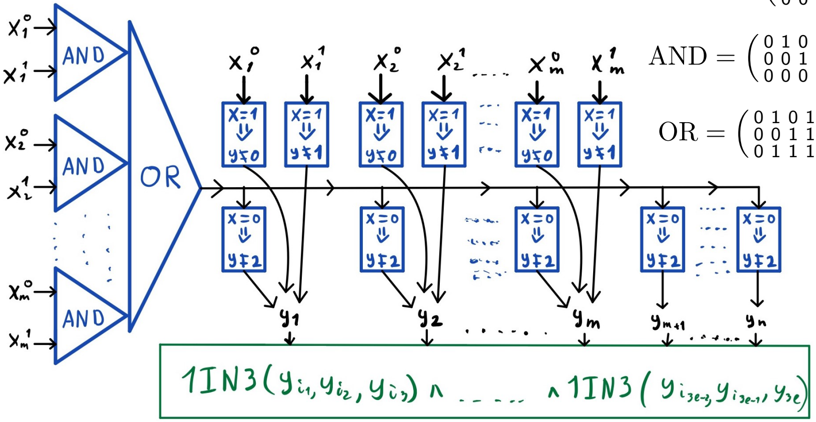
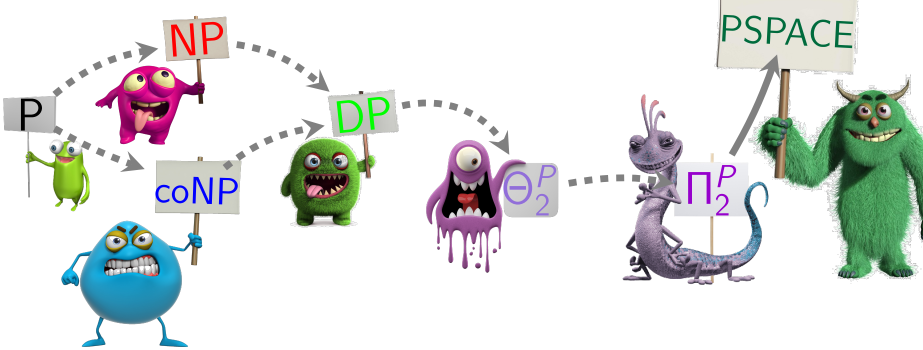
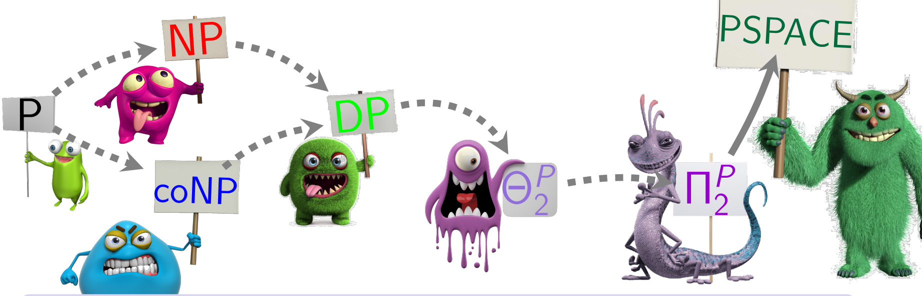




P
NP
coNP
DP
Θ_2^P
Π_2^P
PSPACE

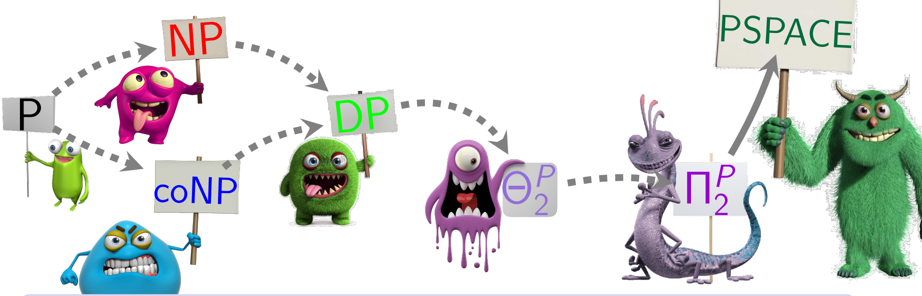


QCSP Hepta-chotomy

P: All moves are trivial.
NP: Only EP plays, the play of UP is trivial.
coNP: Only UP plays, the play of EP is trivial.
DP = **NP** $\wedge$ **coNP**: Each plays its own game. Yes-instance: EP wins and UP loses.
$\mathbf{\Theta_2^P}$ = (**NP** $\vee$ **coNP**) $\wedge \cdots \wedge$ (**NP** $\vee$ **coNP**): Each plays many games (no interaction). Yes-instance: any boolean combination.

$\mathbf{\Pi_2^P}$: First, UP plays, then EP plays.

PSpace: EP and UP play against each other. No restrictions.



QCSP Hepta-chotomy

P: All moves are trivial.

NP: Only EP plays, the play of UP is trivial.

coNP: Only UP plays, the play of EP is trivial.

DP = **NP** $\wedge$ **coNP**: Each plays its own game. Yes-instance: EP wins and UP loses.

$\mathbf{\Theta_2^P}$ = (**NP** $\vee$ **coNP**) $\wedge \cdots \wedge$ (**NP** $\vee$ **coNP**): Each plays many games (no interaction). Yes-instance: any boolean combination.

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Thank you for your attention