On the complexity of describing topological bases for QCB_0-spaces

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(joint work with Matthias Schröder and Victor Selivanov)

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Introduction

- We present some previous joint work with Matthias Schröder and Victor Selivanov that was originally presented at CCC 2014, and published in the paper "Base-complexity classifications of QCB_0-spaces", *Computability* (2016).
- It introduced a way of characterizing non-countably based QCB_0-spaces according to the complexity of a basis for their topologies
  - The method generalizes the basic idea of countably based spaces, where a basis can simply be enumerated
  - It complements the (hyper-)projective hierarchy of QCB_0-spaces introduced earlier by Matthias Schröder and Victor Selivanov (2013), which classifies spaces by the complexity of the partial equivalence relation on ω^ω induced by an admissible representation of the space
- The base-complexity hierarchy is useful for studying non-countably based QCB_0-spaces, in particular when investigating the complexity of subsets of spaces, and for investigating issues of embeddability.

Motivation: Absoluteness of the complexity of subspaces

The extension from Polish spaces to quasi-Polish spaces has been very natural:

- For example, a quasi-Polish subspace of a countably based T_0-space is always a $\mathbf{\Pi}^0_2$-subspace.
- This allows us to characterize quasi-Polish spaces as the $\mathbf{\Pi}^0_2$-subspaces of $\mathcal{P}(\omega)$.

This does not extend to non-countably based spaces:

- The singleton subset $\{\omega^\omega\} \subseteq \mathcal{O}(\omega^\omega)$ is clearly quasi-Polish but it is not even a Borel set! (It is $\mathbf{\Pi}^1_1$-complete; Selivanov 2013).
- The space $\mathcal{O}(\omega^{\omega^\omega})$ has $\mathbf{\Pi}^1_2$-complete singletons, and this phenomenon continues up the entire (hyper-) projective hierarchy by considering spaces with increasingly complicated topologies

To even begin to characterize the complexity of singleton subspaces, we must organize the class of QCB_0-spaces into some hierarchy and consider each level separately.

Motivation: Total representations

- What notions of "completeness" should we look for in QCB_0-spaces?
- A useful characterization of quasi-Polish spaces is that they are precisely the countably based T_0-spaces with a total admissible representation (with ω^ω as domain).
- Having a total representation is a useful completeness property of QCB_0-spaces (V. Selivanov, 2013), since every name has an interpretation as some point of the underlying space.

But how general is the class of spaces with total representations? Can every QCB_0-space be embedded into a space with a total representation?

Motivation: Understanding Embeddability

Definition (Kleene-Kreisel continuous functionals)

Using exponentials within QCB_0 (T_0 quotients of countably based spaces), we define $\mathbb{N}\langle n\rangle$ for $n \in \omega$ as:

- $\mathbb{N}\langle 0\rangle = \omega$ are the natural numbers,
- $\mathbb{N}\langle 1\rangle = \omega^\omega$ is the Baire space,
- $\mathbb{N}\langle n+1\rangle = \omega^{\mathbb{N}\langle n\rangle}$.

Note: $\mathbb{N}\langle 2\rangle = \omega^{\omega^\omega}$ and above are not countably based. The topology is the sequentialization of the compact-open topology.

A natural question is:

- Can $\mathbb{N}\langle n+1\rangle$ be topologically embedded into $\mathbb{N}\langle n\rangle$?
- More generally, is there a subspace of $\mathbb{N}\langle n\rangle$ whose sequentialization is homeomorphic to $\mathbb{N}\langle n+1\rangle$?

Motivation: Understanding Embeddability

It is well known that there is no continuous surjection from $\mathbb{N}\langle n\rangle$ onto $\mathbb{N}\langle n+1\rangle$:

- Given continuous $f\colon \mathbb{N}\langle n\rangle \to \omega^{\mathbb{N}\langle n\rangle}$, define $g\colon \mathbb{N}\langle n\rangle \to \omega$ as $g(p) = f(p)(p) + 1$.
- If $f(q) = g$, then $f(q)(q) = g(q) = f(q)(q) + 1$, contradiction.

But this alone does not eliminate the possibility of a topological embedding.

It is also easy to see that $\mathbb{N}\langle 2\rangle = \omega^{\omega^\omega}$ does not embed into $\mathbb{N}\langle 1\rangle = \omega^\omega$, because $\mathbb{N}\langle 1\rangle$ is countably based but $\mathbb{N}\langle 2\rangle$ is not.

In order to extend the last proof to arbitrary $n \in \omega$, we need a generalization of the notion of the complexity of defining a topological basis for a space.

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The following modified definition of the Borel hierarchy, due to V. Selivanov, is necessary for non-metrizable spaces:

Definition

Let X be a topological space. For $1 \leq \alpha < \omega_1$ define $\mathbf{\Sigma}^0_\alpha(X)$, $\mathbf{\Pi}^0_\alpha(X)$, and $\mathbf{\Delta}^0_\alpha(X)$ inductively as follows:

- $\mathbf{\Sigma}^0_1(X)$ is the set of open subsets of X,
- For $\alpha > 1$, $A \in \mathbf{\Sigma}^0_\alpha(X)$ iff A can be expressed in the form

$$A = \bigcup_{i \in \omega} U_i \setminus V_i,$$

  where $U_i, V_i \in \mathbf{\Sigma}^0_{\beta_i}(X)$ for some $\beta_i < \alpha$.
- $\mathbf{\Pi}^0_\alpha(X) = \{X \setminus A \,|\, A \in \mathbf{\Sigma}^0_\alpha(X)\}$, and
- $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$.

Finally, define the Borel sets as $\mathbf{B}(X) = \bigcup_{\alpha < \omega_1} \mathbf{\Sigma}^0_\alpha(X)$.

Projective hierarchy

Definition

The projective hierarchy on a space X is defined as $(n \in \omega)$:

- $\mathbf{\Sigma}^1_0(X) = \mathbf{\Sigma}^0_2(X)$,
- $A \in \mathbf{\Sigma}^1_{n+1}(X)$ iff A is such that

  $$x \in A \iff (\exists p \in \omega^\omega)[\langle p, x\rangle \in B]$$

  for some $B \in \mathbf{\Pi}^1_n(\omega^\omega \times X)$,
- $\mathbf{\Pi}^1_n(X)$ is the set of complements of sets in $\mathbf{\Sigma}^1_n(X)$,
- $\mathbf{\Delta}^1_n(X) = \mathbf{\Sigma}^1_n(X) \cap \mathbf{\Pi}^1_n(X)$.

This definition can be extended to all countable ordinals to get the so-called hyperprojective hierarchy, which is needed when working with full cartesian closed subcategories of QCB_0 that are closed under countable limits and co-limits (Schröder & Selivanov, 2014). (Our results easily generalize to the hyperprojective hierarchy).

Borel subsets are the subsets that are definable using quantification over ω, boolean operations, and open-valued predicates.

Example

Assume $(B_n)_{n\in\omega}$ is a countable basis for X. Then for $x, y \in X$,

$$x = y \iff (\forall n \in \omega)[x \in B_n \iff y \in B_n],$$

hence the diagonal $\Delta_X \subseteq X \times X$ is Borel (in fact, $\mathbf{\Pi}^0_2$).

Analytic (co-analytic) subsets correspond to subsets that are definable using existential (universal) quantifier over ω^ω, boolean operations, and Borel-valued predicates.

Example

Assume X is countably based and let $f\colon \omega^\omega \to X$ be continuous, then

$$x \in range(f) \iff (\exists p \in \omega^\omega)[f(p) = x].$$

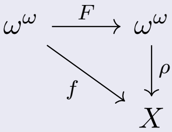
Note that $\{\langle p, x\rangle \in \omega^\omega \times X \mid f(p) = x\}$ is a $\mathbf{\Pi}^0_2$-set.

QCB$_0$-spaces

We will be particularly interested in topological spaces X whose elements can be represented in an appropriate way using infinite strings of natural numbers (i.e., elements of the Baire space ω^ω).

Definition (K. Weihrauch, M. Schröder)

Let X be a topological space. A partial continuous surjection $\rho :\subseteq \omega^\omega \to X$ is an admissible representation of X iff for every partial continuous $f :\subseteq \omega^\omega \to X$ there is a partial continuous $F :\subseteq \omega^\omega \to \omega^\omega$ such that $f = \rho \circ F$.



A topological space is a QCB$_0$-space (Quotient of a Countably Based space) iff it has an admissible representation which is a quotient map.

QCB_0-spaces

Theorem (M. Schröder)

If (X, ρ_X) and (Y, ρ_Y) are admissibly represented QCB_0-spaces, then a function $f\colon X \to Y$ is continuous if and only if there exists a continuous partial function $F :\subseteq \omega^\omega \to \omega^\omega$ such that $f \circ \rho_X = \rho_Y \circ F$ (we say that "F realizes f").

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{F} & \omega^\omega \\ {\scriptstyle \rho_X}\downarrow & & \downarrow{\scriptstyle \rho_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

- The computability (and complexity) of functions $f\colon X \to Y$ between QCB_0-spaces can be defined in terms of the computability (complexity) of realizers $F :\subseteq \omega^\omega \to \omega^\omega$ of f (Weihrauch's Type Two Theory of Effectivity)

Definition (Kleene-Kreisel continuous functionals)

Using exponentials within QCB_0 (T_0 quotients of countably based spaces), we define $\mathbb{N}\langle n\rangle$ for $n \in \omega$ as:

- $\mathbb{N}\langle 0\rangle = \omega$ are the natural numbers,
- $\mathbb{N}\langle 1\rangle = \omega^\omega$ is the Baire space,
- $\mathbb{N}\langle n+1\rangle = \omega^{\mathbb{N}\langle n\rangle}$.

For each $n \in \omega$ we fix an admissible representation $\delta_n \colon D_n \to \mathbb{N}\langle n\rangle$ with $D_n \subseteq \omega^\omega$. We can assume that $D_0 = D_1 = \omega^\omega$ and $D_n \in \mathbf{\Pi}^1_{n-1}(\omega^\omega)$ for $n > 1$ (Schröder & Selivanov, 2013).

Theorem (Schröder & Selivanov, 2013)

Let $n \geq 1$ and $B \subseteq \omega^\omega$ be non-empty. Then $B \in \mathbf{\Sigma}^1_n(\omega^\omega)$ iff there is a continuous function $f \colon \mathbb{N}\langle n\rangle \to \omega^\omega$ with $range(f) = B$.

Projective hierarchy of QCB_0-spaces (Schröder & Selivanov, 2013)

For any representation δ of a space X, define

$$EQ(\delta) := \{\langle p, q\rangle \in \omega^\omega \mid p, q \in dom(\delta) \wedge \delta(p) = \delta(q)\}.$$

Definition

For each family of pointclasses $\mathbf{\Gamma}$ (e.g., $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, etc.), let $\mathsf{QCB}_0(\mathbf{\Gamma})$ be the set of spaces X that have an admissible representation δ with $EQ(\delta) \in \mathbf{\Gamma}(\omega^\omega)$.

The spaces in $\mathsf{QCB}_0(\mathbf{P}) := \bigcup_{n\in\omega} \mathsf{QCB}_0(\mathbf{\Sigma}^1_n)$ are called the projective QCB_0-spaces.

$\mathsf{QCB}_0(\mathbf{P})$ is the smallest (up to equivalence) full cartesian closed sub-category of QCB_0 that contains the Sierpinski space and ω and is closed under finite limits and co-limits (constructed as in QCB_0).

A similar characterization holds for the hyperprojective QCB_0-spaces, but with countable limits and co-limits.

Projective hierarchy of QCB_0-spaces (Schröder & Selivanov, 2013)

Proposition

- If X is countably based and $\mathbf{\Gamma} \in \{\mathbf{\Pi}^1_n, \mathbf{\Sigma}^1_n \mid n \in \omega\}$, then $X \in \mathsf{QCB}_0(\mathbf{\Gamma})$ iff X is homeomorphic to a $\mathbf{\Gamma}$-subset of $\mathcal{P}(\omega)$.
- If X is Hausdorff and $\mathbf{\Gamma} \in \{\mathbf{\Pi}^1_n, \mathbf{\Sigma}^1_n \mid n \in \omega\}$, then $X \in \mathsf{QCB}_0(\mathbf{\Gamma})$ iff X has an admissible representation δ with $dom(\delta) \in \mathbf{\Gamma}(\omega^\omega)$.
- For each $n \in \omega$, $\mathbb{N}\langle n+1\rangle \in \mathsf{QCB}_0(\mathbf{\Pi}^1_n) \setminus \mathsf{QCB}_0(\mathbf{\Sigma}^1_n)$
- For each $n \in \omega$, if $X \in \mathsf{QCB}_0(\mathbf{\Sigma}^1_n)$ and $Y \in \mathsf{QCB}_0(\mathbf{\Pi}^1_n)$, then $Y^X \in \mathsf{QCB}_0(\mathbf{\Pi}^1_n)$.

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Y-based spaces

Note: For any space X, we assume the Scott-topology on the lattice of open sets $\mathcal{O}(X)$. For QCB_0 spaces X, this is isomorphic to the exponential Σ^X, where Σ is the Sierpinski space.

Definition

Let X and Y be topological spaces.

- A Y-indexing of a basis for X is a continuous function $\phi\colon Y \to \mathcal{O}(X)$ such that $range(\phi)$ contains a basis for the topology of X.
- X is Y-based iff there exists a Y-indexing for a basis for X.

Clearly, ω-based is the usual definition of countably based.

Proposition

Every subspace (not subobject!) of a Y-based space is Y-based.

Proof: If $e\colon Z \to X$ is a topological embedding, just compose the Y-indexing with the surjection $e^{-1}\colon \mathcal{O}(X) \to \mathcal{O}(Z)$.

Y-based spaces

Theorem

If X and Y are sequential T_0-spaces and X is Y-based, then X topologically embeds into $\mathcal{O}(Y)$.

Proof (idea): Given a Y-indexed basis $\phi\colon Y \to \Sigma^X$, then the double transpose $\psi\colon X \to \Sigma^Y$, defined as $\psi(x)(y) = \phi(y)(x)$, is a topological embedding. (Very similar to D. Scott's proof that every space embeds into a continuous lattice)

Y-based spaces

Theorem

The following are equivalent for a sequential T_0-space X:

- X is a QCB$_0$-space,
- X is Y-based for some $Y \subseteq \omega^\omega$,
- X topologically embeds into $\mathcal{O}(Y)$ for some $Y \subseteq \omega^\omega$.

If $\phi\colon Y \to \mathcal{O}(X)$ is a Y-indexing of a basis for X and $(B_n)_{n\in\omega}$ is a countable basis for Y, then $A_n := \bigcap\{\phi(p) \mid p \in B_n\}$ for $n \in \omega$ gives a countable pseudo-base for X.

Corollary

Every QCB$_0$-space topologically embeds into a space with a total representation.

Proof: By Selivanov (2013), if Y is countably based then $\mathcal{O}(Y)$ has a total representation.

ω^ω-based spaces

For ω^ω-based spaces we get a complete characterization:

Theorem

A QCB_0-space is ω^ω-based iff it topologically embeds into $\mathcal{O}(\omega^\omega)$.

Open question: Does this extend to $\mathbb{N}\langle n\rangle$ for $n > 1$?

Example

The following are ω^ω-based:

- Every countably based space,
- $\mathcal{O}(X)$ whenever X is quasi-Polish,
- The Gruenhage-Streicher space X:
  - underlying set of X is ω^2
  - basis given by $\beta\colon \omega^2 \times \omega^\omega \to \mathcal{O}(X)$
    $\beta(\langle m,n\rangle, f) = \{\langle m,n\rangle\} \cup \{(i,j) \in \omega^2 \mid i > m \;\&\; j \geq f(i)\}$
    (Exercise: Prove that β is continuous.)

(This is a QCB_0-space but its soberification is not).

Proposition

Every $\mathrm{QCB}_0(\mathbf{\Pi}^1_n)$-space is $\mathbb{N}\langle n+2\rangle$-based.

- Note: M. Hoyrup later showed that this cannot be improved to $\mathbb{N}\langle n+1\rangle$-based: $\mathbb{N}\langle n+1\rangle$ is not $\mathbb{N}\langle n+1\rangle$-based for $n \geq 1$.

Proof (idea): If X is $\mathrm{QCB}_0(\mathbf{\Pi}^1_n)$ then Σ^X is $\mathrm{QCB}_0(\mathbf{\Pi}^1_{n+1})$ so there is a continuous surjection from $\mathbb{N}\langle n+2\rangle$ to Σ^X. □

Recall that $D_n \in \mathbf{\Pi}^1_{n-1}(\omega^\omega)$ is the domain of an admissible representation for $\mathbb{N}\langle n\rangle$.

Corollary

Every $\mathrm{QCB}_0(\mathbf{\Pi}^1_n)$-space topologically embeds into $\mathcal{O}(D_{n+2})$, which has a total representation.

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Sequentially Y-based spaces

Definition

Let X and Y be sequential spaces.

- $\mathcal{B} \subseteq \mathcal{O}(X)$ is a sequential basis for X iff $\mathcal{B}$ is a subbasis for a topology τ on the set X such that the sequentialization of τ is equal to $\mathcal{O}(X)$.
- For $\phi \colon Y \to \mathcal{O}(X)$ define $\mathcal{B}_\phi$ to be the set of all intersections of the form $\bigcap_{n \leq \infty} \phi(p_n)$ where $(p_n)_n$ converges to p_∞ in Y.
- A continuous function $\phi \colon Y \to \mathcal{O}(X)$ is a Y-indexed generating system for X iff $\mathcal{B}_\phi$ is a sequential basis for X.
- X is sequentially Y-based iff there exists a Y-indexing generating system for X.

Proposition

A sequential space X is seq. $\mathbb{N}\langle n \rangle$-based iff there exists continuous $\phi \colon \mathbb{N}\langle n \rangle \to \mathcal{O}(X)$ such that $range(\phi)$ is a sequential basis for X.

Sequentially Y-based spaces

Definition

X sequentially embeds into Y iff there is a subspace of Z of Y such that the sequentialization of Z is homeomorphic to X.

Theorem

Let X and Y be sequential T_0-spaces. X is sequentially Y-based iff X sequentially embeds into $\mathcal{O}(Y)$.

Corollary

If X is a sequential T_0-space then $\mathcal{O}(X)$ is sequentially X-based.

Proposition

Let X_i, Y_i be sequential T_0-spaces such that X_i is sequentially Y_i-based for $i \in \omega$. Then the sequential product $\prod_{i\in\omega} X_i$ is sequentially $(\bigoplus_{i\in\omega} Y_i)$-based.

Proposition

Let X, Y, P be sequential T_0-spaces such that Y is sequentially P-based. Then Y^X is sequentially $(P \times X)$-based.

Corollary

For every $n \in \omega$, $\mathbb{N}\langle n+1\rangle$ is sequentially $\mathbb{N}\langle n\rangle$-based.

Proof: By induction, $\mathbb{N}\langle n+1\rangle = \omega^{\mathbb{N}\langle n\rangle}$ is sequentially $(\omega \times \mathbb{N}\langle n\rangle)$-based.

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Application to Kleene-Kreisel continuous functionals

Given a QCB_0-space X, we let 0_X denote the constantly zero function $\lambda x \in X.0$ in ω^X.

Lemma

Let X be a QCB_0-space, $Y \subseteq \omega^\omega$, $f\colon X \to Y$ a continuous function, and $A = Y \setminus f(X)$ the complement of the range of f. Then there is a continuous function $g\colon Y \to \left(\omega^X\right)^\omega$ such that $g(y)$ is a sequence in ω^X converging to 0_X if and only if $y \in A$.

Proof (idea): Define $g\colon Y \to \left(\omega^X\right)^\omega$ as $g(y)(n)(x) = 0$ if $f(x) \notin \uparrow y[n]$ and $g(y)(n)(x) = 1$, otherwise, where $\uparrow y[n]$ is the clopen subset of Y of elements agreeing with y in the first n places.

Theorem

For each $n \in \omega$, $\mathbb{N}\langle n+2\rangle$ is not sequentially $\mathbb{N}\langle n\rangle$-based.

Proof (idea): Fix $A \in \mathbf{\Pi}^1_{n+1}(\omega^\omega) \setminus \mathbf{\Sigma}^1_{n+1}(\omega^\omega)$ and continuous $f \colon \mathbb{N}\langle n+1\rangle \to \omega^\omega$ such that $A = \omega^\omega \setminus range(f)$. The lemma implies there is continuous $g \colon \omega^\omega \to (\mathbb{N}\langle n+2\rangle)^\omega$ such that $g(y)$ is a sequence in $\mathbb{N}\langle n+2\rangle$ converging to $0_{\mathbb{N}\langle n+1\rangle}$ iff $y \in A$.

Assume for a contradiction that $\mathbb{N}\langle n+2\rangle$ is sequentially $\mathbb{N}\langle n\rangle$-based. Then there is continuous $\phi \colon D_n \to \mathcal{O}(\mathbb{N}\langle n+2\rangle)$ with $range(\phi)$ a sequential basis for $\mathbb{N}\langle n+2\rangle$. Then $y \in A$ iff

$$\forall x \in \omega^\omega . \left[\underbrace{\neg \left(x \in D_n \wedge 0_{\mathbb{N}\langle n+1\rangle} \in \phi(x) \right)}_{\mathbf{\Sigma}^1_{n-1} \text{ because } D_n \text{ is } \mathbf{\Pi}^1_{n-1}} \vee \underbrace{\forall^\infty_n . g(y)(n) \in \phi(x)}_{Borel} \right].$$

Hence A is $\mathbf{\Pi}^1_n$, a contradiction. □

Application to Kleene-Kreisel continuous functionals

Theorem

For each $n \in \omega$, $\mathbb{N}\langle n+1\rangle$ does not sequentially embed into $\mathbb{N}\langle n\rangle$.

Proof: For $n = 0$ it is trivial: an embedding $\omega^\omega \hookrightarrow \omega$ would be absurd!

For $n > 0$, there is an embedding $\mathbb{N}\langle n\rangle \hookrightarrow \mathcal{O}(\mathbb{N}\langle n-1\rangle)$ because $\mathbb{N}\langle n\rangle$ is sequentially $\mathbb{N}\langle n-1\rangle$-based. Then

$$\mathbb{N}\langle n+1\rangle \hookrightarrow \mathbb{N}\langle n\rangle \hookrightarrow \mathcal{O}(\mathbb{N}\langle n-1\rangle)$$

would imply $\mathbb{N}\langle n+1\rangle$ is sequentially $\mathbb{N}\langle n-1\rangle$-based, contradicting the previous theorem. □

Universal spaces

Theorem

There is no QCB_0-space which is universal for all QCB_0-spaces.

Proof (idea): For any QCB_0-space X there is $Y \subseteq \omega^\omega$ such that X embeds into $\mathcal{O}(Y)$. Let π_Y be a total admissible representation of $\mathcal{O}(Y)$.

There is $Z \subseteq \omega^\omega$ such that $\omega^\omega \setminus Z$ is not Wadge reducible to $EQ(\pi_Y)$. However, it can be shown that $\omega^\omega \setminus Z$ Wadge reduces to $EQ(\pi_Z)$, where π_Z is a total admissible representation of $\mathcal{O}(Z)$.

This implies that $\mathcal{O}(Z)$ cannot embed into $\mathcal{O}(Y)$, hence $\mathcal{O}(Z)$ can not embed into X. □

Universal spaces

Proposition

$\mathcal{O}(\omega^\omega)$ is universal (w.r.t. topological embeddings) for ω^ω-based spaces.

Open question: Is $\mathcal{O}(\mathbb{N}\langle n\rangle)$ universal for $n > 1$?

Proposition

For all $n \in \omega$, $\mathcal{O}(\mathbb{N}\langle n\rangle)$ is universal (w.r.t. sequential embeddings) for sequentially $\mathbb{N}\langle n\rangle$-based spaces.

Concluding remarks

- $\mathcal{O}(D_n)$ is also universal for sequentially $\mathbb{N}\langle n\rangle$-based spaces, and has a total ω^ω-representation.
  - Note: If X is a QCB_0-space with a total representation, then $\mathcal{O}(X)$ sequentially embeds into $\mathcal{O}(\omega^\omega)$.
- Since $\mathcal{O}(D_n)$ is $\mathsf{QCB}_0(\mathbf{\Pi}^1_n)$, the complexity of singleton subsets and the equality relation for sequentially $\mathbb{N}\langle n\rangle$-based spaces is at most $\mathbf{\Pi}^1_n$ (relative to the domain of an admissible representation for the space).
  - It makes sense to discuss $\mathbf{\Pi}^1_{n+k}$-absoluteness within the class of sequentially $\mathbb{N}\langle n\rangle$-based spaces.
- There are similar totally represented universal spaces for each level of the hyperprojective hierarchy.
- This provides a nice organization of the category of hyperprojective QCB_0-spaces, which is cartesian closed, countably complete, and countably co-complete.