On Partial Numberings

Dieter Spreen

University of Siegen

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Drachenfels/Bonn, 2005

Altai, 2011

Novosibirsk, 2012

Happy Birthday!!

Victor

Definition

Let S be a non-empty set and $\alpha\colon \omega \rightharpoonup S$ be a partial map. Then α is called a numbering of S if α is surjective.

Let $\mathrm{Num}_{\mathrm{p}}(S)$ denote the set of all numberings of S.

In the textbook literature it is mostly assumed that numberings are total maps. There are, however, important cases where this assumption is not true.

Example

Let me recall that a Cauchy sequence $(z_n)_n$ of real numbers is called fast if for all $m, n \in \omega$ with $m \geqslant n$,

$$|z_m - z_n| < 2^{-n}.$$

Moreover, let κ be a canonical indexing of the rational numbers and φ be a Gödel numbering. Then the following result is known.

Proposition

Let $x\colon \omega \rightharpoonup \mathbb{R}_c$ be a numbering of the computable real numbers so that for any $y \in \mathbb{R}_c$,

- *given $i \in \omega$ with $x_i = y$, we can compute an index j such that $(\kappa_{\varphi_j(n)})_n$ is a fast Cauchy sequence converging to y,*
- *from any index of a fast Cauchy sequence converging to y we can compute an x-name of y,*

Then $\mathrm{dom}(x)$ is Π^0_2-hard. In particular, x cannot be a total map.

1. Degrees

Definition

For $\alpha, \beta \in \mathrm{Num_p}(S)$, α is reducible to β (written $\alpha \leqslant \beta$), if there is some partial computable witness function $f \in P^{(1)}$ so that

- $\mathrm{dom}(\alpha) \subseteq \mathrm{dom}(f)$,
- $f(\mathrm{dom}(\alpha)) \subseteq \mathrm{dom}(\beta)$,
- $\alpha_a = \beta_{f(a)}$, for all $a \in \mathrm{dom}(\alpha)$.

Note that if $\alpha \leqslant \beta$ and α is total, then the witness function will be total as well. Hence, in the case of total numberings the just defined reducibility notion coincides with the well known reducibility notion for such numberings.

Degrees of numberings are defined in the usual way

$$\mathrm{deg_p}(\alpha) = \{\, \beta \in \mathrm{Num_p}(S) \mid \alpha \leqslant \beta \wedge \beta \leqslant \alpha \,\} \quad (\alpha \in \mathrm{Num_p}(S)).$$

Let $\mathcal{L}_p(S) = (\{\, \mathrm{deg_p}(\alpha) \mid \alpha \in \mathrm{Num_p}(S) \,\}, \leqslant)$.

Proposition

$\mathcal{L}_p(S)$ is a distributive lattice. If $\|S\| \geqslant 2$, there are no maximal elements.

Note that the meet of two (partial) numberings $\alpha, \beta \in \mathrm{Num_p}(S)$ exists

$$(\alpha \sqcap \beta)_{\langle m,n \rangle} = \begin{cases} \alpha_m & \text{if } m \in \mathrm{dom}(\alpha), \\ & \quad n \in \mathrm{dom}(\beta), \text{ and } \alpha_m = \beta_n, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

In the case of total numberings and their degrees there is no meet in general and we only have an upper semi-lattice in which the degrees of Friedberg numberings are minimal.

Proposition

Let S be infinite. Then the degree of any (partial) Friedberg numbering is not minimal in $\mathcal{L}_p(S)$. Even more, below each degree in $\mathcal{L}_p(S)$

- *there is an infinite descending chain*
- *as well as an uncountable antichain,*

both generated by (partial) Friedberg numberings.

(The results reported on so far have been obtained jointly with Serikzhan Badaev)

How can all this happen?

a) The reducibility relation behaves more differently on partial numberings than it may seem at the first glance.

- For arbitrary total numbering α the set $\{ \beta \mid \beta \text{ total} \wedge \beta \leqslant \alpha \}$ is infinite, but countable.
- In contrary, if $\alpha \in \mathrm{Num}_{\mathrm{p}}(S)$, then the set $\{ \beta \in \mathrm{Num}_{\mathrm{p}}(S) \mid \beta \leqslant \alpha \}$ is uncountable. Indeed, consider the cylindrification $c(\alpha)$ of α with

$$c(\alpha)_{\langle i,j\rangle} = \begin{cases} \alpha_i & \text{if } i \in \mathrm{dom}(\alpha). \\ \text{undefined} & \text{otherwise,} \end{cases}$$

  then by identifying mappings with their graphs we have uncountably many partial numberings $\beta \subseteq c(\alpha)$ of the same set S. The identity function on ω reduces each of these numberings β to $c(\alpha)$.

b) By definition we have that $\alpha \leqslant \beta$ via $f \in P^{(1)}$ iff for all $s \in S$ and $i \in \omega$,

$$i \in \alpha^{-1}(\{s\}) \Longrightarrow f(i)\downarrow \in \beta^{-1}(\{s\}).$$

That is, if i is an α-name of s, we can conclude that $f(i)$ is defined and a β-name of s, which means that f behaves correctly in this case. But in the converse case that $f(i)$ is a β-name of s, we do not know whether i is an α-name of s: i can be any index in $\omega \setminus \mathrm{dom}(\alpha)$.

In the case of total numberings this problem does not occur!

Definition

Numbering α is strongly reducible to numbering β (written $\alpha \leqslant_s \beta$), if there is some $f \in P^{(1)}$ such that for all $s \in S$ and $i \in \omega$,

$$i \in \alpha^{-1}(\{s\}) \Longleftrightarrow f(i)\downarrow \in \beta^{-1}(\{s\}).$$

Use strong reducibility to introduce strong degrees and let $\mathcal{L}_s(S)$ be the set of all strong degrees of numberings of S.

Lemma

$(\mathcal{L}_s(S), \leqslant_s)$ is an upper semilattice.

Now, let $\alpha \in \mathrm{Num_p}(S)$ and apply Ershov's completion procedure to α. That is, for a universal computable function u let

$$\hat{\alpha}_a = \begin{cases} \{\alpha_{u(a)}\} & \text{if } a \in u^{-1}(\mathrm{dom}(\alpha)), \\ \varnothing & \text{otherwise,} \end{cases}$$

$$\widehat{S} = \{\, \{s\} \mid s \in S \,\} \cup \{\varnothing\}$$

Lemma

For any $\alpha \in \mathrm{Num_p}(S)$, $\hat{\alpha}$ is complete total numbering of $\widehat{S}$ with special element $\varnothing$.

Let $\mathcal{C}_{\varnothing}(\widehat{S})$ be the set of all degrees of total complete numberings of $\widehat{S}$ with special element $\varnothing$. Then $(\mathcal{C}_{\varnothing}(\widehat{S}), \leqslant)$ is an upper semilattice.

Proposition

The two upper semilattices $(\mathcal{L}_s(S), \leqslant_s)$ *and* $(\mathcal{C}_{\varnothing}(\widehat{S}), \leqslant)$ *are isomorphic.*

2. Precompleteness

Definition

A numbering $\alpha \in \mathrm{Num_p}(S)$ is precomplete, if for any computable function $p \in P^{(1)}$ there is a total computable function $g \in R^{(1)}$ with

$$\mathrm{range}(g) \subseteq \mathrm{dom}(\alpha)$$

such that for all $i \in p^{-1}(dom(\alpha))$,

$$\alpha_{p(i)} = \alpha_{g(i)}.$$

Function g is called totalizer of p or said to totalize p.

For total numberings α, Ershov has characterized precompleteness

- in a category-theoretic way
- by having the effective fixed point property.

We only mention the second one.

Theorem

Let $\alpha \in \mathrm{Num_p}(S)$. *Then the next four statements are equivalent:*

1. α *is precomplete.*
2. *There is some function* $h \in R^{(1)}$ *with* $\mathrm{range}(h) \subseteq \mathrm{dom}(\alpha)$ *such that for all* $i \in \omega$ *with* $\varphi_i(h(i)){\downarrow} \in \mathrm{dom}(\alpha)$,
$$\alpha_{\varphi_i(h(i))} = \alpha_{h(i)}.$$
3. *There is some function* $h \in R^{(1)}$ *with* $\mathrm{range}(h) \subseteq \mathrm{dom}(\alpha)$ *such that for all* $i \in \omega$ *with* $\varphi_i \in R^{(1)}$ *and* $\varphi_i(h(i)) \in \mathrm{dom}(\alpha)$,
$$\alpha_{\varphi_i(h(i))} = \alpha_{h(i)}.$$
4. *There is some function* $h \in R^{(1)}$ *with* $\mathrm{range}(h) \subseteq \mathrm{dom}(\alpha)$ *such that for all* $i \in \omega$ *with* $\varphi_i \in R^{(1)}$ *and* $\mathrm{range}(\varphi_i) \subseteq \mathrm{dom}(\alpha)$,
$$\alpha_{\varphi_i(h(i))} = \alpha_{h(i)}.$$

In the preceding theorem it may happen that

$$\varphi_i(h(i))\downarrow \notin \mathrm{dom}(\alpha), \quad \text{but} \quad h(i) \in \mathrm{dom}(\alpha).$$

A similar asymmetry appears already in the precompleteness definition: If g is a totalizer of p we always have that

- $g(n) \in \mathrm{dom}(\alpha)$.
- However, there may exist $n \in \mathrm{dom}(p)$ with $p(n) \notin \mathrm{dom}(\alpha)$.

In the case of total numberings the problem will not occur. When dealing with partial numberings, however, there are situations where a more symmetric notion of precompleteness is needed.

Definition

A numbering $\alpha \in \mathrm{Num_p}(S)$ is correctly precomplete if for any function $p \in P^{(1)}$ there is a function $g \in R^{(1)}$ such that the following two conditions hold, for all $i \in \mathrm{dom}(p)$,

- $p(i) \in \mathrm{dom}(\alpha) \Longleftrightarrow g(i) \in \mathrm{dom}(\alpha)$,
- $p(i) \in \mathrm{dom}(\alpha) \Longrightarrow \alpha_{p(i)} = \alpha_{g(i)}$.

In this case g is called correct totalizer of p or said to correctly totalize p.

In the case of total numberings both precompleteness notions coincide. In the case of proper partial numberings, however, they are incomparable.

Let φ be a Gödel numbering and $\hat{\varphi}$ be its co-restriction to $R^{(1)}$.

Lemma

$\hat{\varphi}$ is correctly precomplete, but not precomplete.

Next, let K be the halting set, and

$$\alpha_i = \begin{cases} 0 & \text{if } i \in K, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Lemma

α is precomplete, but not correctly precomplete.

Theorem

Let $\alpha \in \mathrm{Num_p}(S)$. Then the following three statements are equivalent:

1. *α is correctly precomplete.*
2. *There is some function $h \in R^{(1)}$ such that the subsequent two requirements hold, for all $i \in \omega$ for which $\varphi_i(h(i))$ is defined,*
    - $\varphi_i(h(i)) \in \mathrm{dom}(\alpha) \Longleftrightarrow h(i) \in \mathrm{dom}(\alpha)$,
    - $\varphi_i(h(i)) \in \mathrm{dom}(\alpha) \Longrightarrow \alpha_{\varphi_i(h(i))} = \alpha_{h(i)}$.
3. *There is some function $h \in R^{(1)}$ such that the subsequent two requirements hold, for all $i \in \omega$ with $\varphi_i \in R^{(1)}$,*
    - $\varphi_i(h(i)) \in \mathrm{dom}(\alpha) \Longleftrightarrow h(i) \in \mathrm{dom}(\alpha)$,
    - $\varphi_i(h(i)) \in \mathrm{dom}(\alpha) \Longrightarrow \alpha_{\varphi_i(h(i))} = \alpha_{h(i)}$.

3. Isomorphisms Theorems

For total numberings it is well known that m-equivalent numberings are isomorphic, if at least one of them is precomplete. The proof proceeds in two steps:

- First it is shown that any numbering that is m-equivalent to a precomplete one is already 1-equivalent.
- Then Myhill's Theorem is applied.

As we have seen, in the case of general reducibility the witness function is allowed to map non-names of one numbering to names of the other one. A computable isomorphism, however, can only map non-names to non-names. So, witness functions have to operate in a similar way, i.e. they must witness strong reduction.

The usual proof of Myhill's Theorem requires the computation of finite sequences of alternating compositions

$$f \circ \cdots \circ g \circ f(n)$$

of the two witness functions f and g involved.

In the case of partial numberings witness functions are partial functions mapping names of some element to names of the same element and—in the strong case—non-names to non-names.

If, applied to a non-name n, $f(n) \in \mathrm{dom}(g)$, then we need not have that $g(f(n)) \in \mathrm{dom}(f)$ again. Hence, some of these computations may not terminate.

Since we cannot decide, in general, whether a number is a name, Myhill's construction will not work in the case of partial numberings.

Theorem (First Isomorphism Theorem)

Let $\|S\| \geqslant 2$ *and* $\alpha, \beta \in \mathrm{Num_p}(S)$ *be correctly precomplete. Then, if* α *and* β *are strongly m-equivalent, they are computably isomorphic.*

Note that in the case of total numberings, precompleteness is inherited under *m*-equivalence. Therefore, this result extends Ershov's Isomorphism Theorem for total numberings to the partial case.

We will next introduce a concept which under very natural conditions satisfied by the standard numberings of many important effectively given topological spaces turns out to be weaker than correct precompleteness.

Definition

A numbering $\alpha \in \mathrm{Num_p}(S)$ is uniformly productive, if there is some computably enumerable set $A \supseteq \mathrm{dom}(\alpha)$, called localizer, and a function $k \in R^{(1)}$, the witness generator, so that $\varphi_{k(i)} \in R^{(1)}$, for all $i \in A$, and

- $\varphi_{k(i)}(j) \in \alpha^{-1}(\{\alpha_i\}) \backslash W_j$, for all $i \in \mathrm{dom}(\alpha)$ and $j \in \omega$ with $W_j \subseteq \alpha^{-1}(\{\alpha_i\})$,
- $\varphi_{k(i)}(j) \in \overline{\mathrm{dom}(\alpha)} \backslash W_j$, for all $i \in A \backslash \mathrm{dom}(\alpha)$ and $j \in \omega$ with $W_j \subset \overline{\mathrm{dom}(\alpha)}$.

Note that in general uniform productivity is not implied by precompleteness.

Let χ_K be the characteristic function of the halting set K. Then we have:

- χ_K is a total precomplete numbering of $\{0, 1\}$.
- Obviously, χ_K is not uniformly productive.

Theorem (Second Isomorphism Theorem)

Let $\alpha, \beta \in \mathrm{Num}_{\mathrm{p}}(S)$ be uniformly productive. Then, if α and β are strongly m-equivalent, they are computably isomorphic.

4. Effectively Given Topological Spaces

- Let $\mathcal{T} = (T, \tau)$ be a countable T_0 space with countable basis $\mathcal{B}$,
- $B \colon \omega \to \mathcal{B}$ be a total numbering of $\mathcal{B}$.

Definition

A transitive binary relation $\prec_B$ on ω is a strong inclusion if for all $m, n \in \omega$,

$$m \prec_B n \Longrightarrow B_m \subseteq B_n.$$

- Let $\prec_B$ be a strong inclusion so that $\mathcal{B}$ is a strong basis, i.e. the property of being a basis holds with respect to $\prec_B$.

Notation: $\mathcal{N}(y)$ neighbourhood filter of y.

Definition

A numbering $x \in \mathrm{Num_p}(T)$ is computable if there is some computably enumerable set $L \subseteq \omega$ such that for all $i \in \mathrm{dom}(x)$ and $n \in \omega$,

$$\langle i, n \rangle \in L \iff x_i \in B_n.$$

If in addition L is such that for all $i \in \omega$,

$$\{ B_n \mid \langle i, n \rangle \in L \} \text{ strong base of } \mathcal{N}(y) \text{ for some } y \in T \implies i \in \mathrm{dom}(x) \wedge x_i = y,$$

numbering x is called strongly computable.

- x computable: a base of $\mathcal{N}(x_i)$ can be enumerated, uniformly, in i.
- x strongly computable: x provides as many names as possible, for every point in T,

Definition

If $f : \omega \rightharpoonup \omega$ so that

- $\mathrm{dom}(f)$ is an initial segment of ω and
- f is decreasing with respect to $\prec_B$,

then the enumeration $(B_{f(a)})_{a \in \mathrm{dom}(f)}$ is called *p-normed*.

If f is total, $(B_{f(a)})_{a \in \mathrm{dom}(f)}$ is said to *normed*.

In case f is computable, $(B_{f(a)})_{a \in \mathrm{dom}(f)}$ is called *computable*, and every Gödel number of f is said to be an *index* of $(B_{f(a)})_{a \in \mathrm{dom}(f)}$.

Definition

If for total such $f : \omega \to \omega$, $(B_{f(a)})_{a \in \mathrm{dom}(f)}$ enumerates a strong base of $\mathcal{N}(y)$, for some $y \in T$, we say $(B_{f(a)})_{a \in \mathrm{dom}(f)}$ *converges* to y.

Definition

A numbering $x \in \mathrm{Num_p}(T)$ strongly allows effective limit passing if there is a is a function $\mathrm{pt} \in P^{(1)}$ such that for all $m \in \omega$,

- If m index of a normed computable enumeration of basic open sets converging to some $y \in T$, then $\mathrm{pt}(m)\downarrow \in \mathrm{dom}(x)$ and $x_{\mathrm{pt}(m)} = y$.
- If $\mathrm{pt}(m) \in \mathrm{dom}(x)$ then

$$\{ B_a \mid a \in \mathrm{dom}(\varphi_m) \}$$

  is a base of $\mathcal{N}(x_{\mathrm{pt}(m)})$.

If $\mathrm{pt} \in P^{(1)}$, in addition, admits filters, then x is called strongly approximating.

Definition

A function $g \in P^{(1)}$ admits filters if g is defined for all indices of p-normed computable enumerations of basic open sets.

Definition

Let $x \in \mathrm{Num_p}(T)$. We say that

- x is strongly acceptable if x is strongly computable and strongly allows effective limit passing.
- x is strongly correct if x is strongly computable and strongly approximating.

Recall that a point $y \in T$ is finite if $\mathcal{N}(y)$ has a finite base.

Proposition

- *Let $\mathcal{T}$ have no finite points,*
- *$\prec_B$ be computably enumerable,*
- *x be strongly acceptable.*

Then

$$x \text{ correctly precomplete} \iff \\ x \text{ has a total strong effective limit passing witness.}$$

Theorem (First Isomorphism Theorem for Effective Spaces)

- *Let $\mathcal{T}$ have no finite points,*
- *$\prec_B$ be computably enumerable,*

Then any two strongly acceptable numberings with total strong limit passing witness are computably isomorphic.

Note: Under the above assumptions on $\mathcal{T}$ we have for strongly acceptable $x \in \mathrm{Num_p}(\mathcal{T})$,

$$x \text{ correctly precomplete} \Longrightarrow x \text{ strongly correct.}$$

When will strongly correct numberings be uniformly productive?

Definition

Space $\mathcal{T}$ has the effective Urysohn property if there is a countable family of closed sets $(A_m)_{m\in\omega}$ so that:

- $\overline{A_m}$ is completely enumerable, uniformly in m,

$$\text{(i.e. } (\exists s \in R^{(1)})\, W_{s(m)} \cap \mathrm{dom}(x) = x^{-1}(\overline{A_m})),$$

- $B_m \subseteq A_m$, for all $m \in \omega$,
- If $m \prec_B n$ then $A_m \subseteq B_n$, for all $m, n \in \omega$.

Proposition

- *Let $\mathcal{T}$ have no finite points, be computably separable (i.e. have an enumerable dense subset), and have the effective Urysohn property,*
- $\prec_B$ *be computably enumerable.*

If $x \in \mathrm{Num_p}(T)$ is strongly correct then x is uniformly productive.

Theorem (Second Isomorphism Theorem for Effective Spaces)

- *Let $\mathcal{T}$ have no finite points, be computably separable, and have the effective Urysohn property,*
- $\prec_B$ *be computably enumerable.*

Then any two strongly correct numberings of T are computably isomorphic.

5. Totalization

Can (S, α) be embedded into a numbered set $(\widehat{S}, \widehat{\alpha})$ so that numbering $\widehat{\alpha}$ is total and extends the given numbering α?

It is a common technique in mathematics to enlarge a given set by suitable elements so that the enlarged set has a wanted property. The question is by what kind of elements we should enlarge S to obtain our goal.

> **Ershov:** *Every indexed set (S, α) with total numbering α can be embedded in a numbered set $(\widehat{S}, \widehat{\alpha})$ such that $\widehat{\alpha}$ is both total and complete and α is reducible to $\widehat{\alpha}$.*

$\widehat{S}$ is obtained from S by adjoining a new element $\perp$.

- The same construction can be carried out for partial numberings α. Again $\widehat{\alpha}$ is total and complete.
- But, if S is an effectively given T_0 space. the numbering $\widehat{\alpha}$ need not be acceptable and the embedding need not be effectively homeomorphic.

The example of the total computable functions with acceptable numberings shows that we have to add finite (partial) elements that approximate the given ones.

Lemma

Let $\mathcal{T}$ have no finite points. If $x \in \mathrm{Num_p}(T)$ is acceptable then x cannot be total.

Theorem

- *Let* $\prec_B$ *be computably enumerable, and*
- $x \in \mathrm{Num_p}(T)$ *be acceptable.*

Then there is an algebraic constructive domain $\widehat{T}$ *) *with*

- *a total acceptable complete numbering* $\hat{x}$ *and*
- *an effectively homeomorphic embedding* $F\colon T \to \widehat{T}$ *such that*
  - *both* F *and its partial inverse are morphisms of the numbered sets* (T, x) *and* $(F(T), \tilde{x})$,
  - *and* $F(T)$ *is an enumerable dense subset of* $\widehat{T}$.

*) That is, $\widehat{T}$ consists of the computable elements of an effectively given algebraic directed-complete partial order.

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