In search of a non-Archimedean Schwartz space

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Desired properties

Finding a good substitute for the space of Schwartz functions in a non-Archimedean setting has proved to be a challenge. Such a function space should ideally satisfy the following requirements[1]

1. It should be invariant under the Fourier transform $\mathcal{F}$.
2. It should be invariant under the multiplication operator $(Qf)(x) = |x|_p f(x)$.
3. It should contain the characteristic function of the integers $\mathbf{1}_{\mathbb{Z}_p}$. This is because $\mathbf{1}_{\mathbb{Z}_p}$ in certain contexts plays the role of the Gauss function (it is fixed under the Fourier transform $\mathcal{F}$), and because it is also a basic building block for functions on $\mathbb{Q}_p$, .

[1] In the first part of this talk we confine our discussion to $\mathbb{Q}_p$. With appropriate adjustments, the results are valid for any local field.

Schwartz-Bruhat functions

A much used space of test functions is the Schwartz-Bruhat space of locally constant functions with compact support. This space is useful in many respects and satisfies conditions 1 and 3 above, but fails to satisfy condition 2. This limits its usefulness in connection with differential equations.

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An unsuccessful attempt

At the Seventh International Conference on p-Adic Mathematical Physics and its Applications, Covilhã, Portugal, 2019, we presented the following candidate for a non-Archimedean Schwartz space (over $\mathbb{Q}_p^n$):

$$\mathfrak{X}(\mathbb{Q}_p^n) := \bigcap_{m \in \mathbb{N}_0} \mathbf{S}_{0,m}(\mathbb{Q}_p^n),$$

where the $\mathbf{S}_{0,m}(\mathbb{Q}_p^n)$, $m = 1, 2, 3, \cdots$ are weighted Feichtinger algebras over $\mathbb{Q}_p^n$. This space has many nice properties; in particular, it is a nuclear Fréchet space, and the Schwartz-Bruhat functions are densely contained in it. However, it suffers from the same defects as the Schwartz-Bruhat space: It satisfies conditions 1 and 3 above, but not condition 2.

Other attempts

Among other attempts to remedy the shortcomings of the Schwartz-Bruhat space we mention an article by Zúñiga-Galindo [ZnG17]. Here he introduced a space of test functions which is invariant under the Vladimirov operator $\mathsf{P} = \mathcal{F}\mathsf{Q}\mathcal{F}^*$. However, it is not invariant under the Fourier transform, a circumstance which in some situations can be a problem.

"No-go" theorem

The above unsuccessful attempts made us suspect that there is a "no-go" theorem here. And, indeed, there is.
If we consider $\mathbb{Q}_p$ as a locally compact group with its additive structure, it turns out that it is impossible to find a non-trivial space which satisfies all the requirements listed on slide 2.

Theorem ("No-go")

If $X \subset \mathbf{L}^2(\mathbb{Q}_p)$ is a vector space of functions that is invariant under Q *and $\mathcal{F}$, then X can not contain a function of the form $\mathbf{1}_{p^k\mathbb{Z}_p}$ for any $k \in \mathbb{Z}$. Furthermore, if in addition X is invariant under translation, then X can not contain a function of the form $\mathbf{1}_{x+p^k\mathbb{Z}_p}$ for any $x \in \mathbb{Q}_p$, $k \in \mathbb{Z}$.*

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Proof.

Assume towards a contradiction that $\chi_{p^k\mathbb{Z}_p}$ is an element in X. In [VVZ94, Example 9, p.104] we can find the computation that shows

$$\mathcal{F}\mathrm{Q}\chi_{p^k\mathbb{Z}_p}(t) = (p+1)^{-1}\begin{cases} p^{1-2k} & \text{if } \ |t|_p \leq p^k, \\ -p^2\,|t|_p^{-2} & \text{if } \ |t|_p > p^k, \end{cases} \quad t \in \mathbb{Q}_p.$$

Multiplying this by $|t|_p^2$ shows that $\mathrm{Q}^2\mathcal{F}\mathrm{Q}\chi_{p^k\mathbb{Z}_p}$ is a function with a constant non-zero value on the unbounded set $\{t \in \mathbb{Q}_p \,:\, |t|_p > p^k\}$. We conclude that $\mathrm{Q}^2\mathcal{F}\mathrm{Q}\chi_{p^k\mathbb{Z}_p} \notin \mathbf{L}_2(\mathbb{Q}_p)$. This contradicts the supposed invariance of $X \subseteq \mathbf{L}_2(\mathbb{Q}_p)$ under $\mathcal{F}$ and Q. The furthermore part follows easily. $\square$

Relaxing the requirements

Ways to get around the no-go theorem include giving up invariance under Q, and replace Q with an operator which behaves more favorably around the origin. This has been explored by Haran [Har93] and later by Bechata [Bec04]. They replace

$$\mathrm{Q} : f(x) \mapsto |x| f(x)$$

by the operator

$$f(x) \mapsto \max(1, |x|) f(x),$$

and with the aid of the latter they define a Schwartz space which works well in their setting. However, we are unsure about the physical significance of this operator. But then again, the same must be said about the operators we are about to introduce in a multiplicative setting!

The multiplicative group $\mathbb{Q}_p^\times$

Because of the "no-go" theorem above we have turned our attention to the multiplicative group $\mathbb{Q}_p^\times = \mathbb{Q}_p \setminus \{0\}$. It is still locally compact, but no longer self-dual. Our candidate for a space of test functions will be constructed from a family of weighted modulation spaces (Feichtinger algebras). Although some standard operators – such as the Vladimirov operator – do not have an obvious interpretation in this setting, we define a family of other operators which could be of physical and/or mathematical interest.

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The additive group of a local field K

From now on we will work in a general local field K and follow the notation in Taibleson's book [Tai75].
$\mathfrak{D} = \{x \in K : |x| \leq 1\}$: ring of integers in K.
μ_K: normalized Haar measure on K, $\mu_K(\mathfrak{D}) = 1$.
$\mathfrak{P}$: the unique maximal ideal in $\mathfrak{D}$, $\mathfrak{P} = \{x \in K : |x| < 1\}$.
q: number of elements in the finite field $\mathfrak{D}/\mathfrak{P}$. We have $q = p^f$, a power of a prime p.
$\mathfrak{p}$: fixed element of maximal absolute value in $\mathfrak{P}$. We have $|\mathfrak{p}| = q^{-1}$, and $\mathfrak{P} = \mathfrak{p}\mathfrak{D}$.

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The multiplicative group $K^\times = K \setminus \{0\}$

$\mathfrak{O}^* = \{x \in K \,:\, |x| = 1\}$: the group of units in $\mathfrak{O}$. It is a compact open subgroup of $K^\times$.
Any non-zero $x \in K$ can be written uniquely as a product of the form $x = \mathfrak{p}^k\, x'$ for some $k \in \mathbb{Z}$ and $x' \in \mathfrak{O}^*$. We have $|x| = q^{-k}$.

The valuation $v : K \to \mathbb{Z} \cup \{\infty\}$ is given by $v(x) = k$, with $v(0) = \infty$. For non-zero $x \in K$ one finds $v(x) = -\log_q |x|$.

The representation $x = \mathfrak{p}^k\, x'$ gives rise to an isomorphism of topological groups $K^\times \cong \mathbb{Z} \times \mathfrak{O}^*$ by

$$(k, x') \in \mathbb{Z} \times \mathfrak{O}^* \to \mathfrak{p}^k x' \in K^\times.$$

The normalization of the Haar measure on $(K, +)$ yields $\mu_K(\mathfrak{O}^*) = 1 - q^{-1}$ and $\mu_K(\mathfrak{P}) = q^{-1}$.

The Haar measure on $K^\times$ and $\mathfrak{D}^*$

Up to scaling by a positive constant the Haar measure on $K^\times$, $\mu_{K^\times}$, is given by μ_K divided by $|\cdot|$. We normalize it in the canonical way so that $\mu_{K^\times}(\mathfrak{D}^*) = 1$, and thus, for suitable $f : K^\times \to \mathbb{C}$,

$$\int_{K^\times} f(x)\, d\mu_{K^\times}(x) \overset{\text{(def)}}{=} \frac{q}{q-1} \int_{K^\times} f(x)\, |x|^{-1}\, d\mu_K(x) \,.$$

The Haar measure $\mu_{\mathfrak{D}^*}$ on $\mathfrak{D}^*$ is the restriction of $\mu_{K^\times}$ to $\mathfrak{D}^*$,

$$\int_{\mathfrak{D}^*} f(x)\, d\mu_{\mathfrak{D}^*}(x) \overset{\text{(def)}}{=} \int_{\mathfrak{D}^*} f(x)\, d\mu_{K^\times}(x) = \frac{q}{q-1} \int_{\mathfrak{D}^*} f(x)\, d\mu_K(x) \,.$$

We can equally well describe the measure on $K^\times$ through the mentioned isomorphism $K^\times \cong \mathbb{Z} \times \mathfrak{D}^*$ via the representation $x = \mathfrak{p}^k x'$, $k \in \mathbb{Z}$, $x' \in \mathfrak{D}^*$ and the product measure of $\mu_{\mathbb{Z}}$ (the counting measure) and $\mu_{\mathfrak{D}^*}$ (as just defined):

$$\int_{K^\times} f(x)\, d\mu_{K^\times}(x) = \sum_{k \in \mathbb{Z}} \int_{\mathfrak{D}^*} f(\mathfrak{p}^k x')\, d\mu_{\mathfrak{D}^*}(x') \,.$$

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The dual group of $\mathfrak{D}^*$

Since $\mathfrak{D}^*$ is compact, its dual group $\widehat{\mathfrak{D}^*}$ is discrete. Set $A_0 = \mathfrak{D}^*$ and $A_n = 1 + \mathfrak{p}^n\mathfrak{D}$, $n \in \mathbb{N}$. Then $A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq \ldots$, and the $(A_n)_{n\geq 0}$ form a fundamental system of neighborhoods of 1 in $K^\times$. Let $\xi' \in \widehat{\mathfrak{D}^*}$. Since ξ' is continuous, it attains the constant value 1 on A_n for some $n \in \{0, 1, 2, \ldots\}$.
For each $\xi' \in \widehat{\mathfrak{D}^*}$ we define its *degree*, $\deg(\xi')$, to be the smallest integer n for which this holds. We can thus define a mapping $\deg : \widehat{\mathfrak{D}^*} \to \mathbb{N}_0$. Note that $\deg(\xi'_1 \cdot \xi'_2) = \max(\deg(\xi'_1), \deg(\xi'_2))$. One finds

$$
\begin{aligned}
&|\{\xi' \in \widehat{\mathfrak{D}^*} \,:\, \deg(\xi') = 0\}| = 1, \\
&|\{\xi' \in \widehat{\mathfrak{D}^*} \,:\, \deg(\xi') = 1\}| = q - 2, \\
&|\{\xi' \in \widehat{\mathfrak{D}^*} \,:\, \deg(\xi') = k\}| = q^{k-2}\,(q-1)^2\,, \quad k > 1,\ k \in \mathbb{N}\,.
\end{aligned}
$$

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Fourier analysis on $\mathfrak{D}^*$ and $\widehat{\mathfrak{D}^*}$ I

For an absolutely integrable function on the compact group $\mathfrak{D}^*$ its Fourier series, indexed by the discrete group $\widehat{\mathfrak{D}^*}$, is a sequence that converges to zero at infinity computed via the Fourier transform $\mathcal{F} : \mathbf{L}^1(\mathfrak{D}^*) \to \mathbf{c}_0(\widehat{\mathfrak{D}^*})$, for functions $f \in \mathbf{L}^1(\mathfrak{D}^*)$,

$$\begin{aligned}(\mathcal{F}f)(\xi') &= \int_{\mathfrak{D}^*} f(x')\,\overline{\xi'(x')}\,d\mu_{\mathfrak{D}^*}(x') \\ &= \frac{q}{q-1}\int_{\mathfrak{D}^*} f(x')\,\overline{\xi'(x')}\,d\mu_K(x')\ , \quad \xi' \in \widehat{\mathfrak{D}^*}.\end{aligned}$$

Similarly, to every sequence $c \in \ell^1(\widehat{\mathfrak{D}^*})$ we associate a continuous function on $\mathfrak{D}^*$ through the inverse Fourier transform $\mathcal{F}^{-1} : \ell^1(\widehat{\mathfrak{D}^*}) \to \mathbf{C}(\mathfrak{D}^*)$,

$$(\mathcal{F}^{-1}c)(x') = \sum_{\xi' \in \widehat{\mathfrak{D}^*}} c(\xi')\,\xi'(x'), \quad x' \in \mathfrak{D}^*.$$

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Fourier analysis on $\mathfrak{D}^*$ and $\widehat{\mathfrak{D}^*}$ II

If $f \in \mathbf{L}^1(\mathfrak{D}^*)$ is continuous and such that the sequence $\{\mathcal{F}f(\xi')\}_{\xi' \in \widehat{\mathfrak{D}^*}}$ belongs to $\ell^1(\widehat{\mathfrak{D}^*})$, then we can recover f pointwise via

$$f(x') = \sum_{\xi' \in \widehat{\mathfrak{D}^*}} \mathcal{F}f(\xi')\,\overline{\xi'(x')}\,, \quad x' \in \mathfrak{D}^*\,.$$

As is true for any compact abelian group, and so also for $\mathfrak{D}^*$, the elements of the dual group, i.e., the functions $\{\mathfrak{D}^* \to \mathbb{C},\ x' \mapsto \xi'(x')\}_{\xi' \in \widehat{\mathfrak{D}^*}}$, form an orthonormal basis for $\mathbf{L}^2(\mathfrak{D}^*)$.

The dual group of $K^\times$

The isomorphism of $K^\times$ with $\mathbb{Z} \times \mathfrak{D}^*$ implies that the dual group $\widehat{K^\times}$ of $K^\times$ is isomorphic to $\mathbb{T} \times \widehat{\mathfrak{D}^*}$. We make use of this isomorphism and think of elements $\xi \in \widehat{K^\times}$ as a pair $\xi = (z, \xi')$, where $z \in \mathbb{T}$ and $\xi' \in \widehat{\mathfrak{D}^*}$. We follow [Tai75] and abuse notation and write $\xi = z\,\xi'$. We use multiplication to indicate the group operation of $\widehat{K^\times}$ and $\widehat{\mathfrak{D}^*}$, so that for $\xi_1 = z_1\,\xi_1'$ and $\xi_2 = z_2\,\xi_2'$ we have $\xi_1\xi_2 = z_1\,z_2\,\xi_1'\,\xi_2'$. In particular, this gives us the possibility notationally to understand an element $z \in \mathbb{T}$ as an element in $\widehat{K^\times}$ simply by writing $z \in \widehat{K^\times}$, where we understand $z = z\,\mathrm{e}_{\widehat{\mathfrak{D}^*}}$ and $\mathrm{e}_{\widehat{\mathfrak{D}^*}}$ is the unit of $\widehat{\mathfrak{D}^*}$. The action of an element $\xi = z\,\xi' \in \widehat{K^\times}$ on an $x = \mathfrak{p}^k x' \in K^\times$ is given by $\xi(x) = z^k\,\xi'(x') = |x|^{-i\arg(z)/\ln(q)}\,\xi'(x')$, where $\xi'(x')$ is the action of an element $\xi' \in \widehat{\mathfrak{D}^*}$ on an $x' \in \mathfrak{D}^*$

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The Haar measure on $\widehat{K^\times}$

We equip $\widehat{K^\times}$ with the product measure of the normalized measure on $\mathbb{T}$ and the counting measure on $\widehat{\mathfrak{D}^*}$, so that, for suitable functions $g : \widehat{K^\times} \to \mathbb{C}$,

$$\int_{\widehat{K^\times}} g(\xi)\, d\mu_{\widehat{K^\times}}(\xi) \overset{\text{(def)}}{=} \sum_{\xi' \in \widehat{\mathfrak{D}^*}} \int_{\mathbb{T}} g(z\,\xi')\, d\mu_{\mathbb{T}}(z)$$

$$= \sum_{\xi' \in \widehat{\mathfrak{D}^*}} \int_0^1 g(e^{2\pi i\theta}\,\xi')\, d\mu_{\mathcal{L}}(\theta)$$

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Fourier analysis on $K^\times$ and $\widehat{K^\times}$ I

For functions on $K^\times$ the Fourier transform acts as a continuous operator from $\mathbf{L}^1(K^\times)$ into $\mathbf{C}_0(\widehat{K^\times})$, so that for $f \in \mathbf{L}^1(K^\times)$

$$
\begin{aligned}
(\mathcal{F}f)(\xi) = (\mathcal{F}f)(z\,\xi') &= \int_{K^\times} f(x)\,\overline{\xi(x)}\,d\mu_{K^\times}(x) \\
&= \int_{K^\times} f(x)\,\overline{\xi'(x')}\,|x|^{i\arg(z)/\ln(q)-1}\,d\mu_{K^\times}(x) \\
&(\ x = \mathfrak{p}^k x') \\
&= \frac{q}{q-1}\sum_{k\in\mathbb{Z}} q^{-ik\theta}\int_{\mathfrak{D}^*} f(\mathfrak{p}^k x')\,\overline{\xi'(x')}\,d\mu_K(x'), \quad \xi = (\theta, \xi')
\end{aligned}
$$

Fourier analysis on $K^\times$ and $\widehat{K^\times}$ II

Similarly, for functions on $\widehat{K^\times}$ the inverse Fourier transform maps continuously from $\mathbf{L}^1(\widehat{K^\times})$ into $\mathbf{C}_0(K^\times)$, for $g \in \mathbf{L}^1(\widehat{K^\times})$

$$
\begin{aligned}
(\mathcal{F}^{-1}g)(x) = (\mathcal{F}^{-1}g)(\mathfrak{p}^k x') &= \int_{\widehat{K^\times}} g(\xi)\,\xi(x)\,d\mu_{\widehat{K^\times}}(\xi) \\
&= \frac{\ln(q)}{2\pi} \sum_{\xi' \in \widehat{\mathfrak{D}^*}} \xi'(x') \int_0^{\frac{2\pi}{\ln(q)}} g(\theta, \xi')\,|x|^{-i\theta}\,d\mu_{\mathbb{R}}(\theta)
\end{aligned}
$$

The measures $\mu_{K^\times}$ and $\mu_{\widehat{K^\times}}$ are normalized such that the Fourier inversion formula holds. That is, if $f \in \mathbf{L}^1(K^\times)$ is continuous and such that $\mathcal{F}f$ belong to $\mathbf{L}^1(\widehat{K^\times})$, then $\mathcal{F}^{-1}\mathcal{F}f = f$ pointwise and similarly for $g \in \mathbf{L}^1(\widehat{K^\times})$. The Fourier transform can be extended to a unitary operator between $\mathbf{L}^2(K^\times)$ and $\mathbf{L}^2(\widehat{K^\times})$ in the usual way.

Translation and modulation on $K^\times$ and $\widehat{K^\times}$

For $t \in K^\times$ and $\xi = z\xi' \in \widehat{K^\times}$ we define the translation operator T_t and modulation operator E_ξ that act on functions $f : K^\times \to \mathbb{C}$ through

$$T_t f(x) = f(x\,t^{-1})\ , \quad E_\xi f(x) = \xi(x)\, f(x) = |x|^{-i\theta}\, \xi'(x')\, f(x),$$

$x = \mathfrak{p}^k x'$, $k \in \mathbb{Z}$, $x' \in \mathfrak{D}^*$. Similarly, for $\tau \in \widehat{K^\times}$ and $x \in K^\times$ the translation operator T_τ and modulation operator E_x act on functions $g : \widehat{K^\times} \to \mathbb{C}$ through

$$T_\tau g(\xi) = g(\xi\tau^{-1})\ , \quad E_x g(\xi) = \xi(x)\, g(\xi) = |x|^{-i\theta}\, \xi'(x')\, g(\xi).$$

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If we let $\mathcal{F}$ be the Fourier transform from $\mathbf{L}^1(K^\times)$ to $\mathbf{C}_0(\widehat{K^\times})$ and $\mathcal{F}^{-1}$ be the inverse Fourier transform from $\mathbf{L}^1(\widehat{K^\times})$ to $\mathbf{C}_0(K^\times)$ observe that

$$\mathcal{F}\, E_\xi\, T_x = T_\xi\, E_{x^{-1}}\, \mathcal{F} \;\text{ and }\; \mathcal{F}^{-1}\, E_x\, T_\xi = T_{x^{-1}}\, E_\xi\, \mathcal{F}^{-1}$$
$$\text{for }\; x \in K^\times, \xi \in \widehat{K^\times}\,.$$

Other operators to be studied in the multiplicative setting

Although we are unsure of their physical interpretation, it might be of interset to study the following operators in the multiplicative setting. For functions $f : K^\times \to \mathbb{C}$

$$f(x) \mapsto v(x)^\alpha f(x)\,,\ x \in K^\times\,,\ \alpha > 0\,, \tag{1}$$

$$\text{and}\ \ f(x) \mapsto |x|^\alpha f(x)\,,\ \alpha \in \mathbb{R}\,, \tag{2}$$

and for functions $g : \widehat{K^\times} \to \mathbb{C}$ operators such as

$$g(\xi) \mapsto \deg(\xi)^\alpha g(\xi)\,,\ \xi \in \widehat{K^\times}\,,\ \alpha > 0\,, \tag{3}$$

$$\text{and}\ \ g(\xi) \mapsto e^{\alpha \deg(\xi)} g(\xi)\,,\ \alpha \in \mathbb{R}\,, \tag{4}$$

where $v : K^\times \to \mathbb{Z}$ is the valuation on the field K, $|\cdot| : K \to \mathbb{R}_0^+$ is its absolute value and $\deg : \widehat{K^\times} \to \mathbb{N}_0$ yields <u>the degree of ramification</u> of a character $\xi \in \widehat{K^\times}$.[2]

[2] By the expression $\deg(\xi)$ we mean the composite mapping

$$\xi = (z, \xi') \in \widehat{K^\times} \mapsto \xi' \in \widehat{\mathfrak{D}^*} \mapsto \deg(\xi').$$

But in order to study these operators, we need to define a suitable space of test functions. This will be discussed in the following slides.

The weighted Feichtinger algebra

The construction of weighted Feichtinger algebras for locally compact abelian groups that factor into a discrete and a compact group such as $K^\times$ and $\widehat{K^\times}$ is straight-forward. On a discrete group a weighted Feichtinger algebra coincides with a weighted ℓ^1-space. Similarly, a function on a compact abelian group belongs to the weighted Feichtinger algebra exactly if its Fourier series satisfies a weighted ℓ^1-summability condition.

We combine these requirements to a weighted summability condition for a function $f : K^\times \to \mathbb{C}$ as follows: For each $k \in \mathbb{Z}$ consider the restriction f_k of f to the coset $p^k\mathfrak{D}^*$ of $\mathfrak{D}^*$ in $K^\times$. We can think of f_k as a function on $\mathfrak{D}^*$ and thus consider its Fourier series $\mathcal{F}f_k : \widehat{\mathfrak{D}^*} \to \mathbb{C}$. The function f belongs to the (weighted) Feichtinger algebra on $K^\times$ exactly if the combined (weighted) sum of *all* Fourier coefficients of *all* the functions f_k, $k \in \mathbb{Z}$ is absolutely convergent,

$$\sum_{k\in\mathbb{Z}} \sum_{\xi'\in\widehat{\mathfrak{D}^*}} |\big(\mathcal{F}f_k\big)(\xi')|\, w(k,\xi') < \infty,$$

where $w : \mathbb{Z} \times \widehat{\mathfrak{D}^*} \to [1,\infty)$ is some weight function.

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Definition

For a given weight-function $w : \mathbb{Z} \times \widehat{\mathfrak{D}^*} \to [1, \infty)$ the w-weighted Feichtinger algebra on $K^\times$, $\mathbf{S}_{0,w}(K^\times)$, consists exactly of those functions f in $\mathbf{L}^2(K^\times)$ that satisfy

$$\sum_{k\in\mathbb{Z}} \sum_{\xi'\in\widehat{\mathfrak{D}^*}} |\langle f, E_{\xi'} T_{\mathfrak{p}^k} \chi_{\mathfrak{D}^*} \rangle| \, w(k, \xi') < \infty \, .$$

The construction is mimicked in the setting of $\widehat{K^\times}$.

Definition

For a given weight-function $w : \mathbb{Z} \times \widehat{\mathfrak{D}^*} \to [1, \infty)$ the w-weighted Feichtinger algebra over $\widehat{K^\times}$, $\mathbf{S}_{0,w}(\widehat{K^\times})$, consists exactly of those functions f in $\mathbf{L}^2(\widehat{K^\times})$ that satisfy

$$\sum_{k\in\mathbb{Z}} \sum_{\xi'\in\widehat{\mathfrak{D}^*}} |\langle f, E_{\mathfrak{p}^k} T_{\xi'} \chi_{\mathbb{T}} \rangle| \, w(k, \xi') < \infty \, .$$

The unweighted case, e.g., $w = 1$, defines the Feichtinger algebra $\mathbf{S}_0(K^\times)$. This space in itself exhibits many convenient properties, [Fei81, Jak18]. The weighted cases, $w \neq 1$, inherit these properties if the weight w is *moderate* with respect to a sub-multiplicative weight m, or m-moderate for short, i.e.,

$$w(k_1+k_2, \xi_1' \cdot \xi_2') \leq m(k_1, \xi_1')\, w(k_1, \xi_2') \text{ for all } k_1, k_2 \in \mathbb{Z}\,,\ \xi_1', \xi_2' \in \widehat{\mathfrak{D}^*},$$

where $m : \mathbb{Z} \to \widehat{\mathfrak{D}^*} \to [1, \infty)$ is a *sub-multiplicative* weight:

$$m(k_1+k_2, \xi_1' \cdot \xi_2') \leq m(k_1, \xi_1')\, m(k_1, \xi_2') \text{ for all } k_1, k_2 \in \mathbb{Z}\,,\ \xi_1', \xi_2' \in \widehat{\mathfrak{D}^*}.$$

Note that a sub-multiplicative weight m is itself m-moderate.

Remark

The property of a weight to be sub-multiplicative and to be moderate with respect to a sub-multiplicative weight is the key in order for weighted function spaces on locally compact groups to have invariance properties with respect to translation and convolution [Fei79]. See especially [Grö07] and also [Grö01, GS07].

Sub-multiplicative weights are, for example, products of non-negative powers of the following functions on $\mathbb{Z}$ and $\widehat{\mathfrak{D}^*}$:

$$m_{\mathrm{pol}} : \mathbb{Z} \to [1,\infty),\ m_{\mathrm{pol}}(k) = 1 + |k|\,, \tag{5}$$

$$m_{\mathrm{exp}} : \mathbb{Z} \to [1,\infty),\ m_{\mathrm{exp}}(k) = q^{|k|}\,, \tag{6}$$

$$\widetilde{m}_{\mathrm{pol}} : \widehat{\mathfrak{D}^*} \to [1,\infty),\ \widetilde{m}_{\mathrm{pol}}(\xi') = 1 + \deg(\xi')\,, \tag{7}$$

$$\widetilde{m}_{\mathrm{exp}} : \widehat{\mathfrak{D}^*} \to [1,\infty),\ \widetilde{m}_{\mathrm{exp}}(\xi') = q^{\deg(\xi')}\,. \tag{8}$$

Theorem

For any LCA group G (in particular $K^\times, \widehat{K^\times}$) the unweighted Feichtinger algebra $\mathbf{S}_0(G)$ satisfies the following

1. $\mathbf{S}_0(G)$ *contains the Schwartz-Bruhat functions as a dense subspace.*
2. $\mathbf{S}_0(G)$ *is dense in, and continuously imbedded in,* $\mathbf{L}^1(G)$ *and* $(\mathbf{C}_0(G), ||\cdot||_\infty)$.
3. *The Fourier transform is an isometric isomorphism between* $\mathbf{S}_0(G)$ *and* $\mathbf{S}_0(\widehat{G})$.
4. *It is a Banach algebra w.r.t. convolution and products.*

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In itself, the unweighted Feichtinger algebra presents itself with a plethora of properties that promotes it to a practical Banach space of test-functions. For example, it is invariant under translation, it forms an algebra under convolution and pointwise multiplication and it is continuously embedded into the Lebesgue spaces. Moreover, the Fourier transform, mapping functions on $K^\times$ to functions on $\widehat{K^\times}$, is a continuous bijection between this space on these groups.

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However, it fails to be an advantageous domain for the multiplication operators we already mentioned on slide 22, and which we repeat here: For functions $f : K^\times \to \mathbb{C}$,

$$f(x) \mapsto v(x)^\alpha f(x)\,,\ x \in K^\times\,,\ \alpha > 0\,, \tag{9}$$

$$\text{and}\ \ f(x) \mapsto |x|^\alpha f(x)\,,\ \alpha \in \mathbb{R}\,, \tag{10}$$

and for functions $g : \widehat{K^\times} \to \mathbb{C}$ operators such as

$$g(\xi) \mapsto \deg(\xi)^\alpha g(\xi)\,,\ \xi \in \widehat{K^\times}\,,\ \alpha > 0\,, \tag{11}$$

$$\text{and}\ \ g(\xi) \mapsto e^{\alpha \deg(\xi)} g(\xi)\,,\ \alpha \in \mathbb{R}\,, \tag{12}$$

where $v : K^\times \to \mathbb{Z}$ is the valuation on the field K, $|\cdot| : K \to \mathbb{R}_0^+$ is its absolute value and $\deg : \widehat{K^\times} \to \mathbb{N}_0$ yields the degree of ramification of a character $\xi \in \widehat{K^\times}$.

The failure of the unweighted Feichtinger algebra to be a well-suited domain for these operators is due to the fact that it enforces no condition on the rate of decay towards zero at infinity. An appropriately weighted Feichtinger algebra will enforce control of its functions' decay properties. The resulting Fréchet space, constructed from these weighted spaces, will then become invariant with respect to the multiplication operators mentioned above.

For $\alpha, \beta, \gamma, \delta \geq 0$ consider the weight

$$m_{\alpha,\beta,\gamma,\delta} : \mathbb{Z} \times \widehat{D^*} \to [1, \infty)$$
$$(k, \xi') \mapsto m_{\text{pol}}(k)^{\alpha}\, m_{\exp}(k)^{\beta}\, \widetilde{m}_{\text{pol}}(\xi')^{\gamma}\, \widetilde{m}_{\exp}(\xi')^{\delta}.$$

Theorem

For $\alpha, \beta, \gamma, \delta \geq 0$ the weighted Feichtinger algebras

$$\mathbf{S}_{0,m_{\alpha,\beta,\gamma\,\delta}}(K^{\times}) =$$
$$\{f \in \mathbf{L}^2(K^{\times}) : \sum_{k\in\mathbb{Z}} \sum_{\xi'\in\widehat{D^*}} |\langle f, E_{\xi'} T_{p^k} \chi_{D^*}\rangle| \cdot m_{\alpha,\beta,\gamma,\delta}(k, \xi') < \infty\}$$

$$\mathbf{S}_{0,m_{\alpha,\beta,\gamma\,\delta}}(\widehat{K^{\times}}) =$$
$$\{g \in \mathbf{L}^2(\widehat{K^{\times}}) : \sum_{k\in\mathbb{Z}} \sum_{\xi'\in\widehat{D^*}} |\langle g, E_{\xi'} T_{p^k} \chi_{\mathbb{T}}\rangle| \cdot m_{\alpha,\beta,\gamma,\delta}(k, \xi') < \infty\}$$

satisfy the following:

1. The Fourier transform is isometric isomorphism between $\mathbf{S}_{0,m_{\alpha,\beta,\gamma\,\delta}}(K^{\times})$ and $\mathbf{S}_{0,m_{\alpha,\beta,\gamma\,\delta}}(\widehat{K^{\times}})$.
2. Both spaces form Banach algebras w.r.t. convolution and products.
3. If $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$, $\gamma_1 < \gamma_2$, and $\delta_1 < \delta_2$, then $\mathbf{S}_{0,m_{\alpha_2,\beta_2,\gamma_2,\delta_2}}(K^{\times})$ is a dense subspace of $\mathbf{S}_{0,m_{\alpha_1,\beta_1,\gamma_1,\delta_1}}(K^{\times})$ and similarly for $\widehat{K^{\times}}$. In particular, $\mathbf{S}_{0,m_{\alpha,\beta,\gamma,\delta}}$ is a dense subspace of, and continuously imbedded in, $\mathbf{S}_0$, $\mathbf{L}^1$ and $\mathbf{C}_0$.

Let $a, b, c, d \geq 0$. For functions $f : K^\times \to \mathbb{C}$ consider the operators

$(\mathrm{Q}^a_{\mathrm{pol}} f)(x) = v(x)^a f(x)$, where $v : K^\times \to \mathbb{R}$ is the valuation on $K^\times$.
$(\mathrm{Q}^b_{\mathrm{exp}} f)(x) = |x|^b$, where $|\cdot| : K^\times \to \mathbb{R}$ is the absolute value on $K^\times$.

For functions $g : \widehat{K^\times} \to \mathbb{C}$ consider the operators

$$(\widetilde{\mathrm{Q}}^c_{\mathrm{pol}} g)(\xi) = \mathrm{deg}(\xi)^c g(\xi)$$
$$(\widetilde{\mathrm{Q}}^d_{\mathrm{exp}} g)(\xi) = e^{\mathrm{deg}(\xi) \cdot d} g(\xi)$$

where $\mathrm{deg} : \widehat{K^\times} \to \mathbb{N}_0$ is the degree of ramification of a $\xi \in \widehat{K^\times}$.

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Proposition

Let $\mathcal{F}$ be the Fourier transform from functions on $K^\times$ to functions on $\widehat{K^\times}$.

1. $\mathrm{Q}_{\mathrm{pol}}^{a}$ *is a continuous operator from* $\mathbf{S}_{0,m_{\alpha+a,\beta,\gamma,\delta}}(K^\times)$ *into* $\mathbf{S}_{0,m_{\alpha,\beta,\gamma,\delta}}(K^\times)$.
2. $\mathrm{Q}_{\mathrm{exp}}^{b}$ *is a continuous operator from* $\mathbf{S}_{0,m_{\alpha,\beta+b,\gamma,\delta}}(K^\times)$ *onto* $\mathbf{S}_{0,m_{\alpha,\beta,\gamma,\delta}}(K^\times)$.
3. $\mathcal{F}^{-1}\widetilde{\mathrm{Q}}_{\mathrm{pol}}^{c}\mathcal{F}$ *is a continuous operator from* $\mathbf{S}_{0,m_{\alpha,\beta,\gamma+c,\delta}}(K^\times)$ *into* $\mathbf{S}_{0,m_{\alpha,\beta,\gamma,\delta}}(K^\times)$.
4. $\mathcal{F}^{-1}\widetilde{\mathrm{Q}}_{\mathrm{exp}}^{d}\mathcal{F}$ *is a continuous operator from* $\mathbf{S}_{0,m_{\alpha,\beta,\gamma,\delta+d}}(K^\times)$ *onto* $\mathbf{S}_{0,m_{\alpha,\beta,\gamma,\delta}}(K^\times)$.

Similar statements hold for $\mathbf{S}_{0,m_{\alpha,\beta,\gamma,\delta}}(\widehat{K^\times})$.

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Theorem

Consider $\mathfrak{X}(K^\times) = \cap_{n\in\mathbb{N}} \mathbf{S}_{0,m_{n,n,n,n}}(K^\times)$ *and* $\mathfrak{X}(\widehat{K^\times}) = \cap_{n\in\mathbb{N}} \mathbf{S}_{0,m_{n,n,n,n}}(\widehat{K^\times})$ *with their natural topology induced by the Banach spaces* $\mathbf{S}_{0,m_{n,n,n,n}}(K^\times)$ *(resp.* $\mathbf{S}_{0,m_{n,n,n,n}}(\widehat{K^\times})$*).*

1. *Both spaces are Fréchet spaces, and they form dense subspaces of, and are continuously imbedded in, any* $\mathbf{S}_{0,m_{\alpha,\beta,\gamma\,\delta}}(K^\times)$ *and in* $\mathbf{L}^1$ *and* $\mathbf{C}_0$*.*
2. *They are invariant under convolution and product.*
3. *The Fourier transform is an isometric isomorphism between* $\mathfrak{X}(K^\times)$ *and* $\mathfrak{X}(\widehat{K^\times})$*.*
4. *For any* $a, b, c, d \geq 0$*:*
   $\mathfrak{X}(K^\times)$ *is invariant under* $\mathrm{Q}^a_{\mathrm{pol}}$, $\mathrm{Q}^b_{\mathrm{exp}}$, $\mathcal{F}^{-1}\widetilde{\mathrm{Q}}^c_{\mathrm{pol}}\mathcal{F}$, $\mathcal{F}^{-1}\widetilde{\mathrm{Q}}^d_{\mathrm{exp}}\mathcal{F}$*.*
   $\mathfrak{X}(\widehat{K^\times})$ *is invariant under* $\mathcal{F}\mathrm{Q}^a_{\mathrm{pol}}\mathcal{F}^{-1}$, $\mathcal{F}\mathrm{Q}^b_{\mathrm{exp}}\mathcal{F}^{-1}$, $\widetilde{\mathrm{Q}}^c_{\mathrm{pol}}$, $\widetilde{\mathrm{Q}}^d_{\mathrm{exp}}$*.*

From a topological point of view, the spaces we construct (induced from a norm or a countable family of such) are simpler than the Schwartz-Bruhat space over $K^\times$ and $\widehat{K^\times}$ (the latter requires taking an inductive limit of subspace topologies) [Bru61, Osb75]. Actually, the Schwartz-Bruhat functions are a proper subspace of, and densely contained in, some of the spaces we have constructed. At the same time the convenient properties of the Schwartz-Bruhat functions are all kept by our spaces.

In short, we have employed the framework for test- and generalized- functions that the modulation spaces offer in the setting of $K^{\times}$ and $\widehat{K^{\times}}$ in order to construct convenient Banach and Fréchet spaces of test-functions. In contrast to the Euclidean setting, the obtained Fréchet spaces do *not* coincide with the Schwartz-Bruhat space.

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