Computable families of sets and numberings

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Effectively discrete numberings

- A family $\mathcal{S}$ of c.e. sets is effectively discrete if there is a family of finite sets $\mathcal{D} = \{D_{f(n)} | n \in \omega\}$ for a computable function f such that for every $A \in \mathcal{S}$ there is a $D \subseteq A$, $D \in \mathcal{D}$, and for every $D \in \mathcal{D}$ there is at most one $A \supseteq D$, $A \in \mathcal{S}$.

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- If a family $\mathcal{S}$ of c.e. sets is effectively discrete then for every Friedberg numberings ν and μ of $\mathcal{S}$ we have a computable permutation p such that $\nu = \mu \circ p$, i.e., all Friedberg numberings are computably equivalent.

Families with unique Friedberg numberings

Theorem (Selivanov, 1976). There is a discrete family $\mathcal{S}$ (of graphs) of computable functions which is not effectively discrete such that all Friedberg numberings of $\mathcal{S}$ are computably equivalent.

Corollary (Goncharov, 1977). There is a computable structure which is computably categorical but not relatively computably categorical.

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- (Exercise). There is a T-minimal pair of c.e. sets $A \subseteq B$ such that $A \subseteq X \subseteq B$ for no computable X.

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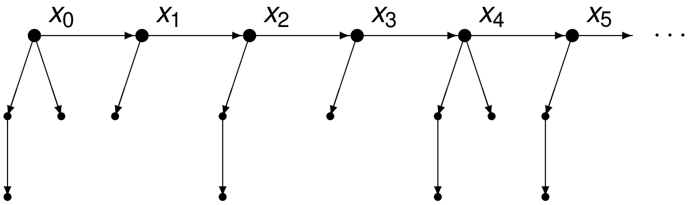

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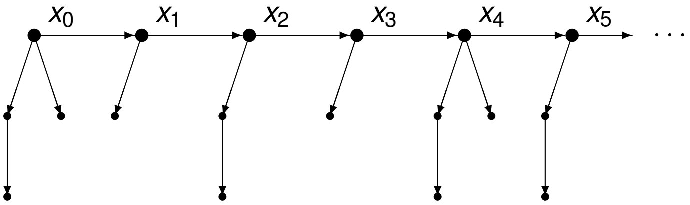

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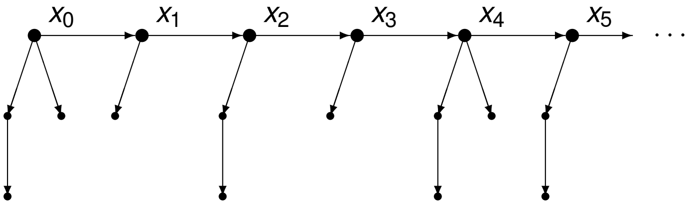

- The structure $\mathcal{M}(A, B)$ is relatively computably categorical only in the degrees of X such that $A \subseteq X \subseteq B$.
- For an isomorphism $p : \mathcal{M}(A, B) \to \mathcal{N}$ we have $p \leq_T A$ and $p \leq_T B$.

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- What are the degrees of Scott families of $\exists$-formulae for computable structures?

Properly 2-c.e. degrees

Theorem (K, 2022). There are c.e. sets $A \subseteq B$ and a 2-c.e. set D, $A \subseteq D \subseteq B$, such that

- $D \not\equiv_T W$ for every c.e. set W;
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In contrast, if $\nu = \mu \circ p$ for Friedberg numberings ν and μ of $\mathcal{S}(A, B)$ then p has a c.e. degree.

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In fact, $D = D_0 \oplus L(D_0)$, where D_0 is from the standard "properly 2-c.e. degree" finite injury construction (Cooper, 1970), and $L(D_0)$ is the Lachlan "pullback".

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The reduction $D \leq_T X$ is not uniform, two cases: $D \setminus X$ is infinite; and, otherwise, $D \subseteq^* X$.

Properly n-c.e. degrees

Theorem (K, 2022). For every $n \in \omega$ there are c.e. sets $A \subseteq B$ and an n-c.e. set D, $A \subseteq D \subseteq B$, such that

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Question. Is it possible for the degree of such D to be non-n-c.e. degree for every n?

Arslanov's criterion

Definition (Ershov, 1970). A numbering ν of a family $\mathcal{S}$ is precomplete if there is a computable function h such that

$$\varphi_n(h(n))\downarrow \Rightarrow \nu(h(n)) = \nu(\varphi_n(h(n))).$$

Theorem (Arslanov, 1972). For a c.e. set A we have $A \equiv_T \emptyset'$ iff there is a function $f \leq_T A$ such that $W_{f(n)} \neq W_n$ for every $n \in \omega$.

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Theorem (Selivanov, 1988). For a c.e. set A and a precomplete computable numbering ν of a family $\mathcal{S}$ of c.e. sets, $|\mathcal{S}| > 1$, we have $A \equiv_T \emptyset'$ iff there is a function $f \leq_T A$ such that $\nu(f(n)) \neq \nu(n)$ for every $n \in \omega$.

Wide numberings

Definition (Selivanov, 1992). A numbering ν of a family $\mathcal{S}$ is wide if there is a sequence of families $\mathcal{S}_k \subseteq \mathcal{S}, k \in \omega$ and a computable function $u(k)$ such that the predicate "$\nu(n) \in S_k$" is c.e. and $\nu(u(k)) \in S_k \setminus \cup_{i \neq k} S_i$.

Theorem (Selivanov, 1992). For a set A and a wide precomplete numbering ν the following is equivalent:

- there is a function $f \leq_T A$ such that $W_{f(n)} \neq W_n$ for every $n \in \omega$;
- there is a function $f \leq_T A$ such that $\nu(f(n)) \neq \nu(n)$ for every $n \in \omega$.

Non-uniformity of fixed points

Theorem (Arslanov, 2021). For any non-computable c.e. set A there is a function $h \leq_T A$ such that for every computable function f we have an $n \in \omega$ with $W_{h(n,f(n))} \neq W_{f(n)}$.

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Corollary (Arslanov, 2021). If ν is a wide precomplete numbering then for any non-computable c.e. set A there is a function $h \leq_T A$ such that for every computable function f we have an $n \in \omega$ with $\nu(h(n, f(n))) \neq \nu(f(n))$.

Computable numberings of A-c.e. sets

Definition (Mal'tcev, 1961). A numbering ν of a family $\mathcal{S}$ is complete relative to $a \in S$, if for every partially computable function ψ there is a computable function f such that $\nu(f(n)) = \nu(\psi(n))$ for $n \in \mathrm{dom}(\psi)$, and $\nu(f(n)) = a$ for $n \notin \mathrm{dom}(\psi)$.

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Theorem (Selivanov, 1982). $\emptyset' \leqslant_T A$ iff every principal computable numbering of a family of A-c.e. sets is complete relative to each element of the familly.

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What about precomplete instead of complete?

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Theorem (Faizrahmanov, 2022). A has a hyperimmune degree iff every principal computable numbering ν of a family of A-c.e. sets satisfies the Recursion Theorem, i.e., for every computable f there is an $n \in \omega$ such that $\nu(f(n)) = \nu(n)$.

Thank you!

Gongratulations to Victor L'vovich!