From p-adic special orthogonal group $SO(3)_p$ to p-adic qubits

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$\boxed{\text{arXiv: 2104.06228 \& 2412.03362}}$

Outline

1. p-Adic space $\mathbb{Q}_p^3$
2. $SO(3)_p$ – geometric structure
3. p-Adic angular momentum via projective representations of $SO(3)_p$
4. $SO(3)_p$ – as inverse limit
5. First examples of 2-dim. irreps
6. Outlook: Haar measure, ...

1. p-Adic space $\mathbb{Q}_p^3$

- Recall p-adic norm on $\mathbb{Q}$,

$$\left| p^n \frac{s}{t} \right|_p = p^{-n}$$ if st and p relatively prime

- $\mathbb{Q}_p :=$ completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$

$$= \left\{ \sum_{k=n}^{\infty} a_k p^k : a_k \in \{0, 1, \dots, p-1\} \right\}$$

- $\mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \le 1 \}$ p-adic integers

$\supset \mathbb{U}_p = \{ x : |x|_p = 1 \}$ unit elements

- $\mathbb{Q}_p^3$ takes on the role of $\mathbb{R}^3$ as configuration space.

- Acts on itself by translations $\longrightarrow$ Haar measure
  $\longrightarrow L^2(\mathbb{Q}_p^3)$ wave functions

  [I. Volovich, V. Vladimirov, CMP 123: 659–676, 1989]

- Here: bring rotation symmetry into play

2. $SO(3)_p$

Define orthogonal transformation via a quadratic form Q on $\mathbb{Q}_p^3$ left invariant.

W.l.o.g $Q(x,y,z) = ax^2 + by^2 + cz^2$

[up to linear isomorphism of $\mathbb{Q}_p^3$]

$$SO(Q) = \{ M : \mathbb{Q}_p^3 \to \mathbb{Q}_p^3 : \forall x \; Q(Mx) = Q(x) \text{ and } \det M = 1 \}$$

$SO(Q) = \{ M \in \mathbb{Q}_p^{3\times 3} : \forall x \;\; Q(Mx) = Q(x) \;\&\; \det M = 1 \}$

same group for scaled $Q' = cQ$;

isomorphic group for similar $Q' = Q \circ T$

$(T \in GL_3(\mathbb{Q}_p))$

Fact: up to equivalence, there are two non-degenerate quadratic forms on $\mathbb{Q}_p^3$, one indefinite (i.e. $\exists x \neq 0 \;\; Q_0(x) = 0$), the other definite (i.e. $Q_+(x) = 0 \Rightarrow x = 0$).

$SO(Q_+)$ compact

$SO(Q_0)$ non-compact

Picking a non-square $u \in \mathbb{U}_p$ (for $p \equiv 1\ (4)$; for $p \equiv 3\ (4)$ use -1)

and letting $v = \begin{cases} -1 & : p \equiv 3\ (4) \\ -u & : p \equiv 1\ (4) \end{cases}$

$$Q_+(x,y,z) = \begin{cases} x^2+y^2+z^2 & : p = 2 \\ x^2 - vy^2 + pz^2 & : p \text{ odd} \end{cases}$$

Btw, unique (up to equivalence) definite form on $\mathbb{Q}_p^4$: $Q_+(x,y,z,w) = \begin{cases} x^2+y^2+z^2+w^2 & : p = 2 \\ x^2 - vy^2 + pz^2 - pvw^2 & : p \text{ odd} \end{cases}$

⚠ All forms on $\mathbb{Q}_p^d$, $d \geq 5$, are indefinite

Thus we choose $SO(3)_p := SO(Q_+)$.

Letting
$$A = \begin{cases} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} & : p = 2, \\ \begin{bmatrix} 1 & & \\ & -v & \\ & & p \end{bmatrix} & : p \text{ odd}, \end{cases}$$

we can write, more explicitly:

$$SO(3)_p = \left\{ L \in \mathbb{Q}_p^{3\times 3} : L^T A L = A \ \&\ \det L = 1 \right\}$$

That this is indeed compact follows from $SO(3)_p \subset SL(3, \mathbb{Z}_p)$, i.e. all $L \in SO(3)_p$ have p-adic integer matrix elements

In 2104.06228 show several interesting geometric features of $SO(3)_p$, similar to $SO(3)_{\mathbb{R}}$:

- All $L \in SO(3)_p$ have a fixed point $Lv = v$, and so L is a rotation around the axis $\mathbb{Q}_p v$.

- $L|_{v^\perp}$ (w.r.t. Q_+) is an element of $SO(2)_p$, of which there exist 3 ($p > 2$) or 7 ($p = 2$) inequivalent forms

- For p odd, $L \in SO(3)_p$ have a nautical angle decomposition into rotations around coordinate axes
[but not as versatile as real case]

3. p-Adic angular momentum

Just as the Heisenberg group and its quantum mechanics arises from the spatial translations in $\mathbb{Q}_p^3$, so we expect the symmetry group $SO(3)_p$ and its irreps to result in a p-adic theory of angular momentum ... spin.

Programme:

- construct all unitary projective irreps of $SO(3)_p$

- describe their tensor products and how they decompose into irreps

4. $SO(3)_p$ as a fractal: inverse limit

Recall $SO(3)_p \subset SL(3,\mathbb{Z}_p) \subset \mathbb{Z}_p^{3\times 3}$

Thus the maps $\pi_k : SO(3)_p \longrightarrow \left(\mathbb{Z}/p^k\right)^{3\times 3}$,

$\pi_k(L) = L \bmod p^k$ (entry-wise), are

well defined and group homomorphisms

$$\pi_k : SO(3)_p \longrightarrow G_{p^k} := \underbrace{\pi_k(SO(3)_p)}_{SO(3)_p \bmod p^k}$$

Fact (in preparation): For all p and k,

$$\mathcal{G}_{p^k} = \widehat{\mathcal{G}}_{p^k} := \{ L \in (\mathbb{Z}/_{p^k})^{3\times 3} : L^T A L \equiv A \bmod p^k \text{ and } \det L \equiv 1 \bmod p^k \}$$

Inclusion "$\subseteq$" obvious, opposite "$\supseteq$" proved by Hensel lifting

$$|\mathcal{G}_{p^k}| = \begin{cases} 3 \cdot 2^{5k-4} & : p = 2 \\ 2(p+1)p^{3k-1} & : p \text{ odd} \end{cases}$$

Thus:

$$SO(3)_p \xrightarrow{\pi_{k+1}} G_{p^{k+1}} \xrightarrow{\pi^{k+1}_{k}} G_{p^k} \longrightarrow \cdots \longrightarrow G_p$$

forms a projective system, and

$$SO(3)_p = \varprojlim G_{p^k},$$ making it a profinite group.

Fibre of p^3 (2^5) over every point in G_{p^k}

$\Longrightarrow$ One of two constructions of the Haar measure on $SO(3)_p$: projective limit of normalized counting measures

[In prep., P. Aniello, S. L'Innocente, S. Mancini, V. Parisi, I. Svampa & A.W.]

5. First simple examples

Because of the onto homomorphisms

$$\pi_k : SO(3)_p \longrightarrow G_{p^k},$$

every irrep of the (finite!) group G_{p^k} becomes one of $SO(3)_p$

So let's start with $k=1$ ...

$p=2$: G_2 has six elements and is non-commutative, so turns out $G_2 \cong S_3$, which is isomorphic to D_3 the dihedral group of rank 3

- One 2-dim. irrep ("qubit") : symmetries of a regular triangle in the plane
- Two 1-dim irreps : trivial and sign

<u>p odd:</u> $|G_p| = 2p^2(p+1)$, and we still don't have the general structure of G_p (except G_3 and G_5 by brute force), but for all p, have onto hom $\delta : G_p \to D_{p+1}$, the dihedral group of rank $p+1$

- Four 1-dim. irreps, corresponding to $D_{p+1}/[D_{p+1}, D_{p+1}] \cong \mathbb{Z}/_2 \times \mathbb{Z}/_2$
- $\frac{p-1}{2}$ 2-dim. irreps (multiple "qubits"?!)

6. Outlook

- Using Haar measure, can construct $L^2(SO(3)_p)$ and regular representations.
  Peter-Weyl yields (in theory) all irreps

⚠ Many Haar-integrals to compute
Hard despite two independent constructions of the invariant measure

- What do the multiple 1- and 2-dim. irreps signify?

1-dim irreps correspond to abelianization of the group, $SO(3)_p / [SO(3)_p, SO(3)_p]$.
Seems to be $\cong \mathbb{Z}/2 \times \mathbb{Z}/2 \ldots$

- p-Adic quantum information processing or other applications?

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