Formal grammars after the Cuban missile crisis

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Part I

Introduction

Formal grammars: a model of syntax

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- Classroom presentation rooted in the early 1960s.

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- Various authors: early recursive descent parsing.

October 1962: the Cuban missile crisis

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- $AB \to CD$: strange nondeterministic computation.

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- Other grammar families: in relation to the ordinary.

Part II

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- "$\to$" denotes implication in the other direction.

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Formal grammar: a logic for describing syntax.

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- Restrictions and extensions of ordinary grammars.

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Several bracketed grammar models

Constituents: well-nested strings;
operation on constituents: $<u>v$.

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Conjunctive grammar: $G = (\Sigma, N, R, S)$, with rules

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Definition (Joshi et al., 1987; Seki et al., 1991)

Multi-component gr.: $G = (\Sigma, N, \dim, R, S)$, with each $A \in N$ having *dimension* $k = \dim A$, and with rules

$$A(\alpha_1, \ldots, \alpha_k) \to B_1(x_{1,1}, \ldots, x_{1,m_1}), \ldots, B_\ell(x_{\ell,1}, \ldots, x_{\ell,m_\ell})$$

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$$\begin{aligned} S(xy) &\to A(x,y) \\ A(axb, cyd) &\to A(x,y) \\ A(\varepsilon,\varepsilon) &\to \text{TRUE} \end{aligned}$$

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- Well-nested 2-component: tree-adjoining, also head.
  - Constituents: substrings u, pairs (u, v).
  - Main operation: wrap (u, v) around (x, y), get (ux, yv).

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- $\alpha_1 \ldots \alpha_k$: all $x_{i,j}$ shuffled.
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  $\ldots x_{i,j} \ldots x_{i',m} \ldots x_{i,k} \ldots x_{i',n} \ldots$.
- Well-nested 2-component: tree-adjoining, also head.
  - Constituents: substrings u, pairs (u, v).
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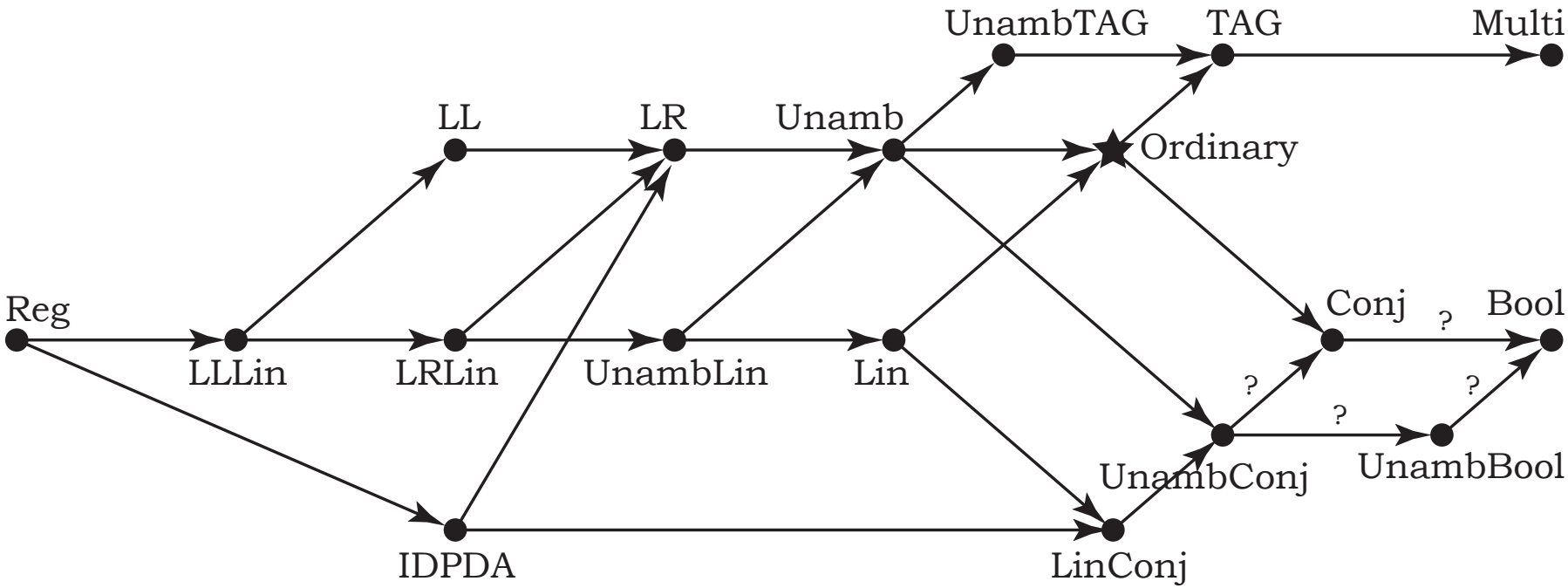
  - linear conjunctive, etc.

The hierarchy



Part III

Formal definitions

Definition by logical derivation

- $S \rightarrow \mathrm{NP\ VP}$

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"If u is NP and v is VP, then uv is a sentence".

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✓ Directly formalizes the intuition.

Logical derivation for generalizations

For conjunctive grammars

$$\frac{A(a^n) \quad B(b^n c^n) \quad D(a^n b^n) \quad C(c^n)}{S(a^n b^n c^n)} \; (S \to AB \,\&\, DC)$$

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Works for all kinds of grammars without negation.

Definition by language equations

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Definition (Ginsburg, Rice, 1962)

Grammar $G = (\Sigma, N, R, S)$ as a least solution of:

$$A = \bigcup_{A \to X_1 \ldots X_\ell \in R} X_1 \cdot \ldots \cdot X_\ell \qquad \text{(for each } A \in N\text{)}$$

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  - Same problem with negation as in logic programming.
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Multi-component: unknown sets of k-tuples.

Applies to all grammar families.

Oh yes, that definition by rewriting...

By string rewriting (Chomsky, 1956)

$S \Longrightarrow \mathrm{NP\ VP} \Longrightarrow \ldots \Longrightarrow \mathrm{NP}\ \texttt{is mortal} \Longrightarrow \ldots \Longrightarrow$
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Works for ordinary grammars, gives wrong ideas.

Part IV

Recurring ideas

Expressibility of operations

- A family of grammars.

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Closure under operation f

If $L_1, \ldots L_n$ are in the family, then so is $f(L_1, \ldots L_n)$.

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- Easily expressed: Kleene star, reversal.
- Any other operations expressible?

Expressibility of operations, II

Theorem (Bar-Hillel et al., 1962)

Ordinary grammars closed under $\cap Reg$.

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- Non-closure for unambiguous; conjunctive.

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Idea and the limits of its applicability.

Negative results using iteration

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Methods based on iteration for grammars with disjunction. Nothing known for conjunctive!

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Parsing time

Ordinary grammars: Cocke–Kasami–Younger

G in CNF, given $w = a_1 \dots a_n$, construct

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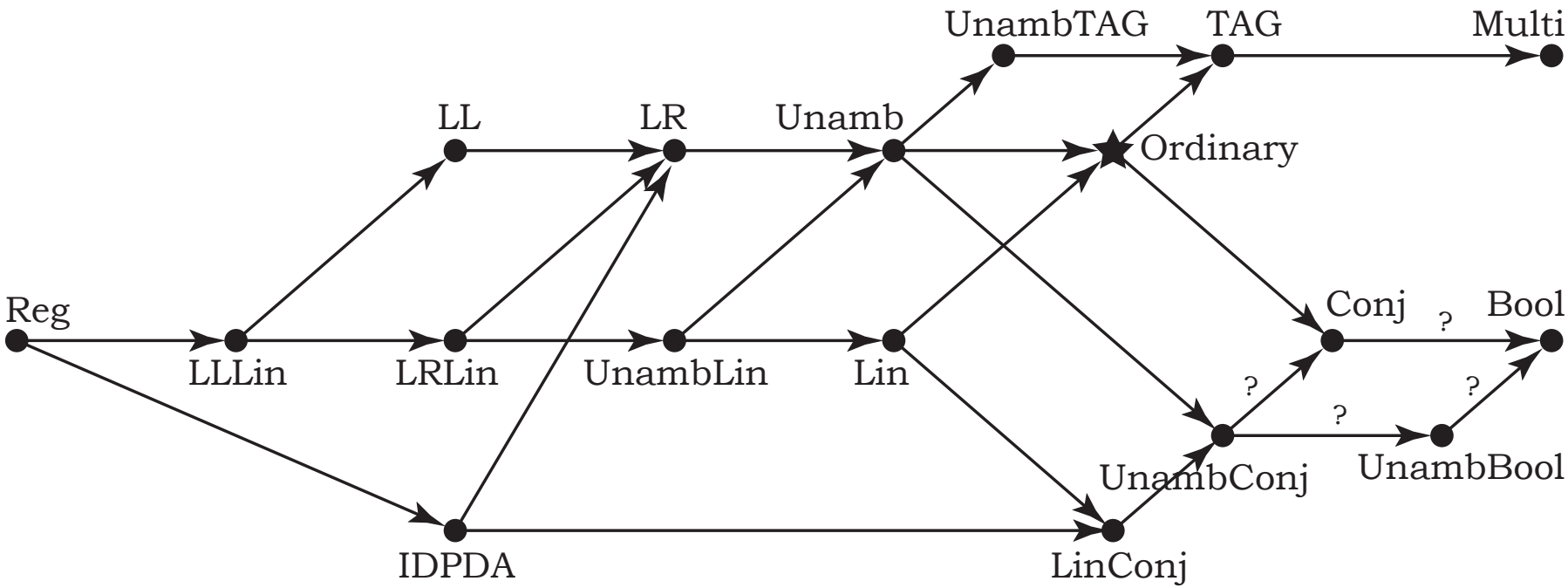
Polynomial-time parsing for each family.

The hierarchy: parsing algorithms



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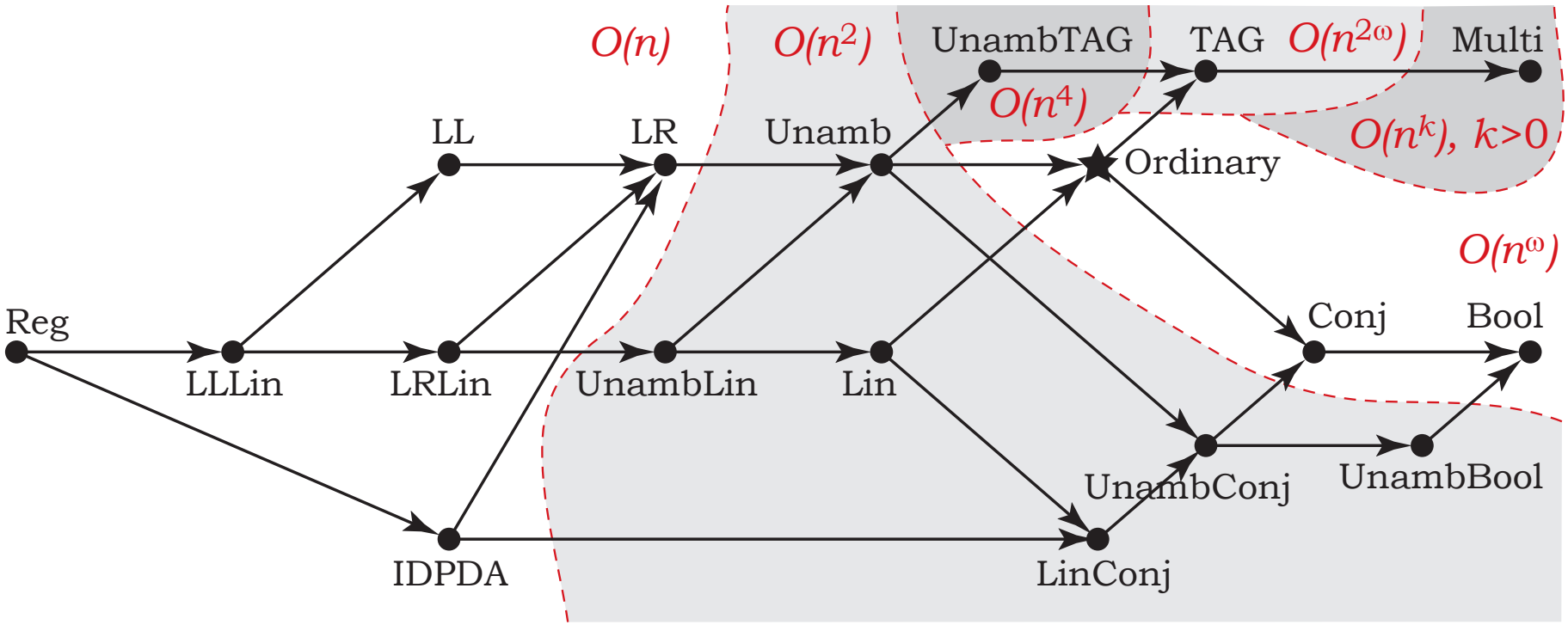

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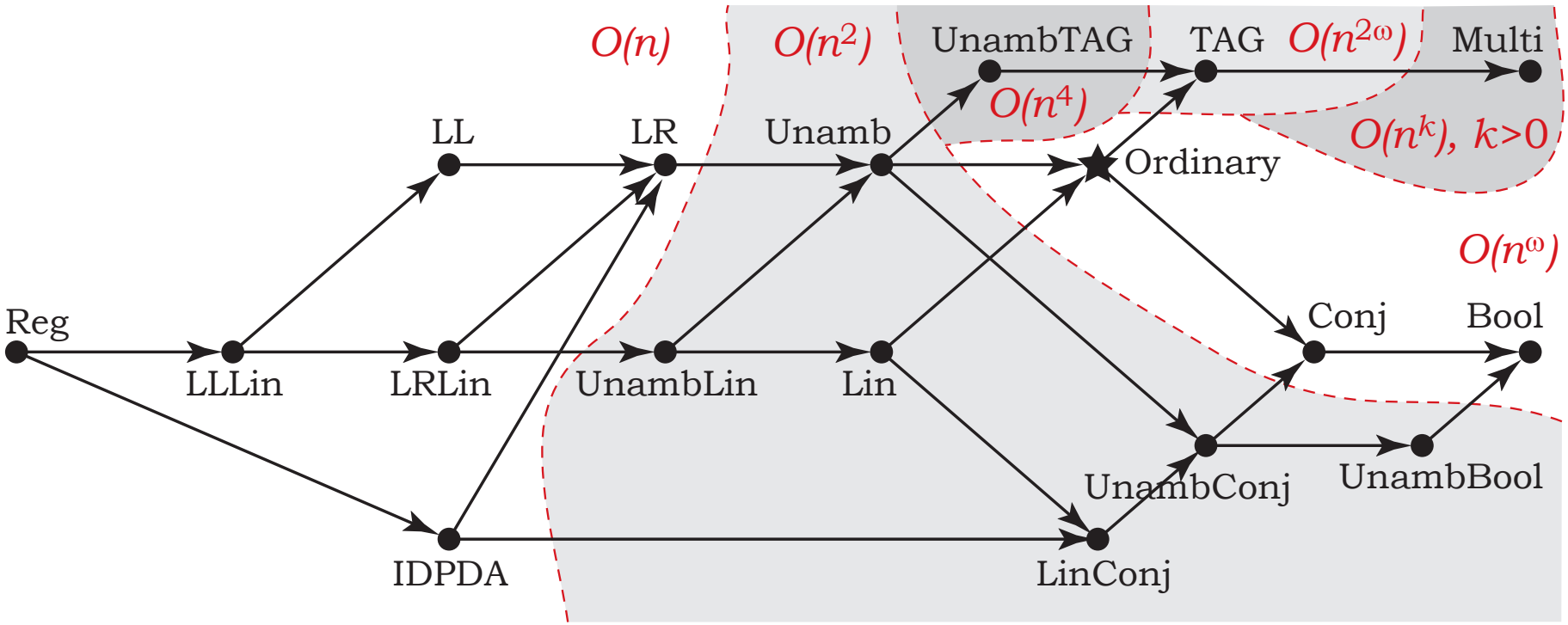

- Why is there always a polynomial-time algorithm?..

Part V

General model: FO(LFP) logic

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Logical contents of grammars

- In 1957: grammars as a special case of TM.
- Not useful at all!

Correct general model: FO(LFP) logic (Rounds, 1988).

- Variables ranging over positions in a string.
- Recursive definitions of predicates.

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- k-component: predicates $A(x_1, y_1, \ldots x_k, y_k)$

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- Membership in P: almost Cocke–Kasami–Younger!

Part VI

Common results

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Theorem

$L \subseteq (\Sigma^2)^*$ as $h(D_k \cap R)$, with length-preserving h.

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Theorem (Greibach, 1973)

$\exists$ *"hardest" ordinary grammar* $G_0 = (\Sigma_0, N_0, R_0, S_0)$,
s.t. for every G *there is* h*:*
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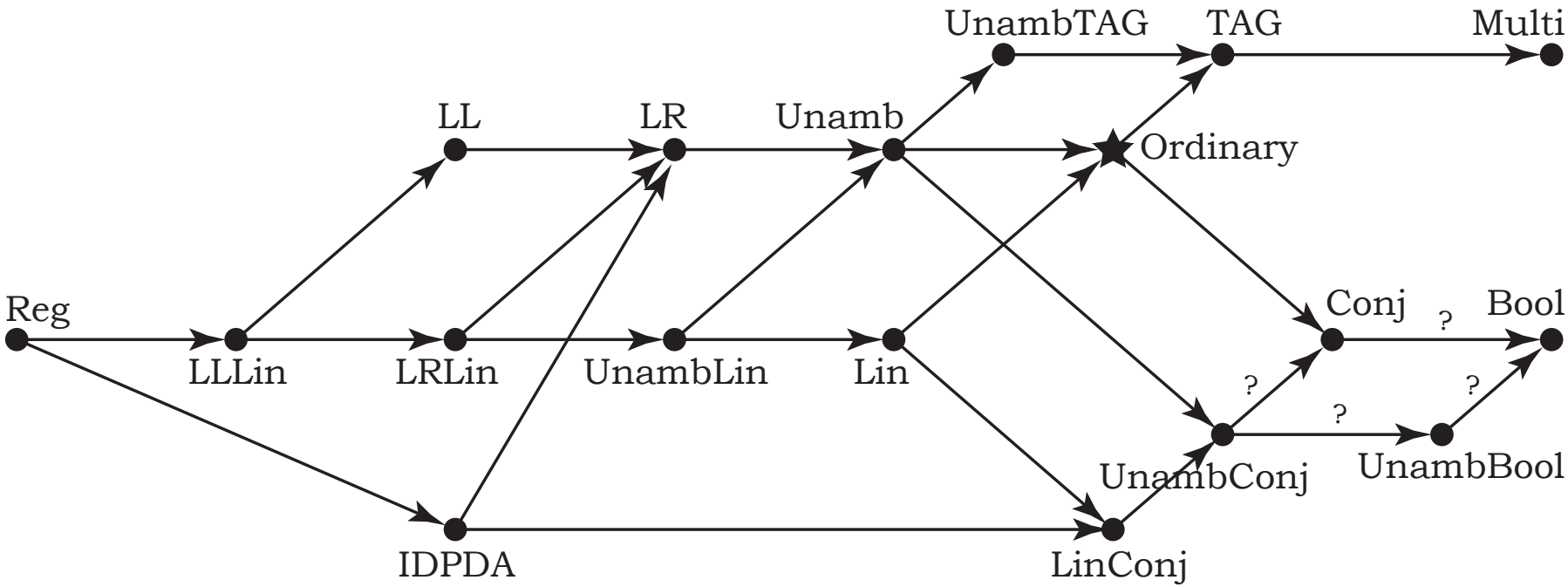
- To unambiguous?

The hierarchy: complexity



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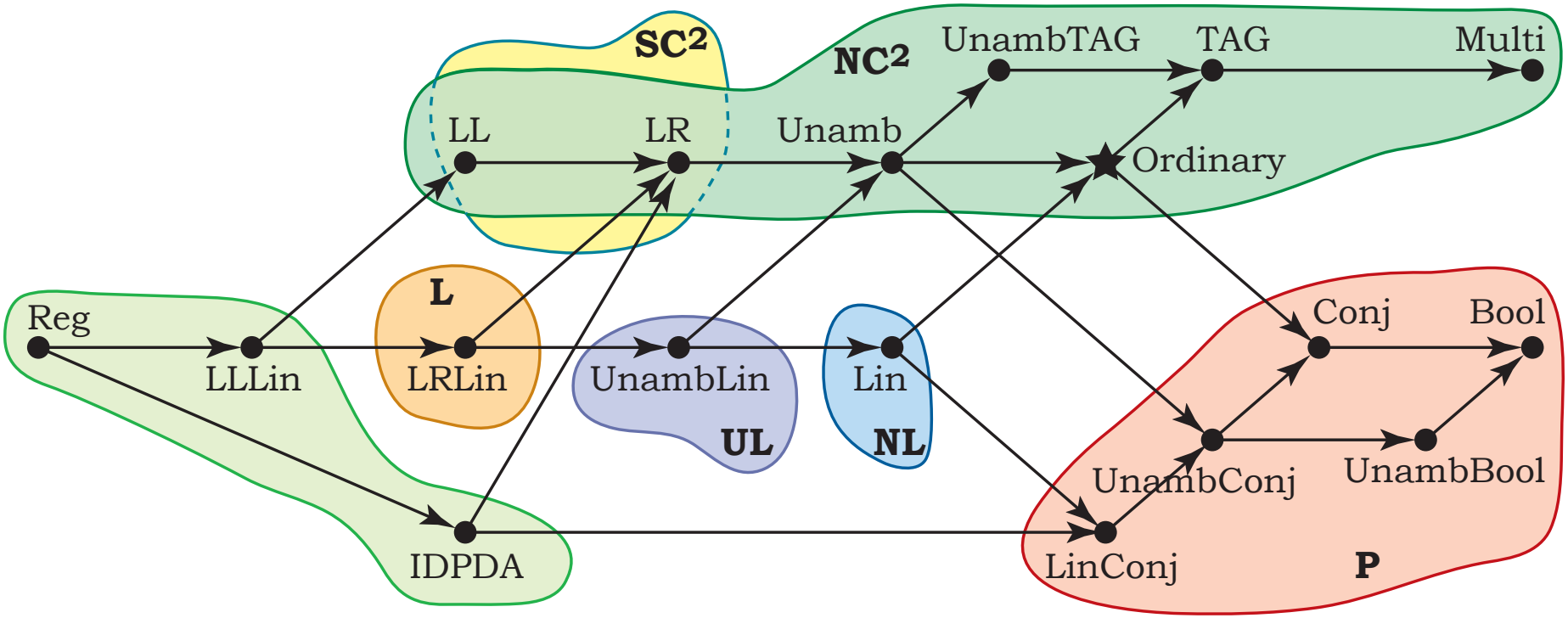

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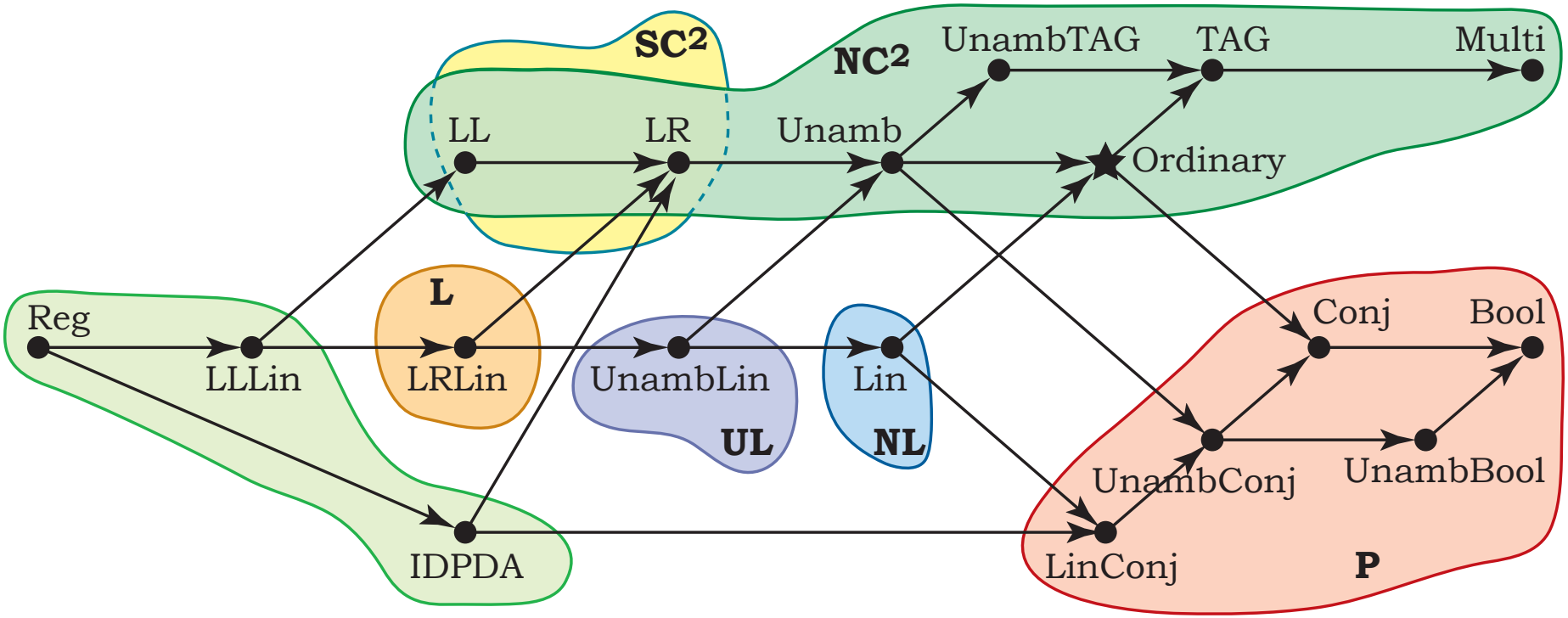

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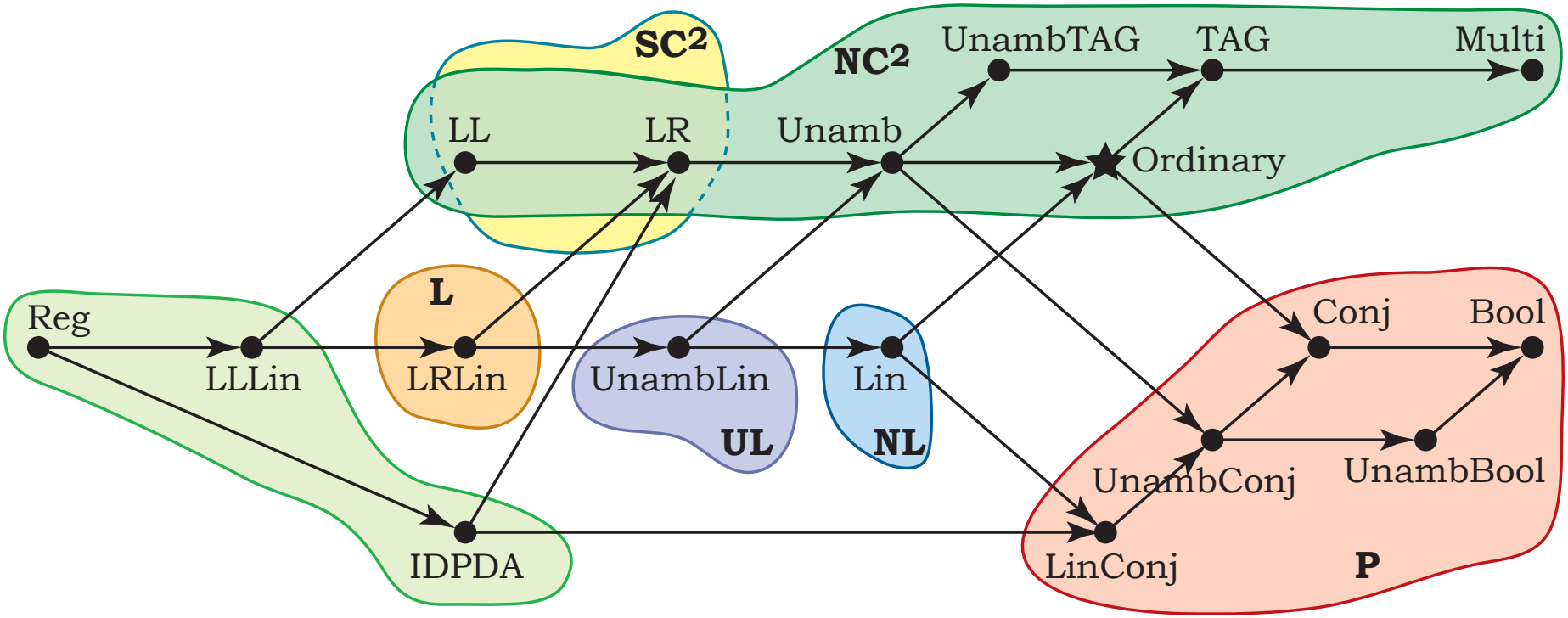

- Complete sets for L, NL, P.
- Ordinary in NC^2:
  depth $O((\log n)^2)$, with $O(n^6)$ gates.

Part VII

Conclusion

Towards further models

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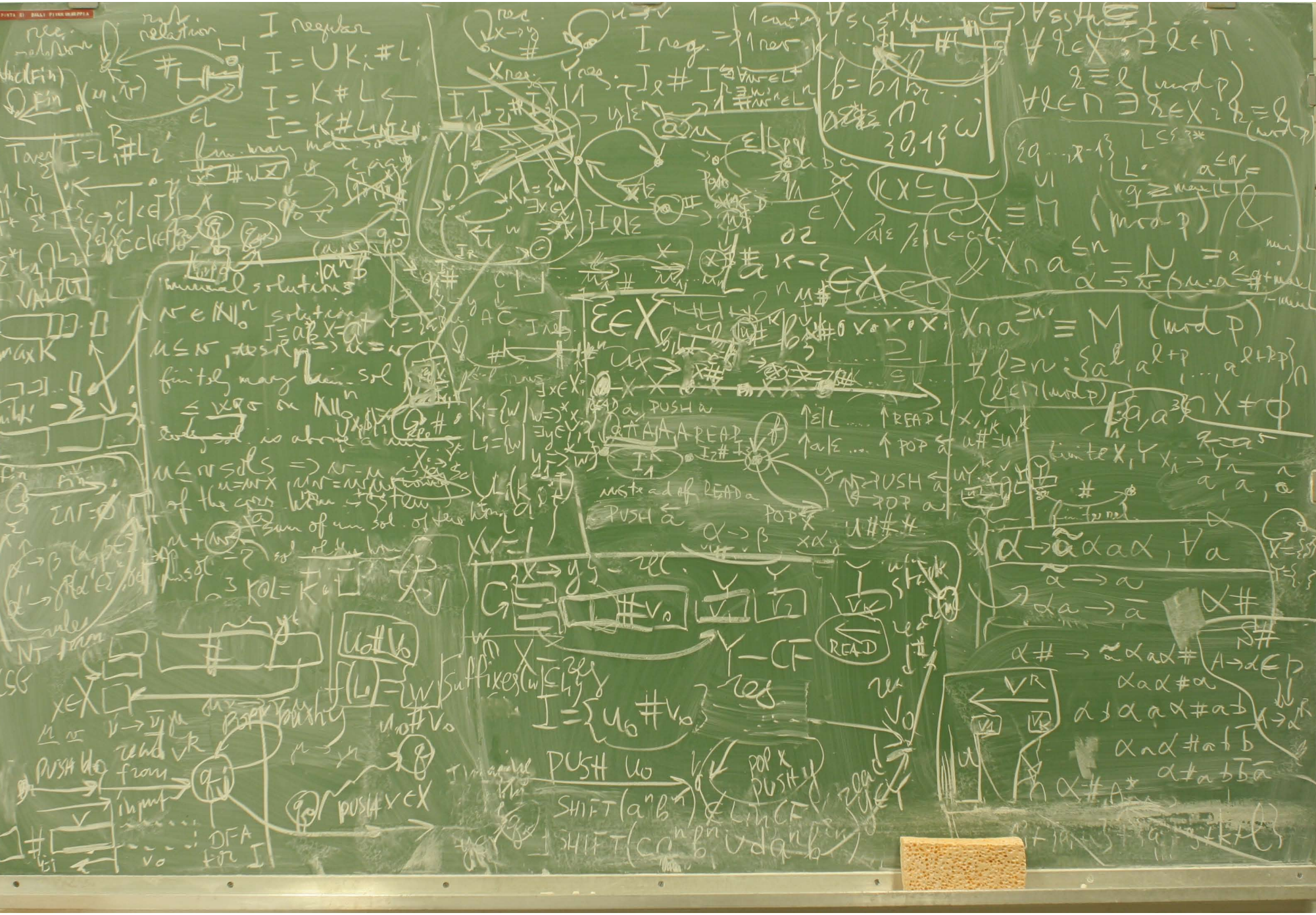
- Understanding the existing models:
  any negative methods for conjunctive grammars?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READ
POP
PUSH u0
SHIFT(a^n b^n)
SHIFT(ca^n b^n ∪ da^n b^n)
DFA for I