Recent developments on the Wadge degrees of Borel functions

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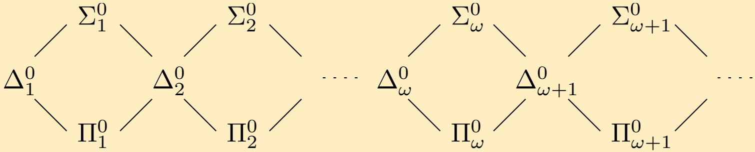
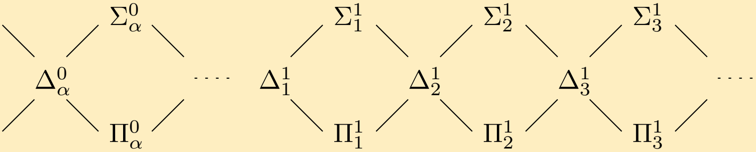
- Classical Wadge theory
- Describing Borel functions
- Homomorphic quasi-order
- Other recent results

Borel hierarchy / arithmetical hierarchy / hyperarithmetical hierarchy

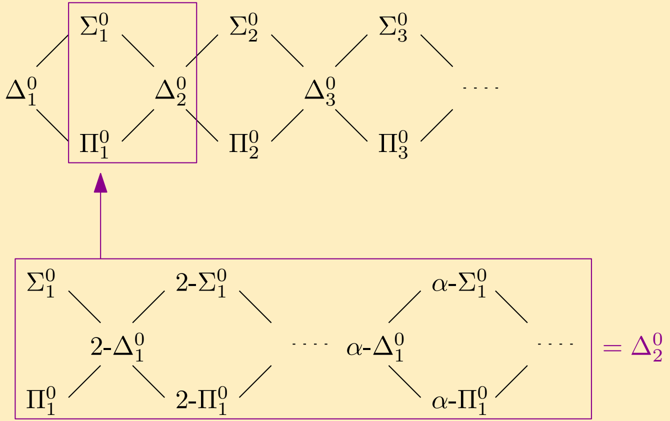


Projective hierarchy / analytical hierarchy



$\mathbf{\Delta^0_1}$ = clopen; $\mathbf{\Sigma^0_1}$ = open; $\mathbf{\Pi^0_1}$ = closed; $\mathbf{\Sigma^0_2} = \boldsymbol{F_\sigma}$; $\mathbf{\Pi^0_2} = \boldsymbol{G_\delta}$;
$\mathbf{\Delta^1_1}$ = Borel; $\mathbf{\Sigma^1_1}$ = analytic; $\mathbf{\Pi^1_1}$ = coanalytic

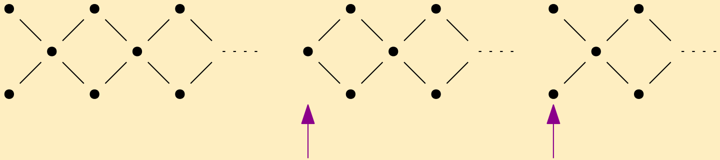
Hausdorff-Kuratowski difference hierarchy / Putnam-Ershov trial-and-error hier.



$\mathbf{2}\text{-}\mathbf{\Sigma^0_1}$: the differences $A \setminus B$ of open sets; $\boldsymbol{n}\text{-}\mathbf{\Sigma^0_1}$: $\boldsymbol{n}$-differences of open sets;
$(<\omega)\text{-}\mathbf{\Sigma^0_1}$: the finite Boolean combinations of open sets;

(The hierarchy obtained by finite Boolean operations and countable coproduct)

Wadge hierarchy (1970s)

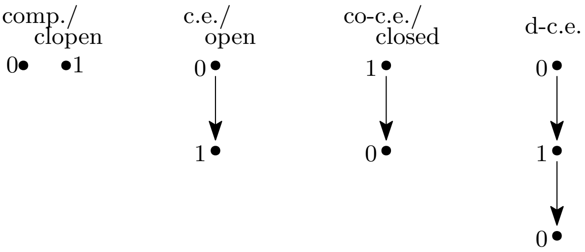


- Wadge degree: Ultimate measure for topological complexity
  Let X and Y be topological spaces, $A \subseteq X$ and $B \subseteq Y$,

$$A \leq_W B \iff \exists \text{ continuous } \theta\colon X \to Y$$
$$\forall x \in X \quad [x \in A \longleftrightarrow \theta(x) \in B]$$

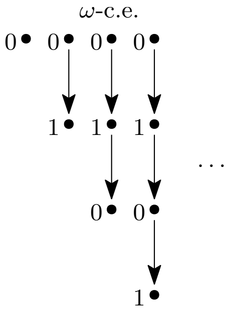
- $A <_W B \iff B$ is topologically more complicated than A.
- (AD) The subsets of ω^ω are semi-well-ordered by $\leq_W$.
- This assigns an ordinal rank to each subset of ω^ω.

Tree/Forest-representation of various Δ^0_2 sets:



- (computable/clopen) Given an input x, effectively decide $x \notin A$ (indicated by $\mathbf{0}$) or $x \in A$ (indicated by $\mathbf{1}$).
- (c.e./open) Given an input x, begin with $x \notin A$ (indicated by $\mathbf{0}$) and later x can be enumerated into A (indicated by $\mathbf{1}$).
- (co-c.e./closed) Given an input x, begin with $x \in A$ (indicated by $\mathbf{1}$) and later x can be removed from A (indicated by $\mathbf{0}$).
- (d-c.e.) Begin with $x \notin A$ (indicated by $\mathbf{0}$), later x can be enumerated into A (indicated by $\mathbf{1}$), and x can be removed from A again (indicated by $\mathbf{0}$).

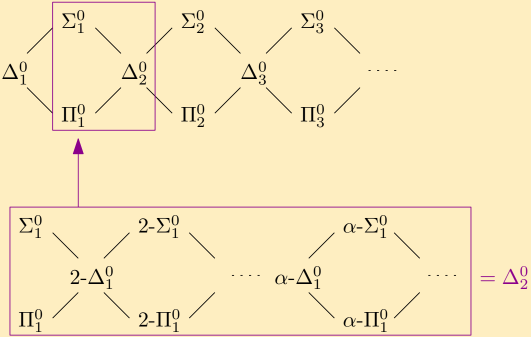
Forest-representation of a complete ω-c.e. set:

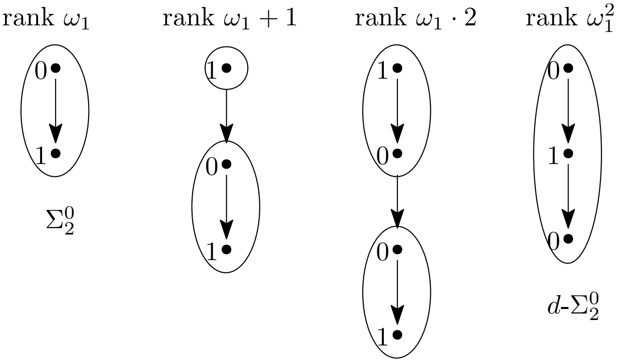


(ω-c.e.) The representation of "ω-c.e." is a forest consists of linear orders of finite length (a linear order of length $n + 1$ represents "n-c.e.").

- Given an input x, effectively choose a number $n \in \omega$ giving a bound of the number of times of mind-changes until deciding $x \in A$.

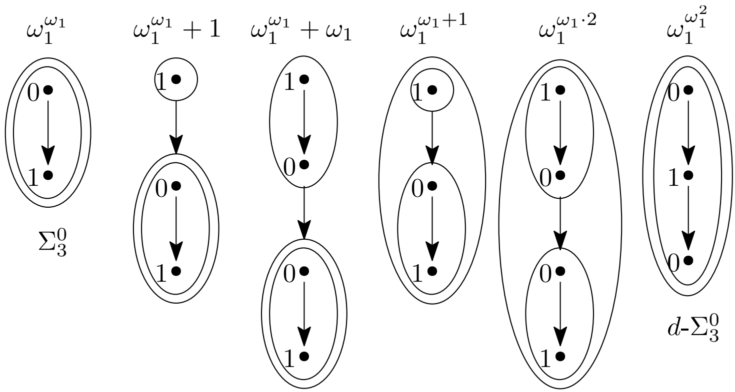
- The term $\mathbf{0}$, $\mathbf{1} = \emptyset$, ω^ω, resp. = rank $\mathbf{0}$
- The term $\mathbf{0}^\to\mathbf{1}$ = Open; $\mathbf{1}^\to\mathbf{0}$ = Closed. (rank $\mathbf{1}$)
- The term $\mathbf{0}^\to\mathbf{1}^\to\mathbf{0}$: Difference of two open sets (rank $\mathbf{2}$)
- The term $\mathbf{0}^\to\mathbf{1}^\to\mathbf{0}^\to\mathbf{1}$: Difference of three open sets (rank $\mathbf{3}$)
- Boolean combination of finitely many open sets (rank finite)
- The α-th level of the difference hierarchy (rank α)





Tree/Forest-representation of $\underset{\sim}{\mathbf{\Delta}}_3^0$ sets

The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}_3^0$ sets are exactly those represented by forests labeled by trees.



Tree/Forest-representation of $\underset{\sim}{\mathbf{\Delta}}^0_4$ sets

The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_4$ sets are exactly those represented by

forests labeled by trees which are labeled by trees.

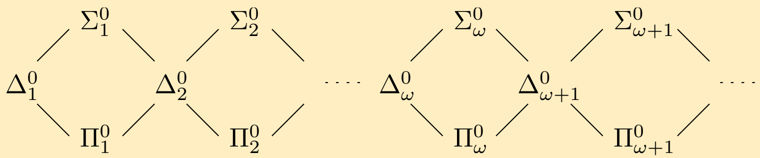
Term operation vs Ordinal operation

$\rightarrow \approx +$; $\sqcup \approx \sup$; $\langle \bullet \rangle \approx \omega_1^{\bullet}$

- The term $\langle 0^{\rightarrow} 1 \rangle = \mathbf{\Sigma}^0_2$; $\langle 1^{\rightarrow} 0 \rangle = \mathbf{\Pi}^0_2$ (rank ω_1)
- The term $\langle\langle 0^{\rightarrow} 1 \rangle\rangle = \mathbf{\Sigma}^0_3$; $\langle\langle 1^{\rightarrow} 0 \rangle\rangle = \mathbf{\Pi}^0_3$ (rank $\omega_1^{\omega_1}$)
- The term $\langle\langle\langle 0^{\rightarrow} 1 \rangle\rangle\rangle = \mathbf{\Sigma}^0_4$; $\langle\langle\langle 1^{\rightarrow} 0 \rangle\rangle\rangle = \mathbf{\Pi}^0_4$ (rank $\omega_1^{\omega_1^{\omega_1}}$)

Let $\omega_1 \uparrow\uparrow n$ be the n-th level of the exponential tower over ω_1.

- The term $\langle 0^{\rightarrow} 1 \rangle^n = \mathbf{\Sigma}^0_{n+1}$; $\langle 1^{\rightarrow} 0 \rangle^n = \mathbf{\Pi}^0_{n+1}$ (rank $\omega_1 \uparrow\uparrow n$)



What's the rank of $\mathbf{\Sigma}^0_\omega$ sets? Is it $\sup_{n<\omega}(\omega_1 \uparrow\uparrow n)$? No!!

- $\varepsilon_{\omega_1+\alpha}$ = the α-th solution of "$\omega_1^x = x$".
- $\varepsilon_{\omega_1+1} = \sup_{n<\omega}(\omega_1 \uparrow\uparrow n)$

Theorem (Wadge)

The Wadge rank of $\underset{\sim}{\Sigma}^0_\omega$ sets is $\varepsilon_{\omega_1+\omega_1}$.

Why? Term presentation:

- The term $\langle 0^{\rightarrow} 1\rangle^n$ represents Σ^0_{n+1} (rank $\omega_1 \uparrow\uparrow n$)
- The term $\sqcup_{n<\omega}\langle 0^{\rightarrow} 1\rangle^n$ represents a disjoint union of Σ^0_n sets (rank ε_{ω_1+1})
- The term $1^{\rightarrow} \sqcup_{n<\omega} \langle 0^{\rightarrow} 1\rangle^n$ represents a more complicated set which is still in Δ^0_ω (rank $\varepsilon_{\omega_1+1} + 1$)
- The term $\langle 1^{\rightarrow} \sqcup_{n<\omega} \langle 0^{\rightarrow} 1\rangle^n\rangle$ corresponds to rank $\omega_1^{\varepsilon_{\omega_1+1}+1}$
- The term $\sqcup_{m<\omega}\langle 1^{\rightarrow} \sqcup_{n<\omega} \langle 0^{\rightarrow} 1\rangle^n\rangle^m$ corresponds to rank ε_{ω_1+2}
- The term $\langle 0^{\rightarrow} 1\rangle^\omega$ represents Σ^0_ω (rank $\varepsilon_{\omega_1+\omega_1}$)

- The term $\langle 0^{\rightarrow}(\langle 1^{\rightarrow}0\rangle^{\omega} \sqcup \langle 0^{\rightarrow}1\rangle^{\omega})\rangle$ = rank $\omega_1^{\varepsilon_{\omega_1\cdot 2}+1}$
- The term $\langle 0^{\rightarrow}1^{\rightarrow}0\rangle^{\omega}$ represents d-Σ^0_{ω} (rank $\varepsilon_{\omega_1\cdot 3}$)
- The term $\langle\langle 0^{\rightarrow}1\rangle\rangle^{\omega}$ represents $\Sigma^0_{\omega+1}$ (rank $\varepsilon_{\omega_1^2}$)
- The term $\langle\langle 0^{\rightarrow}1\rangle^{n}\rangle^{\omega}$ represents $\Sigma^0_{\omega+n}$ (rank $\varepsilon_{\omega_1\uparrow\uparrow n}$)
- The term $\langle\langle 0^{\rightarrow}1\rangle^{\omega}\rangle^{\omega}$ represents $\Sigma^0_{\omega\cdot 2}$ (rank $\varepsilon_{\varepsilon_{\omega_1+\omega_1}}$)
- The term $\langle\langle 0^{\rightarrow}1\rangle^{n}\rangle^{\omega\cdot m}$ represents $\Sigma^0_{\omega\cdot m+n}$ (rank $\phi_1^{(m)}(\phi_0^{(n)}(0))$)

Here $\phi_0(x) = \omega_1^{1+x}$ and $\phi_1(x) = \varepsilon_{\omega_1+1+x}$.
Define $\phi_2(x)$ as the x-th solution of "$\phi_1(x) = x$".

- The term $\sqcup_{n<\omega}\langle 0^{\rightarrow}1\rangle^{\omega\cdot n}$ = rank $\phi_2(0) = \sup_{n<\omega}\phi_1^{(n)}(0)$
- The term $\langle 1^{\rightarrow} \sqcup_{n<\omega} \langle 0^{\rightarrow}1\rangle^{\omega\cdot n}\rangle$ = rank $\phi_0(\phi_2(0)+1)$
- The term $\sqcup_{m<\omega}\langle 1^{\rightarrow} \sqcup_{n<\omega} \langle 0^{\rightarrow}1\rangle^{\omega\cdot n}\rangle^{\omega\cdot m}$ = $\phi_2(1)$
- The term $\langle 0^{\rightarrow}1\rangle^{\omega^2}$ represents $\Sigma^0_{\omega^2}$ (rank $\phi_2(\omega_1)$)

Term operation vs Ordinal operation

$\rightarrow \approx +$; $\sqcup \approx \sup$; $\langle \bullet \rangle \approx \omega_1^{\bullet} = \phi_0(\bullet)$; $\langle \bullet \rangle^{\omega^\alpha} \approx \phi_\alpha(\bullet)$

- Classical Wadge theory
- Describing Borel functions
- Homomorphic quasi-order
- Other recent results

- The processes of approximating functions by finite mind-changes can be represented by

  labeled well-founded trees and forests.

▷ Well-founded forests ≈ terms in the signature $\{\to, \sqcup\}$ over an (infinitary) equational theory.

- (Lebesgue, Hausdorff) The Borel hierarchy ≈ the Baire hierarchy, i.e., the hierarchy of iterated limits.
- (Selivanov, K.-Montalbán) The processes of approximating Borel functions can be described by matryoshkas of trees: Within each node of the tree there is a tree, and within each node of that tree there is a tree, and within each node of that tree there is another tree, and so on.

▷ K.-Montalbán (2019) introduced

  the signature for describing matryoshkas of trees,

  and then, to each term t in the signature, assigned

  a class Σ_t of Borel functions.

- The classes $(\Sigma_t)_{t \text{ term}}$ can be considered as an ultimate refinement of the Borel hierarchy.

- Q: a preordered set.
- $\mathbf{Tree}(Q)$: the set of all Q-labeled well-founded trees.
- $^{\sqcup}\mathbf{Tree}(Q)$: the set of all Q-labeled well-founded forests.
- For any countable ordinal α, we inductively define $^{\sqcup}\mathbf{Tree}^{\alpha}(Q)$ as follows:

$$^{\sqcup}\mathbf{Tree}^{0}(Q) = Q, \quad ^{\sqcup}\mathbf{Tree}^{\alpha}(Q) = {}^{\sqcup}\mathbf{Tree}\left(\bigcup_{\beta<\alpha} {}^{\sqcup}\mathbf{Tree}^{\beta}(Q)\right)$$

$$^{\sqcup}\mathbf{Tree}^{\omega_1}(Q) = \bigcup_{\alpha<\omega_1} {}^{\sqcup}\mathbf{Tree}^{\alpha}(Q)$$

- These are exactly the terms in the following signature:

Definition

For a set Q, consider the signature consists of:

- A constant symbol $[q]$ for each $q \in Q$
- A 2-ary function symbol $^{\rightarrow}$
- An ω-ary function symbol $\bigsqcup$
- A unary function symbol $\langle\cdot\rangle$

Definition (K.-Montalbán 2019)

For each term $t \in {}^{\sqcup}\mathbf{Tree}^{\omega_1}(Q)$, a function $f : Z \to Q$ is in $\underset{\sim}{\Sigma}_t$ $\iff$

- ($t = q \in Q$) f is the constant map $x \mapsto q$.
- ($t = \sqcup_i s_i$) $\exists$ open cover $(U_i)_{i \in \mathbb{N}}$ of Z s.t. $f \upharpoonright U_i \in \underset{\sim}{\Sigma}_{s_i}$ for each $i \in \mathbb{N}$.
- ($t = r^{\to} s$) $\exists$ open $V \subseteq Z$ s.t. $f \upharpoonright V \in \underset{\sim}{\Sigma}_s$ and $f \upharpoonright (X \setminus V) \in \underset{\sim}{\Sigma}_r$.
- ($t = \langle s \rangle$) $\exists$ Baire one $\beta : Z \to Z$ and a $\underset{\sim}{\Sigma}_s$ function $g : Z \to Q$ s.t. $f = g \circ \beta$.

$$\underset{\sim}{\Sigma}^0_3 = \underset{\sim}{\Sigma}_{\langle\langle 0 \to 1 \rangle\rangle} \qquad \underset{\sim}{\Pi}^0_3 = \underset{\sim}{\Sigma}_{\langle\langle 1 \to 0 \rangle\rangle}$$

$$\underset{\sim}{\Sigma}^0_4 = \underset{\sim}{\Sigma}_{\langle\langle\langle 0 \to 1 \rangle\rangle\rangle} \qquad \underset{\sim}{\Pi}^0_4 = \underset{\sim}{\Sigma}_{\langle\langle\langle 1 \to 0 \rangle\rangle\rangle}$$

- The $\underset{\sim}{\Sigma}_t$-functions for terms $t \in {}^{\sqcup}\mathbf{Tree}^n(Q)$ = the $\underset{\sim}{\Delta}^0_{n+1}$-measurable functions.
- The $\underset{\sim}{\Sigma}_t$-functions for terms $t \in {}^{\sqcup}\mathbf{Tree}^{\omega_1}(Q)$ = the $\underset{\sim}{\Delta}^0_{\omega}$-measurable functions.
- There is a huge gap between the finite Borel rank and the Borel rank ω, which is filled by the transfinite tree hierarchy.

$$\overbrace{{}^{\sqcup}\mathbf{Tree}(Q) \subsetneq {}^{\sqcup}\mathbf{Tree}^2(Q) \subsetneq \cdots \subsetneq {}^{\sqcup}\mathbf{Tree}^n(Q) \subsetneq {}^{\sqcup}\mathbf{Tree}^{n+1}(Q) \subsetneq \dots}^{\underset{\sim}{\Delta}_{<\omega}} \subsetneq {}^{\sqcup}\mathbf{Tree}^{\omega}(Q) \subsetneq$$

$$\underbrace{\cdots \subsetneq {}^{\sqcup}\mathbf{Tree}^{\omega+n}(Q) \subsetneq \cdots \subsetneq {}^{\sqcup}\mathbf{Tree}^{\omega\cdot n}(Q) \subsetneq \cdots \subsetneq {}^{\sqcup}\mathbf{Tree}^{\omega^n}(Q) \subsetneq \cdots \subsetneq {}^{\sqcup}\mathbf{Tree}^{\omega_1}(Q)}_{\underset{\sim}{\Delta}_{\omega}}.$$

- The transfinite tree hierarchy ${}^{\sqcup}\mathbf{Tree}^{\omega_1}(Q)$ corresponds to $\mathbf{\Delta}^0_{\omega}$.
- ▷ Then, what terms can be used to express the Borel rank beyond ω?
- ▷ The answer is that $\mathbf{\Delta}^0_{\omega+1}$ corresponds to ${}^{\sqcup}\mathbf{Tree}^{\omega_1}(\mathbf{Tree}(Q))$.
- ▷ Here, $t \in \mathbf{Tree}(Q)$ is treated as a constant symbol.
  In this case, this is a constant symbol of rank ω, so it is written as $\langle t\rangle^{\omega}$.
- What this suggests is that the next levels are expressed by trees having two types of labeling functions.

$$ {}^{\sqcup}\mathbf{Tree}^{\omega_1,0}(Q) = Q, \quad {}^{\sqcup}\mathbf{Tree}^{\omega_1,\alpha}(Q) = {}^{\sqcup}\mathbf{Tree}^{\omega_1}\left(\bigcup_{\beta<\alpha}\mathbf{Tree}^{\omega_1,\beta}(Q)\right) $$

$$ {}^{\sqcup}\mathbf{Tree}^{\omega_1,\omega_1}(Q) = \bigcup_{\alpha<\omega_1} {}^{\sqcup}\mathbf{Tree}^{\omega_1,\alpha}(Q) $$

- ▷ Two labeling functions $\phi_0 = \langle\cdot\rangle$ and $\phi_1 = \langle\cdot\rangle^{\omega}$ are available at this hierarchy.
- ▷ By using them, it is finally possible to describe up to $\mathbf{\Delta}^0_{\omega^2}$.

- The label for the next level, $\langle\cdot\rangle^{\omega^2}$, may be used, and to proceed to the next level, the transfinite hierarchy ${}^{\sqcup}\mathbf{Tree}^{\omega_1,\omega_1,\alpha}$ is defined as:

$$ {}^{\sqcup}\mathbf{Tree}^{\omega_1,\omega_1,0}(Q) = Q, \quad {}^{\sqcup}\mathbf{Tree}^{\omega_1,\omega_1,\alpha}(Q) = {}^{\sqcup}\mathbf{Tree}^{\omega_1,\omega_1}\left(\bigcup_{\beta<\alpha} \mathbf{Tree}^{\omega_1,\omega_1,\beta}(Q)\right) $$

$$ {}^{\sqcup}\mathbf{Tree}^{\omega_1,\omega_1,\omega_1}(Q) = \bigcup_{\alpha<\omega_1} {}^{\sqcup}\mathbf{Tree}^{\omega_1,\omega_1,\alpha}(Q) $$

  - ▷ Three labeling functions $\phi_0 = \langle\cdot\rangle$, $\phi_1 = \langle\cdot\rangle^{\omega}$ and $\phi_2 = \langle\cdot\rangle^{\omega^2}$ are available at this hierarchy.
  - ▷ Now, it is possible to describe up to $\underset{\sim}{\mathbf{\Delta}}^0_{\omega^3}$.

Definition (K.-Montalbán's Veblen signature $\mathcal{L}_{\mathrm{Veb}}$)

For a set Q, the Veblen signature $\mathcal{L}_{\mathrm{Veb}}(Q)$ consists of:

- A constant symbol q for each $q \in Q$.
- A binary function symbol $\rightarrow$.
- An ω-ary function symbol $\sqcup$.
- A unary function symbol ϕ_α for each $\alpha < \omega_1$.

Definition (K.-Montalbán 2019)

For each term $t \in \mathcal{L}_{\mathrm{Veb}}(Q)$, a function $f : Z \to Q$ is in $\underset{\sim}{\Sigma}_t \iff$

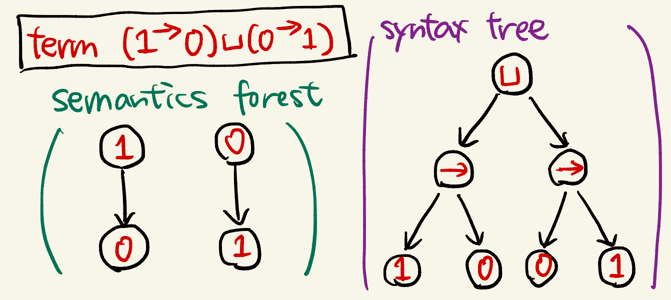
- $(t = q \in Q)$ f is the constant map $x \mapsto q$.
- $(t = \sqcup_i s_i)$ $\exists$ open cover $(U_i)_{i \in \mathbb{N}}$ of Z s.t. $f \upharpoonright U_i \in \underset{\sim}{\Sigma}_{s_i}$ for each $i \in \mathbb{N}$.
- $(t = r^{\rightarrow} s)$ $\exists$ open $V \subseteq Z$ s.t. $f \upharpoonright V \in \underset{\sim}{\Sigma}_s$ and $f \upharpoonright (X \setminus V) \in \underset{\sim}{\Sigma}_r$.
- $(t = \phi_\alpha(s))$ $\exists$ Baire class ω^α function $\beta : Z \to Z$ and a $\underset{\sim}{\Sigma}_s$ function $g : Z \to Q$ s.t. $f = g \circ \beta$.

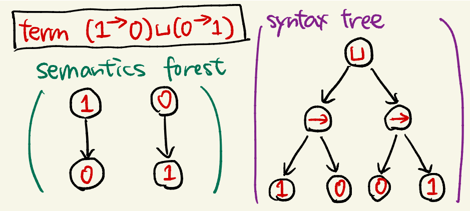
The $\underset{\sim}{\Sigma}_t$-functions for terms $t \in \mathcal{L}_{\mathrm{Veb}}(Q)$ = The Borel functions.

Definition (Syntax tree)

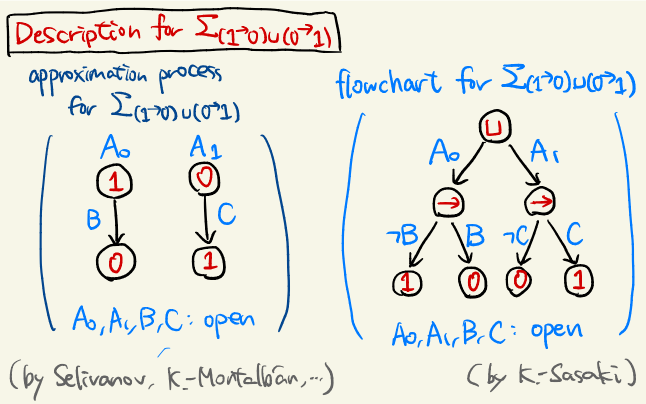
Let $\mathcal{L}$ be a (possibly infinitary) signature. Any $\mathcal{L}$-term t defines a labeled tree $\mathbf{Syn}(t)$ called the *syntax tree* of the term t as follows:

- If t is a constant or variable symbol x, then $\mathbf{Syn}(t) = \langle x \rangle$.
- If t is of the form $f(\langle s_i \rangle_{i \in I})$ for some function symbol $f \in \mathcal{L}$ and $\mathcal{L}$-terms $\langle s_i \rangle_{i \in I}$, then $\mathbf{Syn}(t) = \langle f \rangle^{\to} \sqcup_{i \in I} \mathbf{Syn}(s_i)$.





There are two equivalent ways to describe $\mathbf{\Sigma}_t$:



Definition (K.-Sasaki 2021)

A *flowchart* on an $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t over a space X is a family $(S_\sigma)_{\sigma \in \mathbf{Syn}(t)}$ s.t.:

1. If $\sigma \in \mathbf{Syn}(t)$ is labeled by $\to$, then S_σ is a $\utilde{\Sigma}^0_{\mathrm{rank}(\sigma)}$ subset of X.
2. If $\sigma \in \mathbf{Syn}(t)$ is labeled by $\sqcup$, then S_σ is a sequence $(S_{\sigma,n})_{n \in \omega}$ of $\utilde{\Sigma}^0_{\mathrm{rank}(\sigma)}$ subsets of X.
3. If $\sigma \in \mathbf{Syn}(t)$ is labeled by ϕ_α, then $S_\sigma = X$.

Here, the *Borel rank* of a node $\sigma \in \mathbf{Syn}(t)$ as follows. First, enumerate all proper initial segments of σ which are labeled by Veblen function symbols ϕ_α:

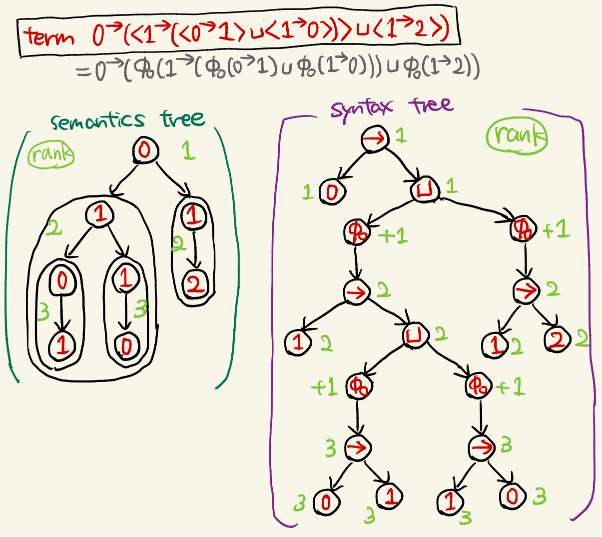
$$\tau_0 \prec \tau_1 \prec \tau_2 \prec \cdots \prec \tau_\ell \prec \sigma,$$

where τ_i is labeled by ϕ_{α_i}. Then, the Borel rank of $\sigma \in \mathbf{Syn}_t$ is defined as:

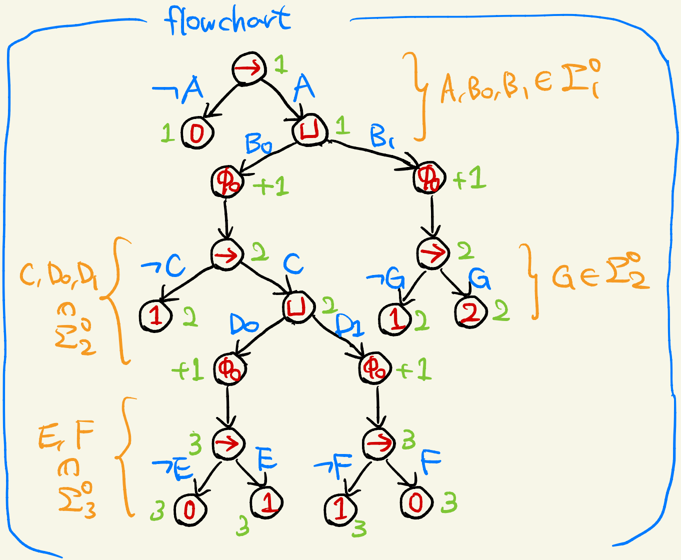
$$\mathrm{rank}(\sigma) = 1 + \omega^{\alpha_0} + \omega^{\alpha_1} + \cdots + \omega^{\alpha_\ell}.$$

Enumerate all proper initial segments of σ which are labeled by Veblen function symbols ϕ_α as $\tau_0 \prec \tau_1 \prec \tau_2 \prec \dots \prec \tau_\ell \prec \sigma$, where τ_i is labeled by ϕ_{α_i}.
Then, the Borel rank of $\sigma \in \mathbf{Syn}_\iota$ is defined as:

$$\mathbf{rank}(\sigma) = 1 + \omega^{\alpha_0} + \omega^{\alpha_1} + \dots + \omega^{\alpha_\ell}.$$



Description for $\Sigma_0^{\to}(\langle 1^{\to}(\langle 0^{\to}1\rangle \sqcup \langle 1^{\to}0\rangle)\rangle \sqcup \langle 1^{\to}2\rangle)$



- Classical Wadge theory
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Let S be a topological space, and Q be a preorder.

- $f : S \to Q$ is Wadge reducible to $g : S \to Q$ (written $f \leq_W g$) if there is a continuous $\theta : S \to S$ s.t. $f(x) \leq_Q g \circ \theta(x)$ for all $x \in S$.

Homomorphic Quasi-Ordering

$\mathcal{A} = (A, \leq_A, c_A)$, $\mathcal{B} = (B, \leq_B, c_A)$: Q-labeled preorders (i.e., $c_X : X \to Q$)

- A morphism $h : \mathcal{A} \to \mathcal{B}$ is a function $h : A \to B$ s.t.
  - $x \leq_A y \implies h(x) \leq_B h(y)$. (order preserving)
  - $c_A(x) \leq_Q c_B(h(x))$. (label increasing)
- Write $\mathcal{A} \leq_{hom} \mathcal{B}$ if there is a morphism $h : \mathcal{A} \to \mathcal{B}$.
- The quasi-ordering $\leq_{hom}$ is called a homomorphic quasi-order.

Theorem (Selivanov 2007)

Let k be a finite discrete order. Then the following are isomorphic:

- The Wadge degrees of k-valued $\underset{\sim}{\Delta}^0_2$ functions on ω^ω
- The poset reflection of the homomorphic quasiorder on countable well-founded k-labeled forests.

- $^{\sqcup}S$: The set of all countable subsets of S, with the dom quasi-order.
- $\mathbf{Tree}(S)$: The set of all S-labeled well-founded countable trees, with the hom quasi-order.
- $^{\sqcup}\mathbf{Tree}(S)$: The set of all S-labeled well-founded countable forests, with the hom quasi-order.

Let $\Gamma(X \to Y)$ be the set of all Γ-measurable functions from X to Y.

Theorem (Selivanov 2007, 2017, K.-Montalbán 2019)

Let Q be a BQO (better-quasi-order).

- The Wadge degrees of $\utilde{\Delta}^0_1(\mathbb{N}^\mathbb{N} \to Q) \simeq {}^{\sqcup}Q$.
- The Wadge degrees of $\utilde{\Delta}^0_2(\mathbb{N}^\mathbb{N} \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(Q)$.
- The Wadge degrees of $\utilde{\Delta}^0_3(\mathbb{N}^\mathbb{N} \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(Q))$.
- The Wadge degrees of $\utilde{\Delta}^0_4(\mathbb{N}^\mathbb{N} \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q)))$.
- The Wadge degrees of $\utilde{\Delta}^0_5(\mathbb{N}^\mathbb{N} \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q))))$.
- and so on...

Theorem (Selivanov 2007, 2017, K.-Montalbán 2019)

Let Q be a BQO (better-quasi-order).

- The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_1(\mathbb{N}^\mathbb{N} \to Q) \simeq {}^{\sqcup}Q$.
- The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_2(\mathbb{N}^\mathbb{N} \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(Q)$.
- The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_3(\mathbb{N}^\mathbb{N} \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(Q))$.
- The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_4(\mathbb{N}^\mathbb{N} \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q)))$.
- The Wadge degrees of $\underset{\sim}{\mathbf{\Delta}}^0_5(\mathbb{N}^\mathbb{N} \to Q) \simeq {}^{\sqcup}\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(\mathbf{Tree}(Q))))$.
- and so on...

- K.-Montalbán (2019) revealed how hom quasi-order can be extended to infinite rank, and proved the similar isomorphism results for infinite Borel ranks.
- The definition of K.-Montalbán appears to be considerably more complicated than the original hom quasi-order.
- Selivanov (2020) introduced an axiomatic paraphrase of infinitary hom quasi-order.
- The axiomatic definition makes little difference between the complexity of ordinary hom quasi-order and infinitary hom quasi-order.

Ordered algebraic theory for hom quasiorder

The algebraic theory $\mathbf{Tr}_{\mathrm{fin}}$ in the signature $\{+,\cdot\}$ consists of the following axioms:

- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$; $(x + y) + z = x + (y + z)$ (associative law)
- $x + y = y + x$ (exchange law)
- $(x + y) \cdot z = x \cdot z + y \cdot z$ (right distributive law)

$\mathbf{Tr}_{\mathrm{fin}}$ can be considered as the theory of labeled finite forests
(by replacing the symbols $+$ and $\cdot$ with $\sqcup$ and $^{\rightarrow}$, respectively).

The ordered algebraic theory $\mathbf{oTr}_{\mathrm{fin}}(Q)$ in the signature $\{+,\cdot,\dot{q}\}_{q\in Q}$ consists of the theory $\mathbf{Tr}_{\mathrm{fin}}$ and the following axioms:

- $x \leq x + y$; $x \leq x \cdot y$; $x \leq y \cdot x$ (weakening)
- $x + x \leq x$; $\dot{q} \cdot \dot{q} \leq \dot{q}$ (contraction)
- $\dot{p} \leq \dot{q}$ for any $p \leq_Q q$ (Q-axiom)

$\mathbf{oTr}_{\mathrm{fin}}(Q)$ characterizes the homomorphic quasiorder on Q-labeled finite forests:
If s, t are closed terms, $\mathbf{oTr}_{\mathrm{fin}} \vdash s \leq t \iff s \leq_{\mathrm{hom}} t$.

Infinitary ordered algebraic theory for hom quasiorder

- $\mathbf{Tr}_{\mathbf{fin}}$: (associative for $+, \cdot$); (exchange for $+$); (right distributive)
- $\mathbf{oTr}_{\mathbf{fin}}(Q)$: (weakening for $+, \cdot$); (contraction for $+$); (const. contraction for $\cdot$); (Q-axiom)

The infinitary algebraic theory $\mathbf{Tr}$ in the signature $\{\cdot, \sum\}$ consists of (associative for $\cdot$); (infinitary associative, exchange for $\sum$); (infinitary right distributive)

$\mathbf{Tr}$ can be considered as the theory of labeled countable well-founded forests (by replacing the symbols $\sum$ and $\cdot$ with $\bigsqcup$ and $^{\rightarrow}$, respectively).

The infinitary ordered algebraic theory $\mathbf{oTr}(Q)$ in the signature $\{\cdot, \sum, \langle q \rangle\}_{q \in Q}$ consists of $\mathbf{Tr}$ and (weakening for $\cdot$); (infinitary weakening, contraction for $\sum$); (const. contraction for $\cdot$); (Q-axiom)

If s, t are closed terms, $\mathbf{oTr} \vdash s \leq t \iff s \leq_{\mathbf{hom}} t$.

$\mathbf{oTr}(Q)$ characterizes the homomorphic quasiorder on Q-labeled finite forests:

Infinitary ordered algebraic theory $\mathbf{oTr}(Q)$, cf. Selivanov 2020

- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$; $\sum_{i \in I} \sum_{j \in J} x_{i,j} = \sum_{(i,j) \in \prod_I J} x_{i,j}$ (associative law)
- $(\sum_{i \in I} x_i) \cdot y = \sum_{i \in I} (x_i \cdot y)$ (right distributive law)
- $x \leq x \cdot y$; $x \leq y \cdot x$ (weakening)
- $\sum_{i \in I} x_{p(i)} \leq \sum_{j \in J} x_j$ for any $p : I \to J$ (exchange, weakening, contraction)
- $\dot{q} \cdot \dot{q} \leq \dot{q}$ (constant contraction)
- $\dot{p} \leq \dot{q}$ for any $p \leq_Q q$ (Q-axiom)

If s, t are closed terms, $\mathbf{oTr} \vdash s \leq t \iff s \leq_{\mathrm{hom}} t$.

So $\leq_{\mathrm{hom}}$ is obtained from the term model of $\mathbf{oTr}(Q)$ by forgetting the algebraic structure (essentially due to Selivanov, although his axioms are slightly different from ours)

Infinitary ordered algebraic theory $\mathbf{oTr}(Q)$, cf. Selivanov 2020

- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$; $\sum_{i \in I} \sum_{j \in J} x_{i,j} = \sum_{(i,j) \in \prod_I J} x_{i,j}$ (associative law)
- $(\sum_{i \in I} x_i) \cdot y = \sum_{i \in I} (x_i \cdot y)$ (right distributive law)
- $x \leq x \cdot y$; $x \leq y \cdot x$ (weakening)
- $\sum_{i \in I} x_{p(i)} \leq \sum_{j \in J} x_j$ for any $p : I \to J$ (exchange, weakening, contraction)
- $\dot{q} \cdot \dot{q} \leq \dot{q}$ (constant contraction)
- $\dot{p} \leq \dot{q}$ for any $p \leq_Q q$ (Q-axiom)

Infinitary ordered algebraic theory $\mathbf{oTr}^{\omega_1}_{\omega_1}(Q)$, cf. Selivanov 2020

- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$; $\sum_{i \in I} \sum_{j \in J} x_{i,j} = \sum_{(i,j) \in \prod_I J} x_{i,j}$ (associative law)
- $(\sum_{i \in I} x_i) \cdot y = \sum_{i \in I} (x_i \cdot y)$ (right distributive law)
- $x \leq x \cdot y$; $x \leq y \cdot x$; $x \leq \phi_\alpha(x)$ (weakening)
- $\sum_{i \in I} x_{p(i)} \leq \sum_{j \in J} x_j$ for any $p : I \to J$ (exchange, weakening, contraction)
- $\dot{q} \cdot \dot{q} \leq \dot{q}$ (constant contraction)
- $\phi_\alpha(x) \cdot \phi_\alpha(x) \leq \phi_\alpha(x)$ (ϕ_α-contraction)
- $\phi_\beta(\phi_\alpha(x)) \leq \phi_\alpha(x)$ (Veblen function axiom)
- $\dot{p} \leq \dot{q}$ for any $p \leq_Q q$ (Q-axiom)

$$^{\sqcup}\mathbf{Tree}^0_\xi(Q) = Q, \quad ^{\sqcup}\mathbf{Tree}^\alpha_\xi(Q) = {}^{\sqcup}\mathbf{Tree}_\xi\left(\bigcup_{\beta<\alpha} {}^{\sqcup}\mathbf{Tree}^\beta_\xi(Q)\right)$$

$$^{\sqcup}\mathbf{Tree}_{\xi+1}(Q) = \bigcup_{\alpha<\omega_1} {}^{\sqcup}\mathbf{Tree}^\alpha_\xi(Q) \qquad ^{\sqcup}\mathbf{Tree}_\lambda(Q) = \bigcup_{\xi<\lambda} {}^{\sqcup}\mathbf{Tree}(Q) \text{ if } \lambda \text{ limit}$$

Theorem (K.-Montalbán 2019)

Let $\alpha = \omega^{\xi_1} \cdot k_1 + \omega^{\xi_2} \cdot k_2 + \cdots + \omega^{\xi_n} \cdot k_n$.
If Q is better-quasi-ordered, the following are order isomorphic:

1. *The Wadge degrees of Q-valued $\mathbf{\Delta}^0_{1+\alpha}$-measurable functions on $\mathbb{N}^\mathbb{N}$.*
2. *The homomorphic order restricted to*

$$^{\sqcup}\mathbf{Tree}^{(k_1 \text{ times})}_{\xi_1}\left(\mathbf{Tree}^{(k_2 \text{ times})}_{\xi_2}\left(\dots \mathbf{Tree}^{(k_n \text{ times})}_{\xi_n}(Q)\right)\right).$$

Corollary (K.-Montalbán 2019)

If Q is better-quasi-ordered, the following are order isomorphic:

1. The Wadge degrees of Q-valued Borel functions on $\mathbb{N}^\mathbb{N}$.
2. The homomorphic order on $^{\sqcup}\mathbf{Tree}_{\omega_1}(Q)$ (i.e., the $\mathcal{L}_{\mathrm{Veb}}(Q)$-terms).

- Classical Wadge theory
- Describing Borel functions
- Homomorphic quasi-order
- Other recent results

A function $f : X \to Y$ is Δ_α if $f^{-1}[A]$ is $\underset{\sim}{\Delta}^0_\alpha$ whenever A is $\underset{\sim}{\Delta}^0_\alpha$.

Definition (cf. Motto Ros 2009)

Let S be a topological space, and Q be a preorder.

- $f : S \to Q$ is Δ_α-Wadge reducible to $g : S \to Q$ if there is a Δ_α-function $\theta : S \to S$ s.t. $f(x) \leq_Q g \circ \theta(x)$ for all $x \in S$.

Theorem (K.-Selivanov 2022)

If Q is better-quasi-ordered, the following are order isomorphic for $\mathbb{N}^\mathbb{N} \to Q$:

1. *The Wadge degrees of Q-valued $\underset{\sim}{\Delta}^0_{1+\alpha}$-measurable functions.*
2. *The Δ_β-Wadge degrees of Q-valued $\underset{\sim}{\Delta}^0_{\beta+\alpha}$-measurable functions.*

- An admissible representation of a topological space X is a partial continuous surjection $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ such that, for any continuous $g :\subseteq \mathbb{N}^{\mathbb{N}} \to X$, $g = \delta_X \circ \theta$ for some continuous θ.
- For such δ_X, a function $f : X \to Q$ is symbolically $\underset{\sim}{\Sigma}_t$ if $f \circ \delta$ is in $\underset{\sim}{\Sigma}_t$.
- Any second-countable T_0 space has an open admissible representation.

Theorem (Selivanov 2022)

Let X be a second-countable T_0-space. For any $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t,

$$f \text{ is in } \underset{\sim}{\Sigma}_t \iff f \text{ is symbolically } \underset{\sim}{\Sigma}_t.$$

- (Pequignot 2015) Write $f \sqsubseteq_W g$ if $f \circ \delta_X$ is Wadge reducible to $g \circ \delta_Y$.

Corollary

Let X be a second-countable T_0-space. For any $\mathcal{L}_{\mathrm{Veb}}(Q)$-term t,

$$\bigcup_{s \leq_{\mathrm{hom}} t} \underset{\sim}{\Sigma}_s(X) = \{f : X \to Q \mid f \sqsubseteq_W g\} \text{ for any/some } g \in \underset{\sim}{\Sigma}_t.$$

The symbolic Wadge reducibility $\sqsubseteq_W$ behaves very well,

even if the space is not zero-dimensional!

- $\Sigma_t(\Delta^1_1)$: the set of $\underset{\sim}{\Sigma}_t$ functions having a hyperarithmetic $\underset{\sim}{\Sigma}_t$-code.
- Louveau (1980) showed that $\underset{\sim}{\Sigma}^0_\xi \cap \Delta^1_1 = \Sigma^0_\xi(\Delta^1_1)$.
- More generally, if a disjoint pair of Σ^1_1 sets is separated by a $\underset{\sim}{\Sigma}^0_\xi$ set, then it is separated by a $\Sigma^0_\xi(\Delta^1_1)$ set.

Theorem (K.-Sasaki 202?)

Let t be a $\mathcal{L}_{\mathrm{Veb}}(Q)$-term.

1. $\underset{\sim}{\Sigma}_t \cap \Delta^1_1 = \Sigma_t(\Delta^1_1)$.
2. $f :\subseteq \omega^\omega \to Q$: *a partial Π^1_1-measurable function with a Σ^1_1 domain*
   f can be extended to a total $\underset{\sim}{\Sigma}_t$ function $g : \omega^\omega \to Q$
   $\Longrightarrow$ *f can be extended to a total $\Sigma_t(\Delta^1_1)$ function $g^\star : \omega^\omega \to Q$.*
3. $\leq_Q$: *a Π^1_1 quasi-order on Q*
   $f :\subseteq \omega^\omega \to Q$: *a partial Π^1_1-measurable function with a Σ^1_1 domain*
   f is $\leq_Q$-dominated by some total $\underset{\sim}{\Sigma}_t$ function $g : \omega^\omega \to Q$
   $\Longrightarrow$ *f is $\leq_Q$-dominated by some total $\Sigma_t(\Delta^1_1)$ function $g^\star : \omega^\omega \to Q$.*

Виктор, с 70-летием!

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