Keplerian trajectories and effective asymptotics of some solutions of the Schrodinger equation with a repulsive Coulomb potential

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The goal: I want to speak about new effective semiclassical formulas based on a modification of the canonical Maslov operator, recently obtained together with my colleagues and illustrate them on two classical problems for **the three-dimensional Schrödinger equation with a repulsive Coulomb potential**.

One of them is the **scattering** problem, and the **second one** is similar to the problem of the **asymptotics of the "individual" Green function**. In these cases, the resulting formulas for wave functions, although long, are absolutely explicit, they are based on well-known **Keplerian trajectories and Airy functions.**

General approaches I will be able to present only simple ideas underlying them.

Formulation of the problem 1: scattering problem for the Schrödinger equation with the Coulomb repulsive potential

$$-h^2\triangle\psi + \frac{\gamma}{|x|} = k^2\psi, \quad \psi \to e^{i\frac{k}{h}x_1} \quad \text{as } x_1 \to -\infty$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \gamma,\, k,\, h \text{ are positive constants, } h \ll 1$$

$+$ the Sommerfeld conditions

Formulation of the problem 2: the inhomogeneous Schrödinger equation with the Coulomb repulsive potential and localised right hand side (the Green function type problem

$$\left(-h^2\Delta + \frac{\gamma}{|x|} - E\right)\psi(x) = F\left(\frac{x - x^0}{h}\right), \quad x \in \mathbb{R}^3,$$

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, γ, E, h are positive constants, $h \ll 1$,

$F(y)$ is a smooth fast-decreasing function $\quad x_0 = \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}, b > 0$

+ absorption conditions at infinity

PROBLEM 1

Faddeev, L.D., Merkuriev, S.P.: Quantum Scattering Theory for Several Particle Systems, Nauka, M. 1985, Kluwer, Dordrecht, 1993

V. S. Buslaev, S. B. Levin, A System of Three Three-Dimensional Charged Quantum Particles: Asymptotic Behavior of the Eigenfunctions of the Continuous Spectrum at Infinity, Funct. Anal. Appl., 46:2 (2012), 147-151

S. B. Levin, Diffraction Approach in the Scattering Problem for Three Charged Quantum Particles, Math. Notes, 108:3 (2020), 457-461

The solution via the degenerate hypergeometric function $F(a,b,t)$:

$$\psi = e^{i\frac{k}{h}x_1}\psi_0, \quad \psi_0 = e^{-\frac{\pi\beta}{2}}\Gamma(1+i\beta)F(-i\beta, 1, i\frac{k}{h}\xi), \quad \beta = \frac{\gamma}{2kh}, \quad \xi = |x|-x_1.$$

The parabolic coordinates: $\xi = |x| - x_1, \quad \zeta = |x| + x_1$, θ is the rotation angle around the axis x_1/

The equation for ψ_0:

$$h^2\Big[-\frac{4}{\xi+\zeta}(\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\zeta}\zeta\frac{\partial}{\partial\zeta}) - \frac{1}{\xi\zeta}\frac{\partial^2}{\partial\theta^2}\Big]\psi + \frac{2\gamma}{\xi\zeta}\psi = k^2\psi$$

The aim is to construct semiclassical asymptotics for $h \to +0$

Semiclassical asymptotics

V.P. Maslov, M.V.Fedoriuk, Semiclassical Approximations in Quantum Mechanics, Reidel, 1981

B.R. Vainberg, Asymptotic Methods in Equations of Mathematical Physics, CRC Press, 1989

S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. I. Shafarevich, "New integral representations of the Maslov canonical operator in singular charts", Izv. Math., 81:2 (2017), 286–328

S. Yu. Dobrokhotov, A. V. Tsvetkova, "Lagrangian Manifolds Related to the Asymptotics of Hermite Polynomials", Math. Notes, 104:6 (2018), 810–822

A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. V. Tsvetkova, "Uniform asymptotic solution in the form of an Airy function for semiclassical bound states in one-dimensional and radially symmetric problems", Theoret. and Math. Phys., 201:3 (2019)

The algorithm:

1) Construction the invariant Lagrangian manifold Λ^3 (via Keplerian trajectories)

2) Construction the phase, the invariant measure, the Maslov indices, the Maslov canonical operator $\psi = K^h_{\Lambda^3} \cdot 1$

3) Simplification and global representation in the form of an Airy function of a complex argumen

Reminder: WKB-asymptotics and Lagrangian manifolds:

$$\psi = a(x)e^{\frac{i}{h}S(x,t)}, \quad h \to +0$$

the (smooth) Lagrangian manifold in $2n - D$ phase space $\mathbb{R}^{2n}_{px}$

$$\Lambda^n = (p = \nabla S) = \{p = P(\alpha),\, x = X(\alpha), \quad \alpha = \alpha_1, \ldots, \ldots, \alpha_n\}$$

$$dS = P\,dX, \quad a = \frac{A(\alpha)}{\sqrt{J(\alpha)}}\bigg|_{\alpha=\alpha(x)}.$$

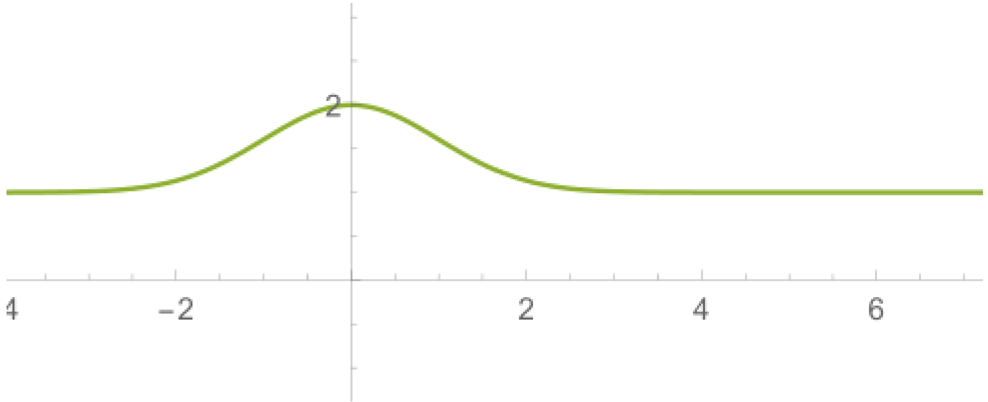
Here J is the Jacobian of projection $\Lambda^n \to \mathbb{R}^n_x$: $J(\alpha) = \det \frac{\partial X}{\partial \alpha}$ and $\alpha = \alpha(x)$ is the solution to the equations

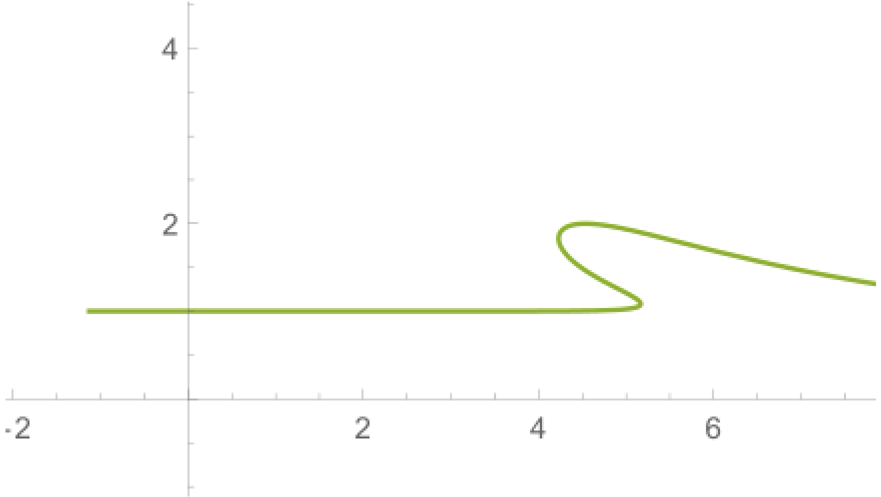
$$X(\alpha) = x \qquad \Longleftarrow \qquad \text{Parametric form}$$

the phase $S(x) = \int_{\alpha_0}^{\alpha(x,t)} PdX$,
(the solution to the Hamilton-Jacobi equation)

The Lagrangian singularities (focal points, caustics).

$$J(\alpha) = \det \frac{\partial X}{\partial \alpha} = 0$$





The Maslov canonical operator:

WKB-type function near the regular points ($J(\alpha) = \det \frac{\partial X}{\partial \alpha} \neq 0$) and the integral representation near the focal points and caustics.

1-D case: the WKB representation in momentum coordinate+ inverse Fourier transform

$$\Psi(x) = \sqrt{\frac{i}{2\pi h}} e^{-i\frac{\pi}{2}m} \int \frac{A(\alpha)}{\sqrt{|\det \frac{\partial P(\alpha)}{\partial \alpha}|}} e^{\frac{i}{h}\left(\int_{\alpha_0}^{\alpha(p)} P dX - P(\alpha)(x - X(\alpha))\right)}\Big|_{\alpha=\alpha(p)} \quad dp,$$

here m is the Maslov index, $\alpha = \alpha(p)$ is the solution to equation:

$$P(\alpha) = p.$$

$$\Downarrow$$

two steps: 1) finding $\alpha(p)$ (could be not trivial), 2) integration over the momentum p

The main idea: let us pass to integration over α $\Longrightarrow$ the first step disappears

$$\Psi(x) = \sqrt{\frac{i}{2\pi h}} e^{-i\frac{\pi}{2}m} \int A(\alpha) \sqrt{|\det \frac{\partial P(\alpha)}{\partial \alpha}|} e^{\frac{i}{h}\left(\int_{\alpha_0}^{\alpha(p)} P dX - P(\alpha)(x - X(\alpha))\right)} \quad d\alpha,$$

n-D case:

mixed representation in the neighborhood of Lagrangian singularities.

Example $n = 2$. Canonical coordinates: (x_1, x_2), (x_1, p_2), (p_1, x_2), (p_1, p_2).

Th. *At least one of Jacobians* $\det \frac{\partial(X_1,X_2)}{\partial\alpha}$, $\det \frac{\partial(X_1,P_2)}{\partial\alpha}$, $\det \frac{\partial(P_1,X_2)}{\partial\alpha}$, $\det \frac{\partial(P_1,P_2)}{\partial\alpha}$ *is not equal to zero.*

Example of mixed representation in the case (p_1, x_2)

$$\Psi(x) = \sqrt{\frac{i}{2\pi h}} e^{-i\frac{\pi}{2}m} \int \frac{A(\alpha)\mathbf{e}(\alpha)}{\sqrt{|\det \frac{\partial(P_1(\alpha),X_2(\alpha))}{\partial\alpha}|}} e^{\frac{i}{h}\left(\int_{\alpha_0}^{\alpha}\langle P,dX\rangle - \langle P(\alpha),x-X(\alpha)\rangle\right)}\Big|_{\alpha=\alpha(p_1,x_2)}$$

here m is the Maslov index, $\alpha = \alpha(p)$ is the solution to equations:

$$P_1(\alpha) = p_1, \quad X_2(\alpha) = x_2.$$

two steps: 1) finding $\alpha(p_1, x_2)$ (could be not trivial),

2) integration over the momentum p_1

$$P_1(\alpha) = p_1, \quad X_2(\alpha) = x_2.$$

The main idea: let us divide α into two sets $\alpha = (\psi, \phi)$, assume that $\frac{\partial X(\psi,\phi)}{\partial \phi}$ and changing p_1 by ψ pass to integration over ψ $\Longrightarrow$ then 1) the first step disappears, 2) the same procedure could be done for the other coordinates e.g. x_1, p_2, and sometimes then we can present the Maslov canonical operator using just one integral. The realisation of this idea in n-D case is ***nontrivial***

S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. I. Shafarevich, New integral representations of the Maslov canonical operator in singular charts, Izv. Math., 81:2 (2017), 286-328

Generalization to n-D case: partial Fourier transform $(x,p) \rightarrow ((x^I, p^{\bar{I}}), (p^I, x^{\bar{I}}))$
Example $n=3$:

$$(x_1,x_2,x_3),(p_1,x_2,x_3),(x_1,p_2,x_3),(x_1,x_2,p_3),$$
$$(p_1,p_2,x_3),(p_1,x_2,p_3),(x_1,p_2,p_3),(p_1,p_2,p_3)$$
$$+$$

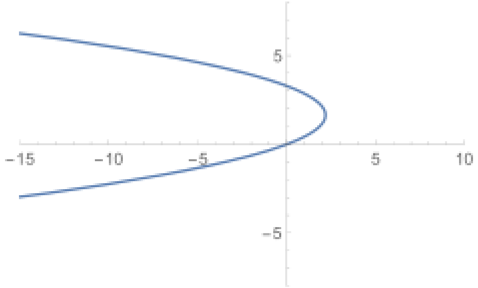
passage from integration by $p^{\bar{I}}$ integration by some coordinates $\alpha_{\bar{I}}$ on Λ^n.

The Maslov canonical operator:

$$K_\Lambda^h A = \sum_{\Omega_I} \psi_I$$

The focal points, “simple caustics” and uniform asymptotics in the form of Airy function: *“naive” constructive approach*

Let $x = X(\alpha^*)$ be nondegenerated focal point ($1D$-case) :



$$\frac{\partial X}{\partial \alpha}(\alpha^*) = 0, \quad \frac{\partial^2 X}{\partial \alpha^2}(\alpha^*) \neq 0.$$

$$\Lambda = \{p = P^* + P'^*(\alpha - \alpha^*) + O((\alpha-\alpha^*)^2), x = X^* + \frac{1}{2}X''^*(\alpha-\alpha^*)^2 + O((\alpha-\alpha^*)^3)\}$$

looks like a “horizontal” parabola.

Then for small $x - X^*$ one can show (at least on the physical level of rigor) that the asymptotic of the integral is expressed in terms of the **Airy function and its derivative**. This implies the following anzats:

$$\psi \approx e^{i\frac{Q(x)}{h}}\big(A_1(x,h)\mathrm{Ai}(\Phi(x,h)) + A_2(x,h)\mathrm{Ai}'(\Phi(x,h))\big),$$

here phases $Q(x)$, $\Phi(x,h)$ and amplitudes $A_j(x,h)$ are unknown functions,

not the matching method but a simplification of a solution

Airy Airy'

Why Ai'? $\alpha^* = 0$ $A(\alpha) = \frac{A(\alpha)+A(-\alpha)}{2} + \alpha\frac{A(\alpha)-A(-\alpha)}{2\alpha} = g_1(X) + \alpha g_2(X)$

We can write for $\Phi(x,h) \ll -1$

$$\psi \approx \frac{e^{i\frac{Q(x)}{h}}}{\sqrt{\pi}}\Big(\frac{A_1(x,h)}{\sqrt[4]{z}}\sin(z+\frac{\pi}{4}) - A_2(x,h)\sqrt[4]{z}\cos(z+\frac{\pi}{4})\Big), \quad z = -\frac{2}{3}(\Phi)^{3/2}.$$

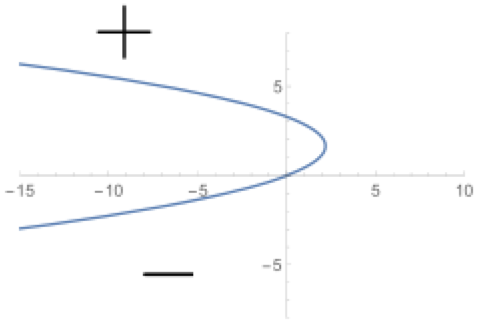
From the other side the Maslov canonical operator gives:

$$\psi \approx a_+(x)e^{i\frac{\pi}{2}}e^{\frac{iS_+(x)}{h}} + a_-(x)e^{\frac{iS_-(x)}{h}} =$$

$$e^{\frac{i\pi}{4}}e^{\frac{i}{2h}(S_+ + S_-)}\Big(i(a_+ - a_-)\sin(\frac{S_+ - S_-}{2h} + \frac{\pi}{4}) + (a_- + a_+)\cos(\frac{S_+ - S_-}{2h} + \frac{\pi}{4})\Big),$$

where

$$S_\pm = \int_{\alpha_0}^{\alpha_\pm(x)} P dX, \quad a_\pm = \frac{A(\alpha_\pm(x))}{\sqrt{|J(\alpha_\pm(x))|}}$$



and $\alpha_\pm(x)$ are two solutions to the equation $X(\alpha) = x$.

The main property is: WKB- asymptotic does not work, but presented functions are defined and smooth in the left (big) neighborhood of the focal point x^*

This gives

$$Q = \frac{1}{2}(S_+ + S_-), \quad \Phi = -\frac{3(S_+ - S_-)^{2/3}}{2h^{2/3}},$$

$$A_1 = -\frac{e^{\frac{-i\pi}{4}}}{\sqrt[6]{h}\sqrt{\pi}}(a_+ - a_-)\sqrt[3]{S_+ - S_-}, \quad A_2 = -\frac{\sqrt[6]{h}e^{\frac{i\pi}{4}}}{\sqrt{\pi}}\frac{(a_+ + a_-)}{\sqrt[3]{S_+ - S_-}}$$

$$\psi \approx e^{i\frac{Q(x)}{h}}\left(A_1(x,h)\mathrm{Ai}(\Phi(x,h)) + A_2(x,h)\mathrm{Ai}'(\Phi(x,h))\right)$$

(Anikin, Dobrokhotov, Nazaikinskii, Tsvetkova, 2019)

Looks like Langer-Kuzmak-Whitham type representation

$$\Psi(x) \approx \frac{\mathbf{a}(x)}{h^\beta} g(\frac{\mathbf{S}(x)}{h^\alpha})$$

(method of reference equations, Langer, Babich, Slavyanov, Kravtsov etc.)

Easy to see that presented formulas are generalised for n-D case (with "simple" caustics=folds). Note also that their derivation is based just only on the fact that the reduction of the phase (in the integral) to its normal form exists and does not require any analytical research. In a sense, we can say that in problems about describing asymptotics in the vicinity of caustics, the Lagrangian manifolds gives an effective solution to the Malgrange theorem (Weierstrass preparatory theorem).

Important remark. We know the answer via the canonical operator and just simplify it (*no matching, no crossing of caustic etc.*)
We need the Lagrangian manifold and special function (if known) with its oscillating asymptotics.

The algorithm:

1) Construction the invariant Lagrangian manifold Λ^3
(via Keplerian trajectories)

2) Construction the phase, the invariant measure, the Maslov indices, the Maslov canonical operator $\psi = K^h_{\Lambda^3} \cdot 1$

3) Simplification and global representation in the form of an Airy function of a complex argumen

Multidimensional case. Keplerian trajectories.

Coordinates: x_1, and ρ, θ are the radius and the angle θ on the plane normal to the axis x_1

The classical Hamiltonian

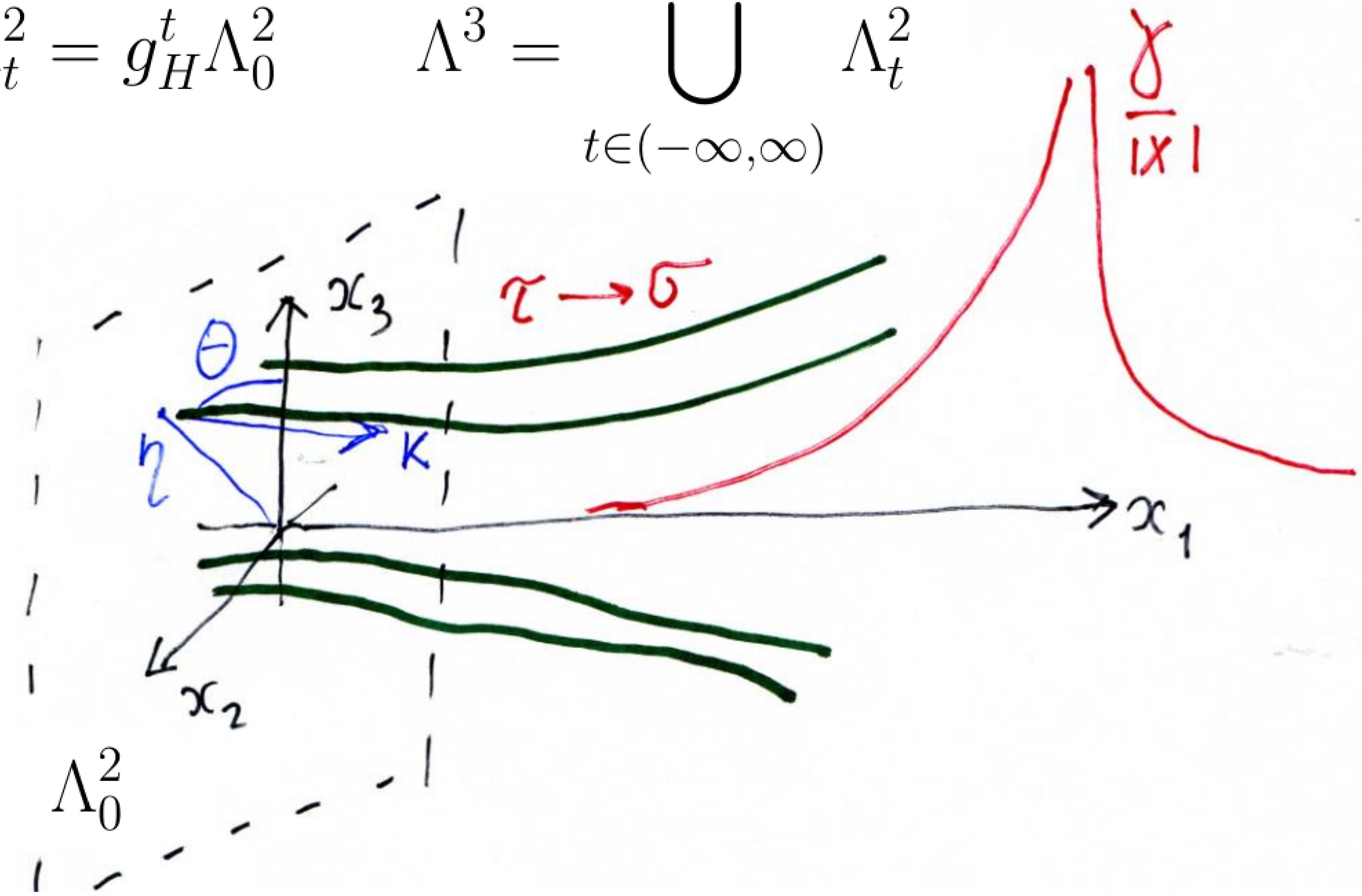
$$H = p^2 + \frac{\gamma}{|x|} = p_\rho^2 + \frac{p_\theta^2}{\rho^2} + p_1^2 + \frac{\gamma}{\sqrt{\rho^2 + x_1^2}}.$$

The Hamilton system

$$\dot{p}_\rho = 2\frac{p_\theta^2}{\rho^3} + \frac{\gamma\rho}{\sqrt{(\rho^2 + x_1^2)^3}}, \quad \dot{p}_\theta = 0, \quad \dot{p}_1 = \frac{\gamma x_1}{\sqrt{(\rho^2 + x_1^2)^3}},$$

$$\dot{\rho} = 2p_\rho, \quad \dot{\theta} = 2\frac{p_\theta^2}{\rho^2}, \quad \dot{x}_1 = 2p_1.$$

$$\Lambda_t^2 = g_H^t \Lambda_0^2 \qquad \Lambda^3 = \bigcup_{t\in(-\infty,\infty)} \Lambda_t^2$$

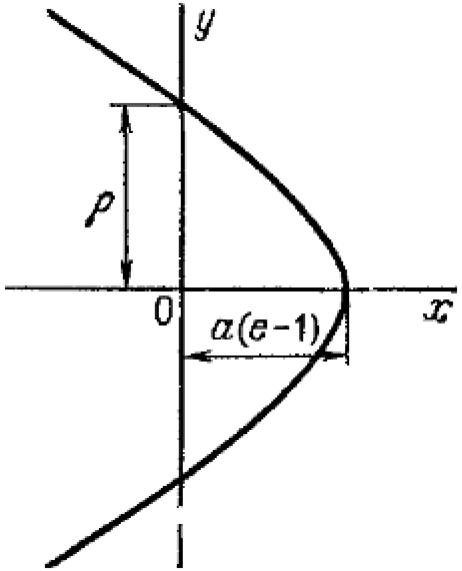


Λ_0^2

The Keplerian trajectories (see e.g. Landau, Lifshits, v.1)

$$X = -a(\cosh\xi + e_1), \quad Y = a\sqrt{e_1^2 - 1}\sinh\xi$$

$$t = q(e_1\sinh\xi + \xi), \quad q = \frac{a^{3/2}}{\sqrt{2\gamma}},$$



a, e_1–are positive constants

"Rotated trajectories"

$$X_1 = \cos\phi X - \sin\phi Y = -\cos\phi a(\cosh\xi + e_1) - \sin\phi a\sqrt{e_1^2 - 1}\sinh\xi,$$

$$R = \sin\phi X + \cos\phi Y = -\sin\phi a(\cosh\xi + e_1) + \cos\phi a\sqrt{e_1^2 - 1}\sinh\xi$$

$$t = q(e_1\sinh\xi + \xi).$$

The invariant $3-D$ Lagrangian manifold

$\Lambda^3 = \{x = X(\sigma,\eta,\theta), p = P(\sigma,\eta,\theta),\}$

$$X = \begin{pmatrix} \frac{\gamma}{2k^2}\left(\frac{\sigma\eta^2}{2} - \frac{(\sigma+1)^2}{2\sigma}\right) \\ \frac{\gamma}{2k^2}\eta(\sigma+1)\cos\theta \\ \frac{\gamma}{2k^2}\eta(\sigma+1)\sin\theta \end{pmatrix}, \quad P = \begin{pmatrix} k\frac{-\sigma+\sigma\eta^2+\frac{1}{\sigma}}{\sigma+\sigma\eta^2+\frac{1}{\sigma}+2} \\ k\frac{2\eta\sigma}{\sigma+\sigma\eta^2+\frac{1}{\sigma}+2}\cos\theta \\ k\frac{2\eta\sigma}{\sigma+\sigma\eta^2+\frac{1}{\sigma}+2}\sin\theta \end{pmatrix},$$

$$t = \frac{\gamma}{4k^3}\left(\frac{1}{2}(\sigma+\sigma\eta^2 - \frac{1}{\sigma}) + \ln\sigma + \frac{1}{2}\ln(1+\eta^2)\right), \quad H(x,p) = |p|^2 + \frac{\gamma}{|x|} = k^2$$

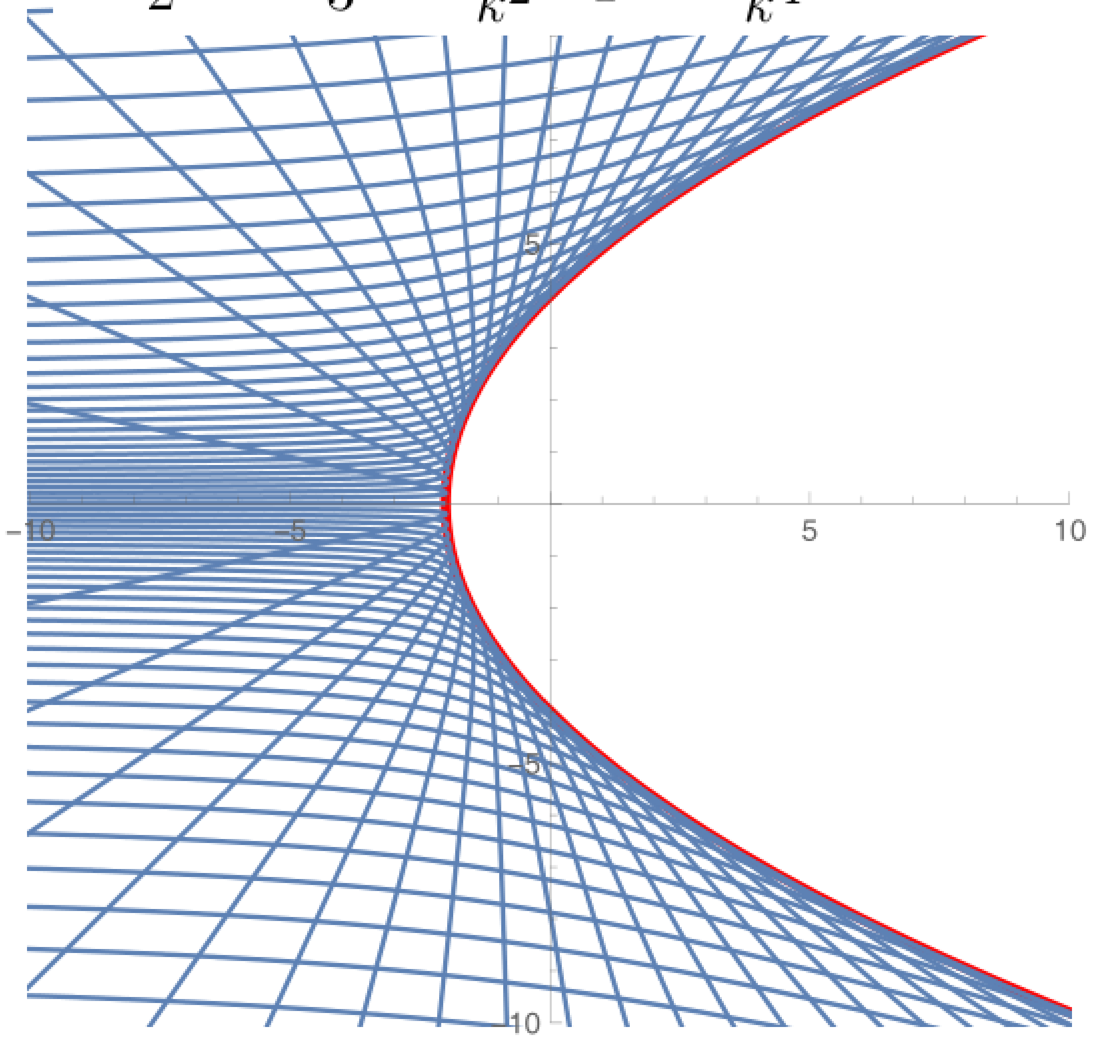
where coordinates $\sigma \in (0,\infty), \eta \in (0,\infty), \theta \in S^1$.

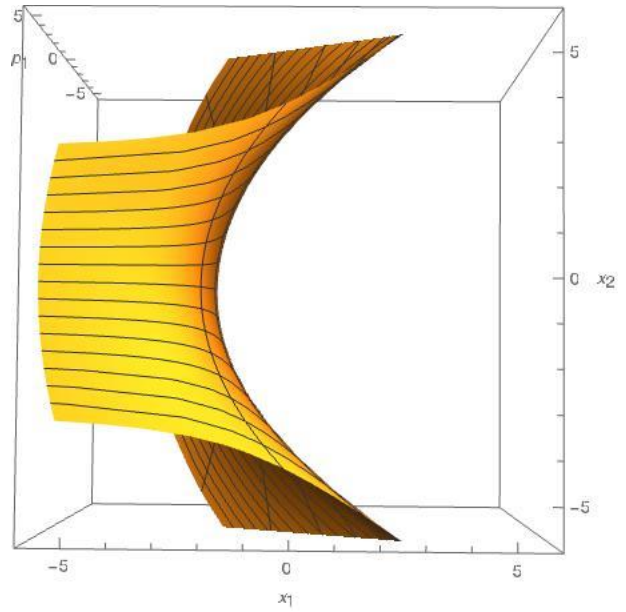
For $t \to -\infty$ $(\Leftrightarrow \sigma \to 0)$:

$$X_1 \to -\infty, \quad X_2 \to \frac{\gamma}{2k^2}\eta\cos\theta = X_2^{lim}, \quad X_3 \to \frac{\gamma}{2k^2}\eta\sin\theta = X_3^{lim}$$

the Jacobian is $J = \frac{dX_1 \wedge dX_2 \wedge dX_3}{d\mu} = 2k(1 - \sigma^2)$

The caustic is the paraboloid of revolution $\sigma = 1 \iff x_2^2 + x_3^2 = \frac{4\gamma}{k^2} x_1 + \frac{4\gamma^2}{k^4}$





Asymptotic solution $\psi_{as} = K^h_{\Lambda^3,\mu}[1]$ in regular points

$$z(x_1, x_2, x_3) = \frac{k^2}{\gamma}\left(-x_1 + \sqrt{x_1^2 + x_2^2 + x_3^2}\right) - 1.$$

And the WKB-aymptotics for $x_2^2 + x_3^2 > \frac{4\gamma}{k^2}x_1 + \frac{4\gamma^2}{k^4}$ has a form

$$\psi_{as}(x_1, x_2, x_3) = e^{\frac{ikx_1}{h}} \sum_{\pm} \frac{e^{-i\pi m_\pm/2}}{\sqrt{2k}(z^2-1)^{\frac{1}{4}}(\sqrt{z+1} \pm \sqrt{z-1})} \times$$

$$\exp\left(\frac{i}{h}\frac{\gamma}{2k}\left[-\ln(z \pm \sqrt{z^2-1}) + z \pm \sqrt{z^2-1} + 1\right]\right)$$

here $m_- = 0, m_+ = 1$.

Asymptotics in the form of the Airy function

Construct the functions:

$$z(x_1,x_2,x_3)=\frac{k^2}{\gamma}\left(-x_1+\sqrt{x_1^2+x_2^2+x_3^2}\right)-1.$$

$$\Theta(x)=\frac{1}{2}(S_+ + S_-)=\frac{k}{2}(x_1+|x|) \qquad \text{for all} \quad x,$$

$$\Psi=\frac{1}{2}(S_+ - S_-)=\frac{\gamma}{2k}\left[\sqrt{z^2-1}-\ln(z+\sqrt{z^2-1})\right]$$

in the projection of Λ^3 to $\mathbb{R}^3$,

One has $\Psi\sim\frac{\gamma}{12k}(2(z-1))^{\frac{3}{2}}$ as $z\to 1+0$,
then

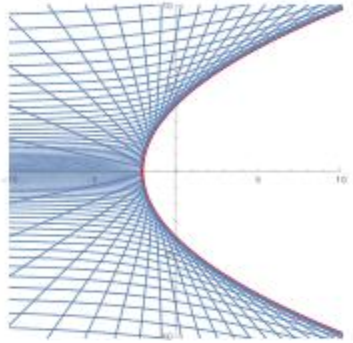
$$\psi_{as} = e^{\frac{i\Theta}{h}}\sqrt{\pi}\left[e^{-\frac{i\pi}{4}}\mathrm{Ai}\left(-\Phi(x)\right)Q_+(x) - e^{\frac{i\pi}{4}}\mathrm{Ai}'\left(-\Phi(x)\right)Q_-(x)\right],$$

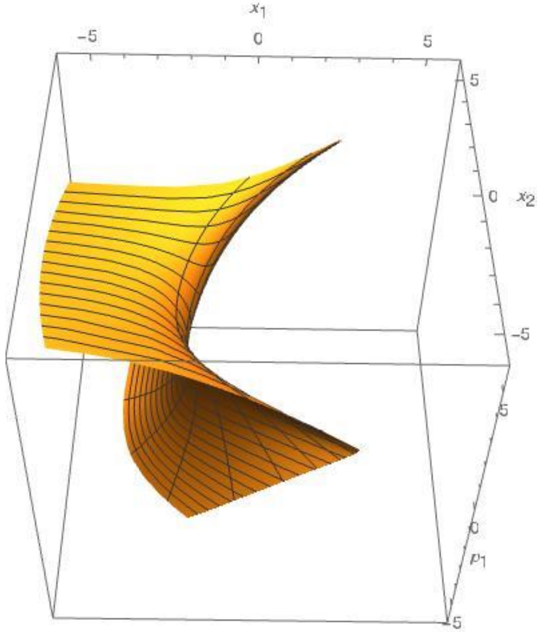
here

$$\Phi(x) = \begin{cases} \left(\frac{3\Psi(x)}{2h}\right)^{\frac{2}{3}}, \ \text{as } z(x) > 1 \\ \left(\frac{\gamma}{kh}\right)^{\frac{2}{3}} \frac{z-1}{2}, \text{as } z(x) \leq 1 \end{cases}.$$

$$Q_\pm(x) = \begin{cases} \left(\frac{1}{\sqrt{2k|\sigma_\pm^2 - 1|}} + \frac{1}{\sqrt{2k|\sigma_\mp^2 - 1|}}\right)\left(\frac{3\Psi(x)}{2h}\right)^{\frac{1}{6}}, \ \text{as } z(x) > 1 \\ \frac{1}{\sqrt[4]{2}\sqrt{k}}\left(\frac{\gamma}{kh}\right)^{\frac{1}{6}}, \text{as } z(x) \leq 1 \end{cases}$$

$$\sigma_\pm = z \pm \sqrt{z^2 - 1}$$





x_1
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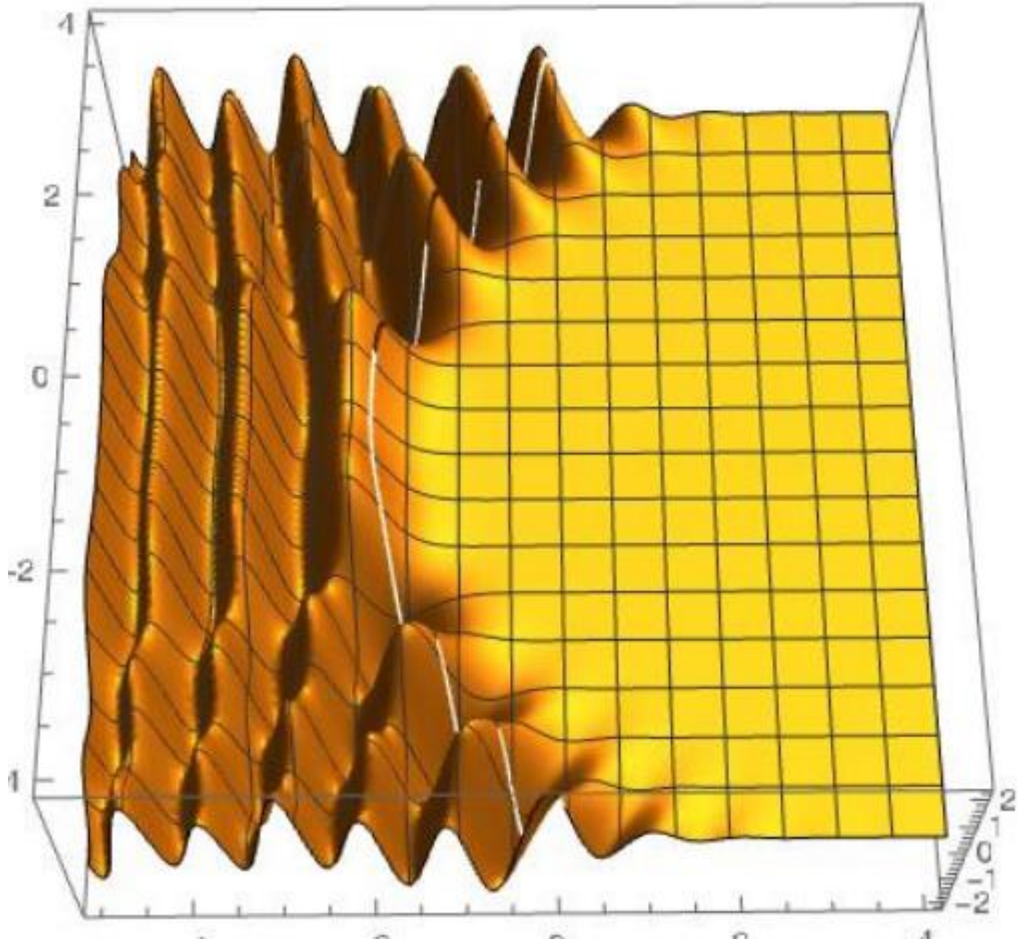
5
5
0
x_2
−5
5
0
p_1
5



2
0
-2
2
1
0
−1
−2

PROBLEM 2 (Green function type asymptotics)

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The algorithm:

1) Construction the invariant Lagrangian manifold

$$\Lambda^3_+ = g^t_H\{p^2 + \frac{\gamma}{|x_0|} = E, \quad t \geq 0\}, \qquad H = p^2 + \frac{\gamma}{|x|}$$

(via Keplerian trajectories)

2) Construction the phase, the invariant measure, the Maslov indices, the Maslov canonical operator $\psi = K^h_{\Lambda^3} \cdot 1$

3) Simplification and global representation in the form of an Airy function of a complex argument

The Lagrangian manifold via Keplerian trajectories

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{\gamma^2}{4bE^2} \begin{pmatrix} (\operatorname{ch}\xi_0 + \operatorname{ch}\beta)(\operatorname{ch}\xi + \operatorname{ch}\beta) + \operatorname{sh}\xi_0 \operatorname{sh}\xi \operatorname{sh}^2\beta \\ [\operatorname{sh}\xi_0 \operatorname{sh}\beta(\operatorname{ch}\xi + \operatorname{ch}\beta) - \operatorname{sh}\xi \operatorname{sh}\beta(\operatorname{ch}\xi_0 + \operatorname{ch}\beta)]\cos\theta \\ [\operatorname{sh}\xi_0 \operatorname{sh}\beta(\operatorname{ch}\xi + \operatorname{ch}\beta) - \operatorname{sh}\xi \operatorname{sh}\beta(\operatorname{ch}\xi_0 + \operatorname{ch}\beta)]\sin\theta \end{pmatrix}$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{\gamma}{2b\sqrt{E}} \frac{1}{\operatorname{ch}\beta \operatorname{ch}\xi + 1} \begin{pmatrix} (\operatorname{ch}\xi_0 + \operatorname{ch}\beta)\operatorname{sh}\xi + \operatorname{sh}\xi_0 \operatorname{ch}\xi \operatorname{sh}^2\beta \\ [\operatorname{sh}\xi_0 \operatorname{sh}\xi \operatorname{sh}\beta - (\operatorname{ch}\xi_0 + \operatorname{ch}\beta)\operatorname{ch}\xi \operatorname{sh}\beta]\cos\theta \\ [\operatorname{sh}\xi_0 \operatorname{sh}\xi \operatorname{sh}\beta - (\operatorname{ch}\xi_0 + \operatorname{ch}\beta)\operatorname{ch}\xi \operatorname{sh}\beta]\sin\theta \end{pmatrix}$$

$$t = q(\operatorname{ch}\beta \operatorname{sh}\xi + \xi - t_0) \qquad q = \frac{a^{\frac{3}{2}}}{\sqrt{2\gamma}},\ t_0 = \operatorname{ch}\beta \operatorname{sh}\xi_0 + \xi_0.$$

$$\operatorname{sh}\beta = -A\sin\psi,\ \operatorname{ch}\beta = \sqrt{1 + A^2\sin^2\psi}$$

$$A = \frac{2Fb\sqrt{E}}{\gamma}$$

$$\operatorname{sh}\xi_0 = \frac{A\cos\psi}{\sqrt{1 + A^2\sin^2\psi}},\ \operatorname{ch}\xi_0 = \frac{\sqrt{1 + A^2}}{\operatorname{ch}\beta}$$

Action function, measure and Jacobian on Λ_+

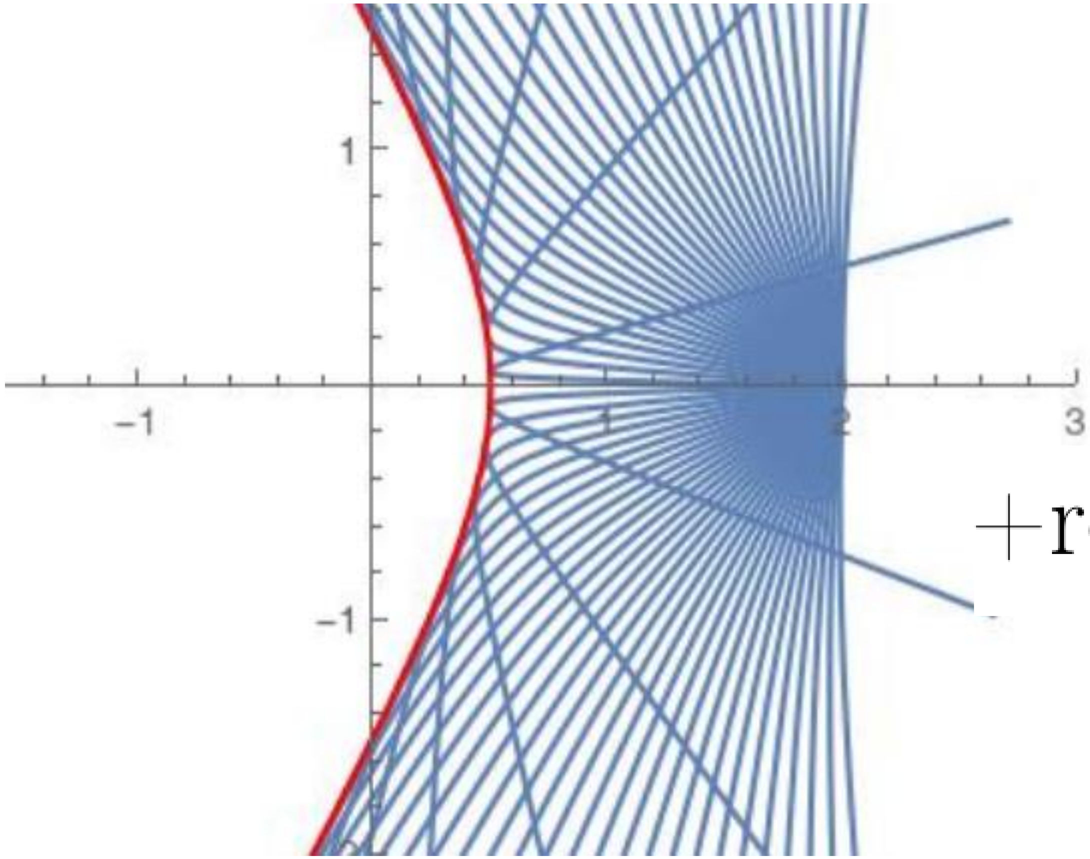
$$S(\xi,\psi)=\int_{\xi_0}^{\xi} p\,dx=\int_{\xi_0}^{\xi}\frac{1}{2t'_\xi}|y'_\xi|^2 d\xi=\frac{\gamma}{2\sqrt{E}}\left(\operatorname{ch}\beta(\operatorname{sh}\xi-\operatorname{sh}\xi_0)-(\xi-\xi_0)\right)$$

$$d\mu_+=\frac{1}{2}F\sin\psi\, d\psi\wedge d\theta\wedge dt=\frac{qF}{2}\sin\psi\left(\operatorname{ch}\beta\operatorname{ch}\xi+1\right)d\xi\wedge d\psi\wedge d\theta$$

$$J=\frac{dx_1\wedge dx_2\wedge dx_3}{d\mu_+}=\frac{F\gamma^2}{E^2}\Big(\operatorname{ch}^2\beta(\operatorname{sh}\xi-\operatorname{sh}\xi_0)^2-\operatorname{sh}^2(\xi-\xi_0)\Big)$$

The caustic = the half of a bicuspid hyperboloid

$$\frac{\left(x_1-\frac{b}{2}\right)^2}{\left(\frac{b}{2}-\frac{\gamma}{E}\right)^2}-\frac{x_2^2}{\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}-\frac{\gamma}{E}\right)^2}=1$$



$+$rotation around the axis x_1

The global uniform asymptotics via Airy function outside of some neighborhood of the point x_0 (“**Far field** ”)

$$\psi = K^h_{\Lambda_+}\widetilde{V} = \sqrt{\pi}e^{-\frac{i\pi}{4}}e^{\frac{i\Theta}{h}}\left[h^{-\frac{1}{6}}Ai\left(-\frac{\Phi}{h^{\frac{2}{3}}}\right)B_+ + i\,h^{\frac{1}{6}}Ai'\left(-\frac{\Phi}{h^{\frac{2}{3}}}\right)B_-\right],$$

Parabolic coordinates

$$\sigma = |x| - x_1,\ \eta = |x| + x_1,\ v = (|x|(1-c^2) + (1+c)^2 x_1 - 2bc,\ c = \frac{\gamma}{E}\frac{1}{b - \frac{\gamma}{E}}$$

$\Theta, \Psi, B_{\pm}$ are defined to the right of caustics ($v > 0$)

$$\Theta = \frac{1}{2}\left(S_{+} + S_{-}\right), \; \Phi = \left(\frac{3\Psi}{2}\right)^{\frac{2}{3}}, \; \Psi = \frac{1}{2}\left(S_{+} - S_{-}\right)$$

$$B_{+} = \left(\frac{3\Psi}{2}\right)^{\frac{1}{6}}\left(\frac{\widetilde{V}_{+}}{\sqrt{|J_{+}|}} + \frac{\widetilde{V}_{-}}{\sqrt{|J_{-}|}}\right), \; B_{-} = \left(\frac{3\Psi}{2}\right)^{\frac{1}{6}}\left(\frac{\widetilde{V}_{+}}{\sqrt{|J_{+}|}} - \frac{\widetilde{V}_{-}}{\sqrt{|J_{-}|}}\right)$$

$$J_{\pm} = \mp 4F\frac{\sqrt{\eta\, v}}{c^2(1+c)}\left(\left((1+c)\sqrt{\eta} \pm \sqrt{v}\right)^2 - 2bc^2\right).$$

$$S_{\pm} = \frac{\gamma}{2\sqrt{E}}\left[\frac{1}{bc^2}\left(\sqrt{\eta}(1+c) \mp \sqrt{v}\right)\sqrt{-2bc^2 + \left((1+c)\sqrt{\eta} \pm \sqrt{v}\right)^2}\right.$$

$$\left. -\operatorname{arcsh}\left(\frac{1}{bc^2}\left(\sqrt{\eta}(1+c) \pm \sqrt{v}\right)\sqrt{-2bc^2 + \left((1+c)\sqrt{\eta} \pm \sqrt{v}\right)^2}\right)\right]$$

To the left of caustics ($v < 0$)

$$\Psi = \frac{2\gamma}{3\sqrt{E}} \left(-2bc^2 + (1+c)^2\eta\right)^{-\frac{3}{2}} v^{\frac{3}{2}} + O(v^{\frac{5}{2}})$$

$$\Phi = \frac{\gamma^{\frac{2}{3}}}{E^{\frac{1}{3}}} \left(-2bc^2 + (1+c)^2\eta\right)^{-1} v + O(v^2)$$

$$B_+ \to \frac{\gamma^{\frac{7}{6}} E^{\frac{5}{12}} b^{\frac{3}{4}}}{(Eb-\gamma)^{\frac{7}{4}}} \left(-2bc^2 + (1+c)^2\eta\right)^{-\frac{3}{4}} \eta^{-\frac{1}{4}} \widetilde{V}_0$$

$$B_- \to -\frac{E^{\frac{19}{12}} b^{\frac{7}{4}} \gamma^{\frac{5}{6}}}{(Eb-\gamma)^{\frac{11}{4}}} (-2bc^2 + (1+c)^2\eta)^{-\frac{5}{4}} \eta^{\frac{1}{4}} \widetilde{V}_0,$$

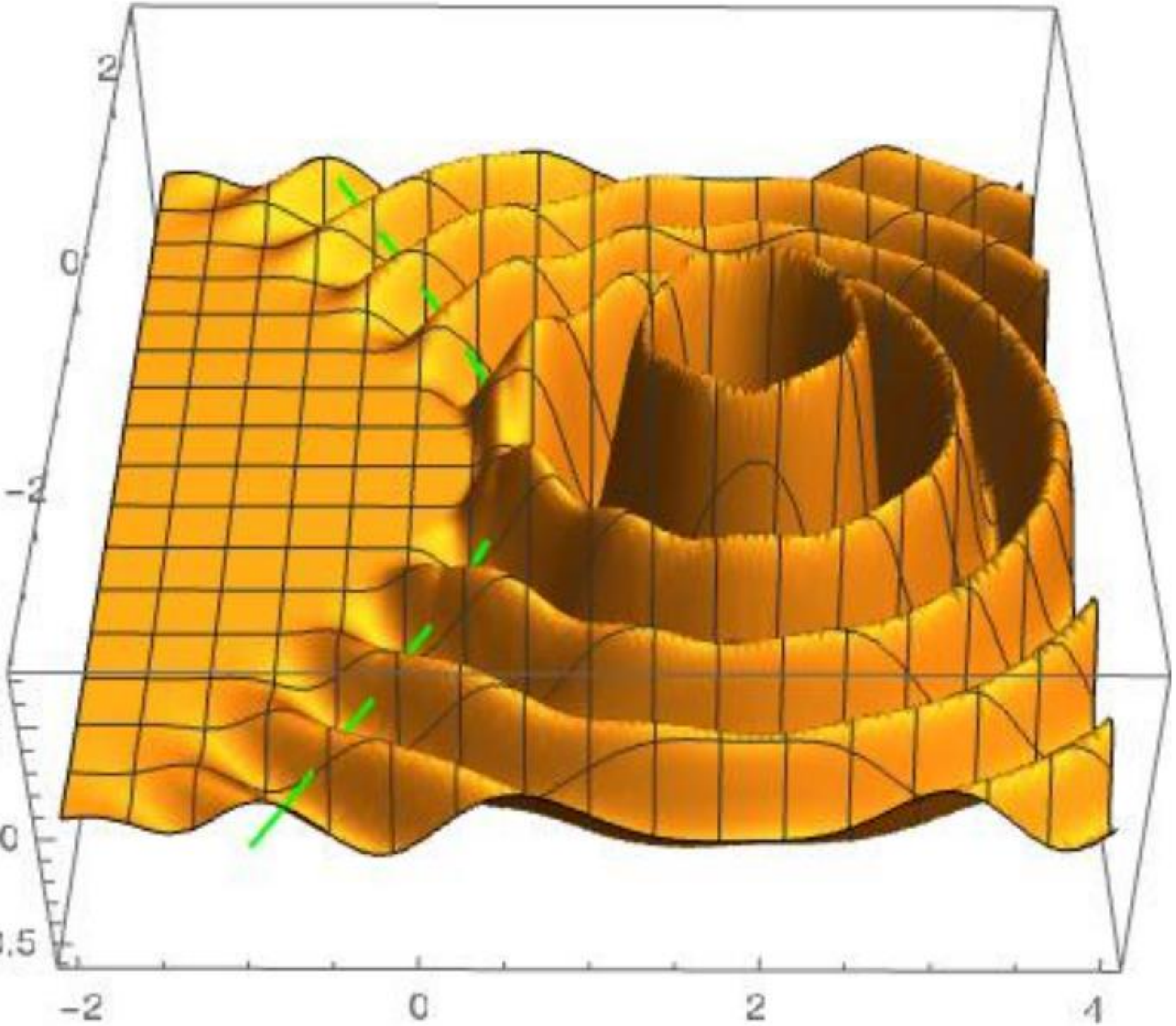
$$\widetilde{V}_0 = \widetilde{V}\left(\sqrt{E - \tfrac{\gamma}{b}}\cos\psi, \sqrt{E - \tfrac{\gamma}{b}}\sin\psi\cos\theta, \sqrt{E - \tfrac{\gamma}{b}}\sin\psi\sin\theta\right)|_{v=0}$$

$$\Phi = \begin{cases} \left(\frac{3}{4}\left(S_+ - S_-\right)\right)^{\frac{2}{3}} \text{ for } v > 0, \\ \gamma^{\frac{2}{3}} E^{-\frac{1}{3}} \left(-2bc^2 + (1+c)^2\eta\right)^{-1} v \text{ for } -\varepsilon < v \le 0 \end{cases}$$

$$B_+ = \begin{cases} \left(\frac{3\Psi}{2}\right)^{\frac{1}{6}} \left(\frac{\widetilde{V}_+}{\sqrt{|J_+|}} + \frac{\widetilde{V}_-}{\sqrt{|J_-|}} \right) \text{ for } v > 0, \\ \frac{\gamma^{\frac{7}{6}} E^{\frac{5}{12}} b^{\frac{3}{4}}}{(Eb-\gamma)^{\frac{7}{4}}} \left(-2bc^2 + (1+c)^2\eta\right)^{-\frac{3}{4}} \eta^{-\frac{1}{4}} \widetilde{V}_0 \text{ for } -\varepsilon < v \leq 0 \end{cases}$$

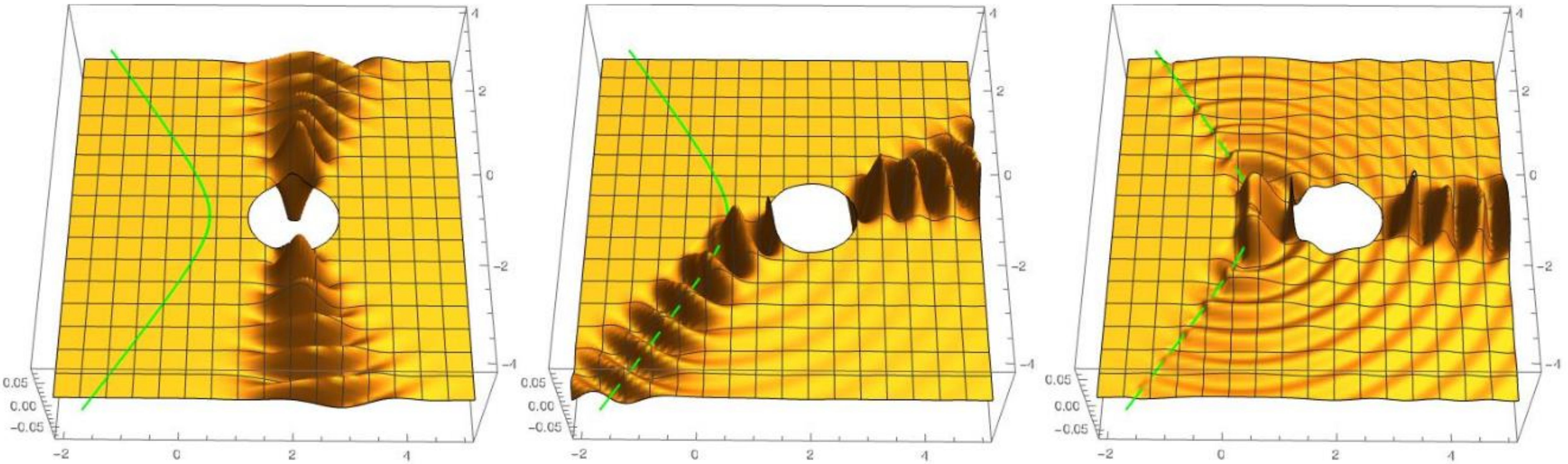
$$B_- = \begin{cases} \left(\frac{3\Psi}{2}\right)^{\frac{1}{6}} \left(\frac{\widetilde{V}_+}{\sqrt{|J_+|}} - \frac{\widetilde{V}_-}{\sqrt{|J_-|}} \right) \text{ for } v > 0, \\ -\frac{E^{\frac{19}{12}} b^{\frac{7}{4}} \gamma^{\frac{5}{6}}}{(Eb-\gamma)^{\frac{11}{4}}} (-2bc^2 + (1+c)^2\eta)^{-\frac{5}{4}} \eta^{\frac{1}{4}} \widetilde{V}_0 \text{for } -\varepsilon < v \leq 0. \end{cases}$$

Example $V = e^{-|x|^2},\ b = 2,\ E = 2,\ \gamma = 1, x_3 = 0.$



Example

$$V(x) = e^{-\left(\frac{x_1 \cos\alpha + x_2 \sin\alpha}{a_1}\right)^2 - \left(\frac{-x_1 \sin\alpha + x_2 \cos\alpha}{a_2}\right)^2 - x_3^2}, \; a_1/a_2 = 1/3, \; \alpha \in [0, 2\pi]$$



THANK YOU FOR YOUR ATTENTION!

Спасибо за внимание!