Fine hierarchy relative to Turing reducibility

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Definitions and notations

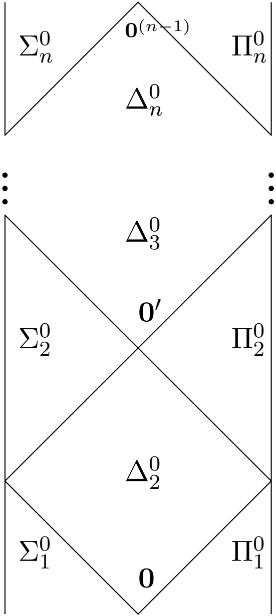
In the talk we consider only subsets of $\omega = \{0, 1, 2, 3, ...\}$

- A set A is a Σ^0_1-set, if $A = \{x | \exists y \ R(x, y)\}$ for some computable relation R.
- A set A is a Π^0_1-set, if $\overline{A}$ is Σ^0_1. A set A is Δ^0_1, if A is Σ^0_1 and Π^0_1.
- The sets from $\Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1$ are computable sets.

Definitions and notations

- The same definitions are for Σ_2^0, Π_2^0 and Δ_2^0, where $A = \{x|\exists y \forall z\ R(x,y,z)\}$ is a Σ_2^0-set, and $B = \{x|\forall y \exists z\ R(x,y,z)\}$ is a Π_2^0-set.
- Also $\Sigma_1^0, \Pi_1^0 \subset \Delta_2^0$.
- By increasing number of quantifiers we obtain Σ_n^0 and Π_n^0 for all n.
- The arithmetical hierarchy is formed as $\cup_{n\in\omega}(\Sigma_n^0 \cup \Pi_n^0)$

Diagram of the arithmetical hierarchy



The Ershov difference hierarchy

- Ershov [1968,1970] showed that all Δ_2^0-sets can be exhausted with a simpler sets, which are described with help of a special classification. This classification can be considered as a refinement of the level Δ_2^0 of the arithmetical hierarchy.

- Note that $\Sigma_1^{-1} = \Sigma_1^0$.
- Note that Σ_2^{-1}-set is a difference of two Σ_1^{-1}-sets, i.e. 2-c.e. sets, numbers of changes bounded by two.
- Gold [1965], Putnam [1965]. Hierarchy of n-c.e. sets, a boolean combination of c.e. sets.

Turing degrees

- A degree $\mathbf{a}$ is Σ_n^{-1}, if it contains a Σ_n^{-1}-set.
- A degree $\mathbf{a}$ is Σ_n^0, if it contains a Σ_n^0-set.
- A degree $\mathbf{a}$ is a proper Σ_n^{-1}, if it is Σ_n^{-1}, but not Σ_{n-1}^{-1}.
- A degree $\mathbf{a}$ is a proper Σ_n^0, if it is Σ_n^0, but not Σ_{n-1}^0.
- The definitions has a natural generalizations if we put computable ordinals instead of natural numbers (by default we consider computable ordinals in Kleene's notation system).
- The same definitions hold for other degrees, e.g. m-degrees.

Proper levels of the Ershov hierarchy

- Cooper [1971]: There exists a properly Σ_2^{-1}-degree.
- Jockusch and Shore [1984], Selivanov [1985]: There exists a properly Σ_α^{-1}-degree, where α is a computable ordinal.
- Thus, each level of the Ershov hierarchy contains a set with proper Turing degree.

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The Ershov hierarchy and the fine hierarchy

- The Ershov hierarchy gives a classification of all Δ^0_2-sets.
- Turing degrees between $\mathbf{0}'$ and $\mathbf{0}''$ can be classified in the same way as the Ershov hierarchy if we use a relativized Ershov hierarchy (relative to $\mathbf{0}'$). In particularly, differences of Σ^1_0-sets become is the same to differences of Σ^2_0-sets
- The same is applied for any oracle $\mathbf{0}^{(n)}$.

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The Ershov hierarchy and the fine hierarchy

- The fine hierarchy was introduced by Selivanov [1983] and it is a refinement of relativizations of all levels of the arithmetical hierarchy.
- In particular, is allows to classify all Δ^0_2, Δ^0_3, ... -sets.
- However, at Δ^0_2-level, the both hierarchies are coincide.

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- However, at Δ^0_2-level, the both hierarchies are coincide.

Definition of the fine hierarchy

- Selivanov [1983, 1989, 2005] gave three different definitions of the fine hierarchy, and also proved their equivalence.
- Kihara and Montalban [2019]: The fine hierarchy can be considered as a finite and effective version of the Wadge hierarchy [1984].

Definition of the fine hierarchy

- Consider the ordinal $\varepsilon_0 = \lim\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$.
- Denote the levels of the fine hierarchy as Σ_α, where $\alpha < \varepsilon_0$ (note that here Σ_α has no a superscript).
- In particular, we have $\Sigma^0_1 = \Sigma_1$, $\Sigma^0_2 = \Sigma_\omega$, $\Sigma^0_3 = \Sigma_{\omega^\omega}$, $\Sigma^0_4 = \Sigma_{\omega^{\omega^\omega}}$ etc.
- Also, for any $1 \leq k < \omega$ we have $\Sigma^{-1}_k = \Sigma_k$, $\Sigma^{-1,\emptyset'}_k = \Sigma_{\omega^k}$, $\Sigma^{-1,\emptyset^2}_k = \Sigma_{\omega^{\omega^k}}$, $\Sigma^{-1,\emptyset^3}_k = \Sigma_{\omega^{\omega^{\omega^k}}}$ etc.

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The formal definition

Consider one of the definitions. For any $\alpha < \varepsilon_0$ we define the following sequence $\{\mathcal{S}^n_\alpha\}_{n<\omega}$ of classes of sets by induction on α:

- $\mathcal{S}^n_0 = \{\emptyset\}$;
- $\mathcal{S}^n_{\omega^\gamma} = \mathcal{S}^{n+1}_\gamma$ for any $\gamma > 0$;
- $\mathcal{S}^n_{\beta+1} = Bisep(\Sigma^0_{n+1}, \mathcal{S}^n_\beta, \check{\mathcal{S}}^n_\beta, \mathcal{S}^n_0)$ for any $\beta < \varepsilon_0$,
- $\mathcal{S}^n_{\beta+\omega^\gamma} = Bisep(\Sigma^0_{n+1}, \mathcal{S}^n_\beta, \check{\mathcal{S}}^n_\beta, \mathcal{S}^n_{\omega^\gamma})$, where $\gamma > 0$ and $\beta = \omega^\gamma \cdot \beta_1 > 0$.

The formal definition

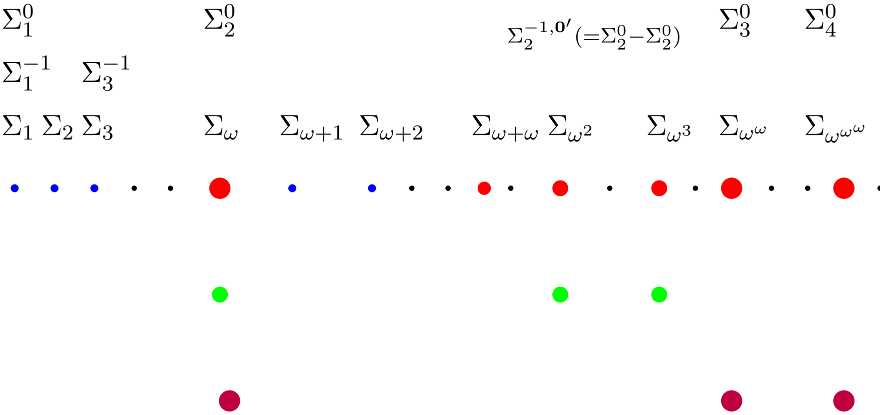
Here $Bisep$ is variant of an operation introduced by Wadge [1984]. It is defined on the class of sets as follows:

- $Bisep(\mathcal{A}, \mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)$ is the class of all sets of the form $(A_0 \cap B_0) \cup (A_1 \cap B_1) \cup (\overline{A_0} \cap \overline{A_1} \cap B_2)$, where $A_0, A_1 \in \mathcal{A}, B_j \in \mathcal{B}_j$, and $A_0 \cap A_1 = \emptyset$.

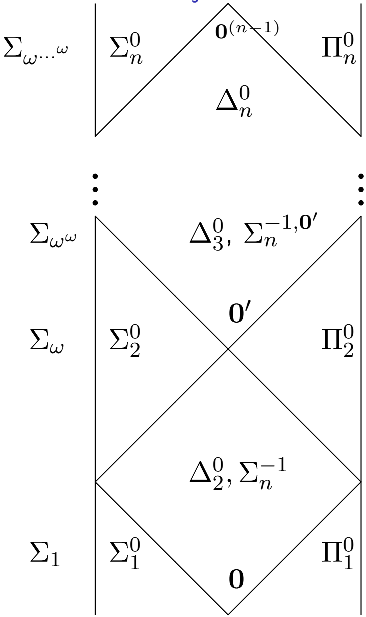
The classes $\mathcal{S}^n_\alpha$, where $n > 0$, are used for technical reasons for this definition and are contained in the sequence $\{\Sigma_\alpha\}_{\alpha<\varepsilon_0}$. A class $\check{\mathcal{S}}^n_\alpha$ consists of complements of sets from $\mathcal{S}^n_\alpha$.

- **Hence**, $\Sigma_\alpha := \mathcal{S}^0_\alpha$

Diagram of the fine hierarchy



The fine hierarchy and the arithmetical hierarchy



Goals and problems

- To find out proper levels of the fine hierarchy
- To describe ordinals $\alpha < \varepsilon_0$ such that $\Sigma_\alpha \not\approx_T \Delta_\alpha$,

where $\Delta_\alpha = \Sigma_\alpha \cap \Pi_\alpha$, and Π_α consists of complement of sets from Σ_α

Example and question

- $\Sigma_\omega \not\approx_T \Sigma_{\omega^2}$ by relativizing Cooper's theorem with oracle $\mathbf{0}'$.

- **Question**: is it true that any level Σ_α contains a proper Σ_α Turing degree?

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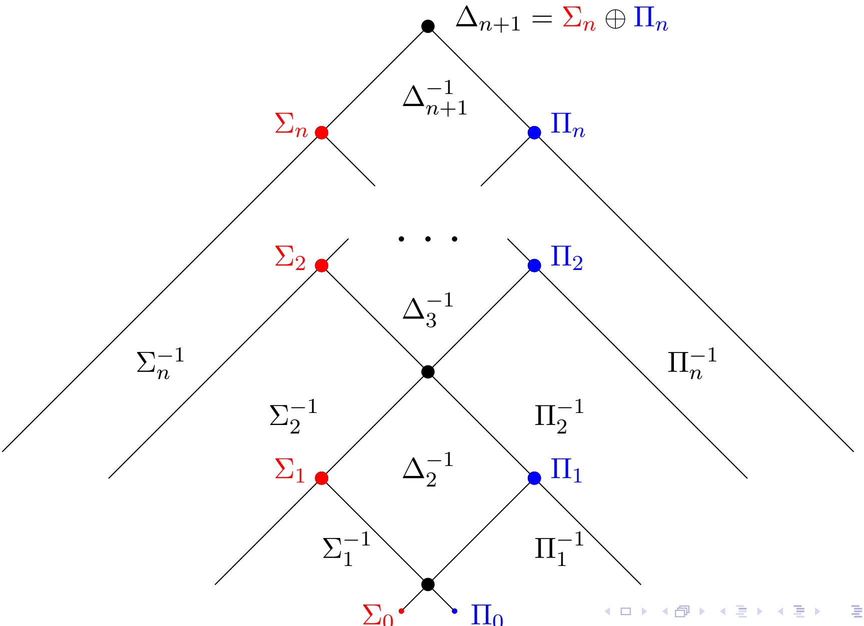
m-degrees

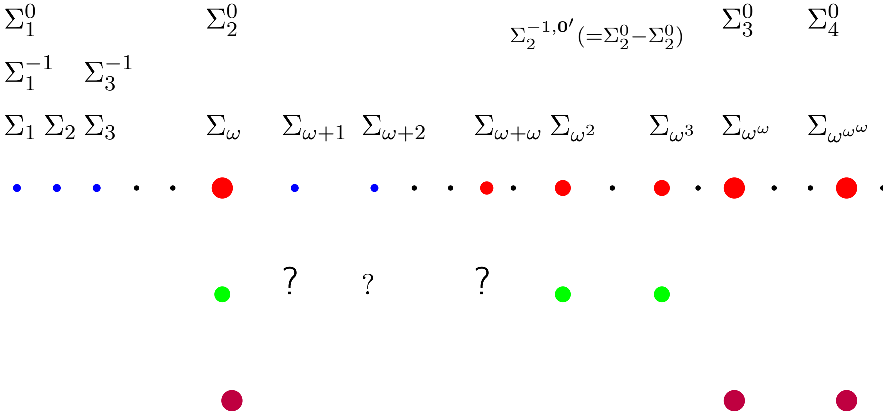
- Selivanov [1983]: All levels are proper relative to m-reducibility.

Picture



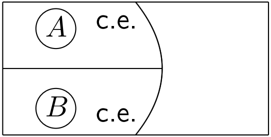
$\Delta_{n+1} = \Sigma_n \oplus \Pi_n$
Δ_{n+1}^{-1}
Σ_n
Π_n
$\cdots$
Σ_2
Π_2
Δ_3^{-1}
Σ_n^{-1}
Π_n^{-1}
Σ_2^{-1}
Π_2^{-1}
Σ_1
Δ_2^{-1}
Π_1
Σ_1^{-1}
Π_1^{-1}
Σ_0
Π_0

Question


Σ^0_1
Σ^0_2
$\Sigma_2^{-1,\mathbf{0}'} (= \Sigma^0_2 - \Sigma^0_2)$
Σ^0_3
Σ^0_4
Σ^{-1}_1
Σ^{-1}_3
Σ_1 Σ_2 Σ_3
Σ_ω
$\Sigma_{\omega+1}$
$\Sigma_{\omega+2}$
$\Sigma_{\omega+\omega}$
Σ_{ω^2}
Σ_{ω^3}
Σ_{ω^ω}
$\Sigma_{\omega^{\omega^\omega}}$
?
?
?

Examples

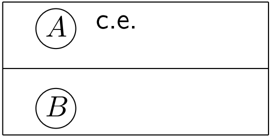
$$A \cup B \in \Sigma_{\omega+1}$$



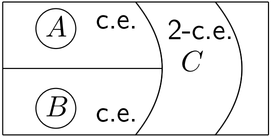
$$A \in \Sigma^0_2$$

$$B \in \Pi^0_2$$

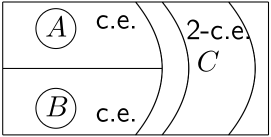
$$A \cup B \in \Sigma_{\omega+\omega}$$



$$A \cup B \cup C \in \Sigma_{\omega+2}$$



$$A \cup B \cup C \in \Sigma_{\omega+3}$$



Proposition

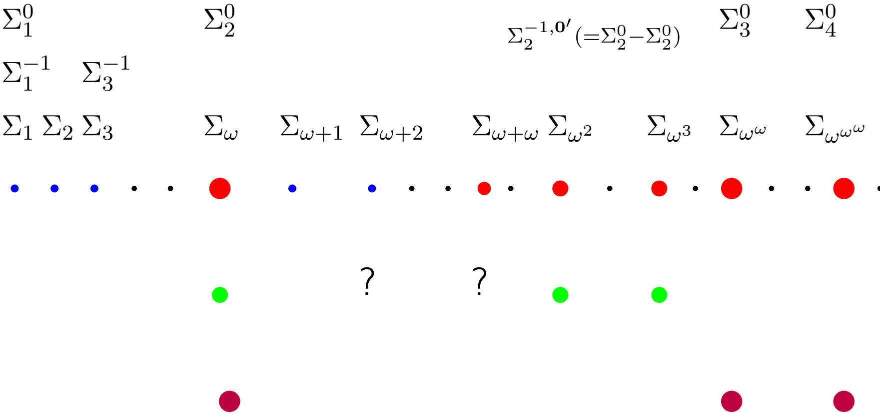
- **Proposition**: If α is a limit ordinal, then $\Sigma_\alpha \approx_T \Sigma_{\alpha+1}$.

- Indeed, if $X \in \Sigma_{\alpha+1}$, then $X = Y \cup Z$, where $Y \in \Sigma_\alpha$, and $Z \in \Pi_\alpha$, also Y and Z are subsets of disjoint c.e. sets. Then $X \equiv_T Y \oplus \overline{Z}$ and, clearly, $Y \oplus \overline{Z}$ is Σ_α.
- Holds for tt-reducibility as well.

Question

- What about other levels? Will other "new" levels collapse or not?

Question


Σ^0_1 Σ^0_2 $\Sigma_2^{-1,\mathbf{0}'}(=\Sigma^0_2-\Sigma^0_2)$ Σ^0_3 Σ^0_4
Σ^{-1}_1 Σ^{-1}_3
Σ_1 Σ_2 Σ_3 Σ_ω $\Sigma_{\omega+1}$ $\Sigma_{\omega+2}$ $\Sigma_{\omega+\omega}$ Σ_{ω^2} Σ_{ω^3} Σ_{ω^ω} $\Sigma_{\omega^{\omega^\omega}}$
?
?

Examples

$A \cup B \in \Sigma_{\omega+1}$

$A \in \Sigma^0_2$

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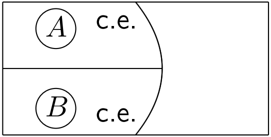
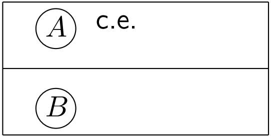
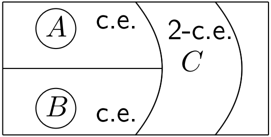
$A \cup B \in \Sigma_{\omega+\omega}$





$A \cup B \cup C \in \Sigma_{\omega+2}$



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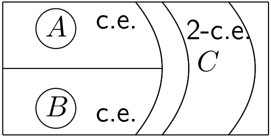


The results

Theorem (Selivanov and Yamaleev, 2018)

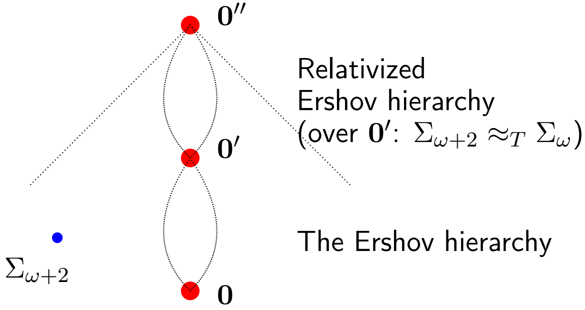
$\Sigma_\omega \not\approx_T \Sigma_{\omega+2}$, *namely:*
There exists a c.e. set E, *there exist disjoint c.e. sets* $E_0, E_1 \subset E$, *there exist* Σ^0_2*-set* $A \subset E_0$, *there exists* Π^0_2*-set* $B \subset E_1$ *such that the Turing degree of* $A \cup B \cup C$ *is not* Σ^0_2, *where*
$C = E - (E_0 \cup E_1)$.

Corollary

Turing degree of the set $A \cup B \cup C$ *is not comparable with* $\mathbf{0}'$.

Note: $\Sigma_{\omega+2}$ is the least "new" proper level in the fine hierarchy.

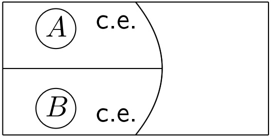
The results



$\mathbf{0}''$
Relativized
Ershov hierarchy
(over $\mathbf{0}'$: $\Sigma_{\omega+2} \approx_T \Sigma_\omega$)
$\mathbf{0}'$
The Ershov hierarchy
$\Sigma_{\omega+2}$
$\mathbf{0}$

Other levels

$$A \cup B \in \Sigma_{\omega+1}$$



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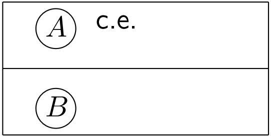
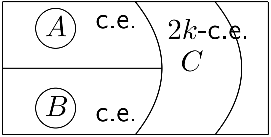
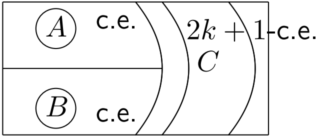
$$A \cup B \in \Sigma_{\omega+\omega}$$



$$A \cup B \cup C \in \Sigma_{\omega+2k}$$



$$A \cup B \cup C \in \Sigma_{\omega+2k+1}$$



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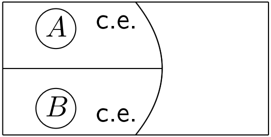

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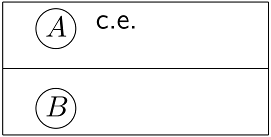
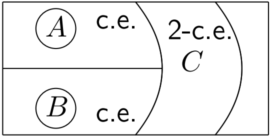
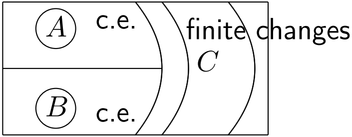


$A \cup B \cup C \in \Sigma_{\omega+2}$



$A \cup B \cup C \in \Delta_{\omega+\omega}$



The results

Theorem (Selivanov and Yamaleev, 2018)

$\Sigma_{\omega+n} \not\approx_T \Sigma_{\omega+n+1}$, *for* $n > 0$.

Corollary

$\Sigma_{\omega+n} \not\approx_T \Sigma_{\omega+\omega}$, *for* $n > 0$.

Theorem (Selivanov and Yamaleev, 2018)

$\Sigma_{\omega+\omega} \not\approx_T \Delta_{\omega+\omega}$.

The results

Theorem (Melnikov, Selivanov and Yamaleev, 2020)

$\Sigma_{\omega^\omega} \not\approx_T \Sigma_{\omega^\omega+2}$.

Theorem (Selivanov and Yamaleev)

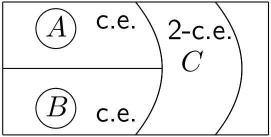
$\Sigma_\alpha \not\approx_T \Sigma_{\alpha+1}$ *for all non-limit* $\alpha < \omega^\omega$.

Picture

$$D \in \Sigma_{\omega^\omega} = \Sigma_3^0$$

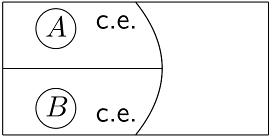
$$A, B \in \Sigma_{\omega^\omega} = \Sigma_3^0$$

$$A \cup B \cup C \in \Sigma_{\omega^\omega+2}$$



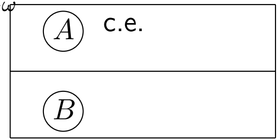
General levels

$A \cup B \in \Sigma_{\omega^2+\omega+1}$

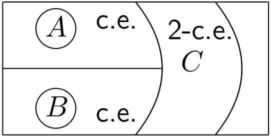


$A \in \Sigma_{\omega^2+\omega}$ $\quad A \cup B \in \Sigma_{\omega^2+\omega+\omega}$

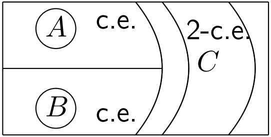
$B \in \Pi_{\omega^2+\omega}$



$A \cup B \cup C \in \Sigma_{\omega^2+\omega+2}$



$A \cup B \cup C \in \Sigma_{\omega^2+\omega+3}$

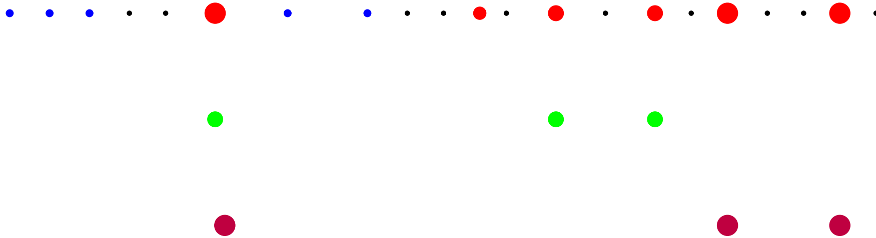


Question

Σ^0_1 Σ^0_2 $\Sigma_2^{-1,\mathbf{0}'}(=\Sigma^0_2-\Sigma^0_2)$ Σ^0_3 Σ^0_4

Σ^{-1}_1 Σ^{-1}_3

Σ_1 Σ_2 Σ_3 Σ_ω $\Sigma_{\omega+1}$ $\Sigma_{\omega+2}$ $\Sigma_{\omega+\omega}$ Σ_{ω^2} Σ_{ω^3} Σ_{ω^ω} $\Sigma_{\omega^{\omega^\omega}}$



Open questions

- Is it true that $\Sigma_\alpha \not\approx_T \Sigma_{\alpha+1}$ for any non-limit ordinal $\alpha < \varepsilon_0$?
- Is it true that $\Sigma_\alpha \not\approx_T \Delta_\alpha$ for any $\alpha < \varepsilon_0$, where $\alpha \neq \lambda + 1$ and λ is limit ordinal?

Conjecture and discussion

- The solution may require $\mathbf{0}^{(n)}$-priority argument, where n depends on the considered level Σ_{n+1} of the arithmetical hierarchy.

- We are forced to work with Σ_n-sets in the oracles of Turing functionals. Thus, need a good understanding of the fine hierarchy and a nice way to deal with the corresponding sets (convenient approximations, good presentations, etc.)

- The fine hierarchy provides a detalized set from the arithmetical hierarchy. Thus, instead of a Σ_{n+1}-set one can consider "Σ_n-set plus something".

- The properness results holds for any reducibility between m- and T-reducibility.

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Conjecture and discussion

- The solution may require $\mathbf{0}^{(n)}$-priority argument, where n depends on the considered level Σ_{n+1} of the arithmetical hierarchy.

- We are forced to work with Σ_n-sets in the oracles of Turing functionals. Thus, need a good understanding of the fine hierarchy and a nice way to deal with the corresponding sets (convenient approximations, good presentations, etc.)

- The fine hierarchy provides a detalized set from the arithmetical hierarchy. Thus, instead of a Σ_{n+1}-set one can consider "Σ_n-set plus something".

- The properness results holds for any reducibility between m- and T-reducibility.

Proof sketch

- Requirements:
- $\mathcal{R}_{\Phi,\Psi,D}: \quad A \cup B \cup C \neq \Phi^D \vee D \neq \Psi^{A \cup B \cup C}$
- The strategy is the Cooper adapted strategy, where A, D are Σ^0_2-sets, and B is a Π^0_2-set.
- Earlier we had an examples of working of with approximation of such sets in the oracles of e-functionals.

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Proof sketch. Strategy

- (1) Choose a "big" x for the attacking the equality
- $A \cup B \cup C = \Phi^D \wedge D = \Psi^{A \cup B \cup C}$
- (2) Using it we obtain an auxiliary string τ
- (3) Put x into C and obtain an auxiliary string σ
- (4) The strings τ and σ must be different at some element. Assume it is z_0. Knowing z_0, and infinitely enumerating x into $A \cup B \cup C$ and extracting it, we force z_0 infinitely often go in and out from D. Thus, we obtain that $z_0 \notin D$.

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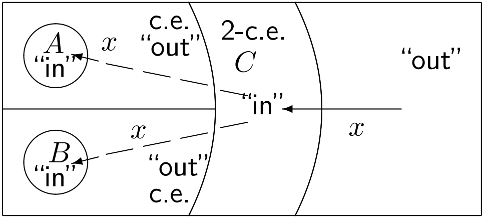
Picture



The set A is Σ^0_2 and the set B is Π^0_2.

Proof sketch. Strategy

- (5) Depending on $\tau(z_0) = 0$ or $\tau(z_0) = 1$ we have a choice for our attack after (3): we can attack either through A or through B.
- (6) Thus, if x wasn't in C and we got $\tau(z_0) = 1$, then infinitely attacking through A we obtain $\Psi^{A \cup B \cup C}(z_0) = 1 \neq D(z_0)$. If x wasn't in C and we got $\tau(z_0) = 0$ (thus, $\sigma(z_0) = 1$), then infinitely attacking through B we also obtain $\Psi^{A \cup B \cup C}(z_0) = 1 \neq D(z_0)$. Recall that $A \in \Sigma^0_2$ and $B \in \Pi^0_2$.

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Proof sketch. Outcomes

- Σ-outcome. There is an infinite attack using through A.
- Π-outcome. There is an infinite attack using through B.
- fin-outcome. There is a diagonalization.

- The tree of strategies $T = 3^{<\omega}$.
- The tree of strategies allows to correctly predict the true placement of witnesses of higher priority strategies, and, sometimes, of witnesses of lower priority strategies. .
- We use $\mathbf{0}''$-priority argument and argue that all requirement are satisfied along the true path.

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Thank you for your attention!