ON THE DIAGONALIZABILITY AND FACTORIZABILITY OF QUADRATIC BOSON FIELDS

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Outline:

- Standard form of a quadratic boson field
- Vacuum characteristic function of a boson quadratic field
- Diagonalizable/ quadratic vacuum factorizable/ vacuum decomposable quadratic boson fields
- Main result: A necessary and sufficient condition for diagonalizability
- Connections

1. Quadratic Boson Fields

For $i, j = 1, 2, ...,$ let $a_i, a_j^\dagger$ be the generators of the **multi-dimensional Heisenberg algebra** with

$$[a_i, a_j^\dagger] = \delta_{i,j} \ , \ [a_i^\dagger, a_j^\dagger] = [a_i, a_j] = 0 \, , \, (a_i^\dagger)^* = a_i \ . \tag{1.1}$$

and for $n \in \mathbb{N}$ let $M_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices. A **homogeneous quadratic boson field in standard form**, is an element X of the multi-dimensional Lie algebra generated by the a's and $a^\dagger$'s, i.e.,

$$X = \sum_{i,j=1}^{n} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) = a^\dagger A a^\dagger + a \bar{A} a + a^\dagger C a = (A, C) \tag{1.2}$$

where the matrix $A = (A_{ij}) \in M_n(\mathbb{C})$ is symmetric and the matrix $C = (C_{ij}) \in M_n(\mathbb{C})$ is Hermitian, i.e.,

$$A_{ij} = A_{ji} \ , \ \overline{C_{ij}} = C_{ji} \ , \ \forall i, j \in \{1, 2, ..., n\} \tag{1.3}$$

Note: A representation of the multi-dimensional Heisenberg algebra on polynomials $f(x) = f(x_1, x_2, ..., x_n)$ is provided by

$$a_j = \frac{1}{\sqrt{2}} (X_j + iP_j) \ , \ a_j^\dagger = \frac{1}{\sqrt{2}} (X_j - iP_j) \ , \ j = 1, 2, ..., n$$

where

$$X_j f(x) = x_j f(x) \, , \, P_j f(x) = -i \frac{\partial}{\partial x_j} f(x)$$

with

$$[P_j, X_k] = -i\delta_{j,k} \ , \ [P_j, P_k] = [X_j, X_k] = 0 \, , \, j, k = 1, 2, ..., n \ .$$

2. Vacuum characteristic function of a boson quadratic field

Definition 1. *The vacuum characteristic function of a homogeneous boson quadratic field X in standard form, in Fock representation, is*

$$f(t) := \langle \Phi, e^{itX} \Phi \rangle$$

where Φ is the normalized (i.e., $\|\Phi\| = 1$) Fock vacuum vector.

Note: A continuous function $f : \mathbb{R} \mapsto \mathbb{C}$ is *positive definite* if

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(t-s) \phi(t) \bar{\phi}(s) \, dt \, ds \geq 0$$

for every continuous function $\phi : \mathbb{R} \mapsto \mathbb{C}$ with compact support.

Bochner's theorem states that such a function can be represented as

$$f(t) = \int_{\mathbb{R}} e^{it\lambda} \, dv(\lambda)$$

where v is a non-decreasing right-continuous bounded function. If $f(0) = 1$ then such a function v defines a probability measure on $\mathbb{R}$ and Bochner's theorem says that f is the Fourier transform of a probability measure, i.e., the characteristic function of a random variable X that follows the probability distribution defined by v. Moreover, the condition of positive definiteness of f is necessary and sufficient for such a representation.

For example, in one dimension, if

$$[a, a^\dagger] = 1\,,\ \mathcal{H} : \text{Heisenberg Fock space}\,,\ a\Phi = 0\,,\ (a^\dagger)^* = a\ ,$$
$$X = (a^\dagger)^2 + a^2\ ,$$

then

$$\langle \Phi, e^{itX}\Phi\rangle = \sqrt{\text{sech2t}}$$

so X follows a **hyperbolic secant distribution**.

Computing $e^{itX}\Phi$ typically utilizes a **splitting lemma** for the exponential, in order to use

$$e^{g(t)a}\Phi = e^{g(t)a^2}\Phi = e^{g(t)a^\dagger a}\Phi = \Phi\ .$$

The splitting lemma for the quadratic field

$$X = m(a^\dagger)^2 + \bar{m}a^2 + l a^\dagger a\,,\ m \neq 0\,,\ l \in \mathbb{R}$$

is

$$e^{i\,t\,X}\,\Phi = p(t)e^{q(t)(a^\dagger)^2}\,\Phi$$

where

$$p(t) = e^{-\frac{ilt}{2}}\sqrt{\frac{\cos L}{\cos(iKt + L)}}\,,\ q(t) = \frac{K\tan(iKt + L) - l}{4\bar{m}}$$
$$K = \sqrt{4\,|m|^2 - l^2}\ ;\ L = \arctan\left(\frac{l}{K}\right)\ .$$

The vacuum characteristic function is

$$\langle \Phi, e^{itX}\Phi\rangle = p(t) = e^{-\frac{ilt}{2}}\sqrt{\frac{\cos L}{\cos(iKt + L)}}$$

which for $l = 0$ and $m = 1$ reduces to the hyperbolic secant, while for $4\,|m|^2 - l^2 = 0$ it corresponds to a degenerate distribution .

The general form of the quadratic characteristic function :

Theorem 1. *For*

$$X = \sum_{i,j=1}^{n} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) = a^\dagger A a^\dagger + a \bar{A} a + a^\dagger C a$$

define v and P, Q, R, S by

$$v := \begin{pmatrix} C & 2A \\ -2\bar{A} & -C^T \end{pmatrix} , \; e^{tv} := \begin{pmatrix} P(t) & Q(t) \\ -R(t) & S(t) \end{pmatrix} .$$

Then, for $t \in \mathbb{R}$ sufficiently close to 0, adapting the Feinsilver-Pap splitting formula:

$$e^{itX} = e^{-\frac{it}{2}\operatorname{Tr}(\bar{C}) + \frac{1}{2}\operatorname{Tr}(g_t(A,C))} e^{\frac{1}{2} a^\dagger \hat{f}_t(A,C) a^\dagger} e^{a^\dagger g_t(A,C) a} e^{\frac{1}{2} a \hat{h}_t(A,C) a}$$

and

$$\langle \Phi, e^{itX} \Phi \rangle = e^{-\frac{it}{2}\operatorname{Tr}(\bar{C}) + \frac{1}{2}\operatorname{Tr}(g_t(A,C))}$$

where

$$f_t(A,C) = Q(it) S(it)^{-1} , \; g_t(A,C) = -\log S(it)^T , \; h_t(A,C) = S(it)^{-1} R(it)$$

and $\hat{f} = (f + f^T)/2, \hat{h} = (h + h^T)/2$ denote the symmetric parts of f, h.

The problem is that e^{tv}, therefore $S(it)$, is not easy to compute explicitly. So we must look for simplifying cases and that leads to the concept of a **diagonalizable field**

Remark If $A = 0$ i.e if

$$X = \sum_{i,j=1}^{n} C_{ij} a_i^\dagger a_j = a^\dagger C a$$

then

$$\langle \Phi, e^{itX} \Phi \rangle = 1$$

corresponding to a degenerate distribution.

In the literature, quadratic fields that can, with the use of a linear transformation, be put in the form

$$b^\dagger L b + \text{ constant}$$

where L is a real diagonal matrix and $b, b^\dagger$ are still bosons, are called **diagonalizable**.

We extend:

3. Diagonalizable/factorizable/decomposable quadratic boson fields

Definition 2. *A homogeneous quadratic boson field (in standard form)*

$$X=\sum_{i,j=1}^{n}\left(A_{ij}a_i^{\dagger}a_j^{\dagger}+\bar{A}_{ij}a_ia_j+C_{ij}a_i^{\dagger}a_j\right)=a^{\dagger}Aa^{\dagger}+a\bar{A}a+a^{\dagger}Ca=(A,C)$$

*with A symmetric and C Hermitian is called **diagonalizable** if there exists a unitary matrix U such that $M:=UAU^T\in M_n(\mathbb{C})$ and $L:=UCU^*\in M_n(\mathbb{R})$ are diagonal matrices.*

Theorem 2. *A diagonalizable quadratic field*

$$X=\sum_{i,j=1}^{n}\left(A_{ij}a_i^{\dagger}a_j^{\dagger}+\bar{A}_{ij}a_ia_j+C_{ij}a_i^{\dagger}a_j\right)=a^{\dagger}Aa^{\dagger}+a\bar{A}a+a^{\dagger}Ca \tag{3.1}$$

takes the form

$$X=\sum_{i=1}^{n}\left(m_{ii}\left(b_i^{\dagger}\right)^2+\bar{m}_{ii}\left(b_i\right)^2+l_{ii}b_i^{\dagger}b_i\right)=b^{\dagger}Mb^T+b\bar{M}b^T+b^{\dagger}Lb^T$$

where

$$b_i^{\dagger}=\sum_{k=1}^{n}\overline{u_{ik}}a_k^{\dagger}\ ,\ b_i=\sum_{k=1}^{n}u_{ik}a_k$$

and the matrices

$$U=(u_{ij})\ ,\ L=(l_{ij})\ \ M=(m_{ij})$$

are as in Definition 2. Moreover, for $i,j\in\{1,2,...,n\}$,

$$[b_i,b_j^{\dagger}]=\delta_{i,j}\ ,\ [b_i^{\dagger},b_j^{\dagger}]=[b_i,b_j]=0$$

i.e., the b's and $b^{\dagger}$'s satisfy the multi-dimensional Heisenberg algebra commutation relations.

Proof. We notice that

$$X=a^{\dagger}A\left(a^{\dagger}\right)^T+a\bar{A}a^T+a^{\dagger}Ca^T$$

where

$$a^{\dagger}=\begin{pmatrix}a_1^{\dagger}&\cdots&a_n^{\dagger}\end{pmatrix}\ ,\ a=\begin{pmatrix}a_1&\cdots&a_n\end{pmatrix}$$

Since the field $X=(A,C)$ is diagonalizable,

$$A=U^*M\bar{U}\ ,\ C=U^*LU$$

so

$$\begin{aligned}X=&a^{\dagger}U^*M\bar{U}\left(a^{\dagger}\right)^T+a\overline{U^*M\bar{U}}a^T+a^{\dagger}U^*LUa^T\\=&a^{\dagger}U^*M\bar{U}\left(a^{\dagger}\right)^T+aU^T\bar{M}Ua^T+a^{\dagger}U^*LUa^T\\=&b^{\dagger}Mb^T+b\bar{M}b^T+b^{\dagger}Lb^T\end{aligned}$$

where

$$b^{\dagger}=a^{\dagger}U^*\ ,\ b=aU^T$$

i.e.,

$$b^\dagger = \begin{pmatrix} b_1^\dagger & \cdots & b_n^\dagger \end{pmatrix} \ , \ b = \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix}$$

where, for $i \in \{1, 2, ..., n\}$,

$$b_i^\dagger = \sum_{k=1}^{n} \overline{u_{ik}} a_k^\dagger \ , \ b_i = \sum_{k=1}^{n} u_{ik} a_k$$

Since L, M are diagonal, we obtain

$$X = \sum_{i=1}^{n} \left(m_{ii} \left(b_i^\dagger\right)^2 + \bar{m}_{ii} \left(b_i\right)^2 + l_{ii} b_i^\dagger b_i \right)$$

Moreover

$$[b_i, b_j^\dagger] = \sum_{p,q=1}^{n} u_{ip} \overline{u_{jq}} [a_p, a_q^\dagger] = \sum_{p,q=1}^{n} u_{ip} \overline{u_{jq}} \delta_{p,q} = \sum_{p=1}^{n} u_{ip} \overline{u_{jp}} = (UU^*)_{ij} = \delta_{i,j}$$

and

$$[a_p^\dagger, a_q^\dagger] = [a_p, a_q] = 0 \implies [b_i^\dagger, b_j^\dagger] = [b_i, b_j] = 0 \ .$$

□

Definition 3. *The quadratic field X is* ***quadratic (vacuum) factorizable*** *if there exist quadratic fields X_j, $j = 1, 2, ..., k$, $k \geq 2$, such that for all s sufficiently close to zero,*

$$\langle \Phi, e^{i\,s\,X} \, \Phi \rangle = \prod_{j=1}^{k} \langle \Phi, e^{i\,s\,X_j} \, \Phi \rangle$$

and at least two of the X_j's have a non-degenerate (i.e. non-delta) vacuum distribution.

Special case of:

Definition 4. *The quadratic field X is* ***(vacuum) decomposable*** *if its vacuum characteristic function can be written as the product of at least two non-trivial (i.e. not corresponding to degenerate distributions) characteristic functions, i.e.,*

$$\langle \Phi, e^{i\,s\,X} \, \Phi \rangle = \prod_{j=1}^{k} f_j(s) \ .$$

Theorem 3. *Let Φ be a Fock vacuum unit vector, i.e., $\|\Phi\| = 1$ and $a_j \Phi = 0$ for all $j \in \{1, 2, ..., n\}$. Let*

$$X = \sum_{i,j=1}^{n} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) \tag{3.2}$$

be diagonalizable, i.e.,

$$X = \sum_{j=1}^{n} X_j$$

where, for each $j=1,2,...,n$*,*

$$X_j = m_{jj}\left(b_j^\dagger\right)^2 + \bar{m}_{jj}\left(b_j\right)^2 + l_{jj} b_j^\dagger b_j$$

Then, for suitably small $t \in \mathbb{R}$ *and* $i^2=-1$*,*

$$\langle \Phi, e^{i\,t\,X}\,\Phi\rangle = \prod_{j=1}^{n} \langle \Phi, e^{i\,t\,X_j}\,\Phi\rangle = \prod_{j=1}^{n} \sqrt{\frac{\cos L_j}{\cos(iK_j t + L_j)}}$$

where, for each $j=1,2,...,n$*,*

$$K_j = \sqrt{4\,|m_{jj}|^2 - l_{jj}^2}\ ;\ L_j = \arctan\left(\frac{l_{jj}}{K_j}\right)\ .$$

If, for at least two values of j*,* $4\,|m_{jj}|^2 - l_{jj}^2 \neq 0$*, then* X *is quadratic vacuum factorizable.*

Proof. As mentioned earlier, for each $j=1,2,...,n$:

$$e^{i\,t\,X_j}\,\Phi = p_j(t) e^{q_j(t)(b_j^\dagger)^2}\,\Phi\ \ ,\ \ \langle \Phi, e^{i\,t\,X_j}\,\Phi\rangle = p_j(t)$$

where

$$p_j(t) = e^{-\frac{i l_{jj} t}{2}} \sqrt{\frac{\cos L_j}{\cos(iK_j t + L_j)}}\ ,\ q_j(t) = \frac{K_j \tan(iK_j t + L_j) - l_{jj}}{4\bar{m}_{jj}}$$

$$K_j = \sqrt{4\,|m_{jj}|^2 - l_{jj}^2}\ ;\ L_j = \arctan\left(\frac{l_{jj}}{K_j}\right)\ .$$

We have $X = \sum_{j=1}^{n} X_j$ where $[X_j, X_k] = 0$ for $j \neq k$. Thus

$$\begin{aligned}
\langle \Phi, e^{i\,t\,X}\,\Phi\rangle &= \langle \Phi, e^{i\,t\,\sum_{j=1}^{n} X_j}\,\Phi\rangle = \langle \Phi, \prod_{j=1}^{n} e^{i\,t\,X_j}\,\Phi\rangle \\
&= \langle e^{-i\,t\,X_1}\Phi, \prod_{j=2}^{n} e^{i\,t\,X_j}\,\Phi\rangle = p_1(t)\langle e^{q_1(t)(b_1^\dagger)^2}\Phi, \prod_{j=2}^{n} e^{i\,t\,X_j}\,\Phi\rangle \\
&= p_1(t)\langle \Phi, e^{\overline{q_1(t)}(b_1)^2} \prod_{j=2}^{n} e^{i\,t\,X_j}\,\Phi\rangle = p_1(t)\langle \Phi, \prod_{j=2}^{n} e^{i\,t\,X_j} e^{\overline{q_1(t)}(b_1)^2}\,\Phi\rangle \\
&= p_1(t)\langle \Phi, \prod_{j=2}^{n} e^{i\,t\,X_j}\,\Phi\rangle = p_1(t)\langle e^{-i\,t\,X_2}\Phi, \prod_{j=3}^{n} e^{i\,t\,X_j}\,\Phi\rangle \\
&= p_1(t)p_2(t)\langle \Phi, \prod_{j=3}^{n} e^{i\,t\,X_j}\,\Phi\rangle = ... = \prod_{j=1}^{n} p_j(t)\langle \Phi, \Phi\rangle = \prod_{j=1}^{n} p_j(t) \\
&= \prod_{j=1}^{n} \langle \Phi, e^{i\,t\,X_j}\,\Phi\rangle = \prod_{j=1}^{n} \sqrt{\frac{\cos L_j}{\cos(iK_j t + L_j)}}\ .
\end{aligned}$$

□

A kind of converse to Theorem 3 is provided in the following.

Theorem 4. *For $j \in \{1, 2, ..., n\}$ let l_{jj}, m_{jj} be complex numbers and let*

$$K_j = \sqrt{4\,|m_{jj}|^2 - l_{jj}^2} \quad , \quad L_j = \arctan\left(\frac{l_{jj}}{K_j}\right) \ .$$

There exist quadratic fields

$$X = \sum_{i,j=1}^{n} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right)$$

whose vacuum characteristic function, for suitably small $s \in \mathbb{R}$, is

$$\langle \Phi, e^{i\,s\,X}\,\Phi \rangle = \prod_{j=1}^{n} f_j(s)$$

where,

$$f_j(s) = \begin{cases} e^{-\frac{i l_{jj} s}{2}} \sqrt{\frac{\cos L_j}{\cos(i K_j s + L_j)}} & \text{if } m_{jj} \neq 0 \\ 1 & \text{if } m_{jj} \neq 0 \end{cases} \ .$$

The quadratic field X is diagonalizable if and only if l_{jj} is real for all $j \in \{1, 2, ..., n\}$.

Proof. Let M and L be diagonal matrices with diagonal entries m_{jj}, l_{jj}, respectively, where $j \in \{1, 2, ..., n\}$. Define $n \times n$ matrices A, C by

$$A = U^* M \overline{U} \quad , \quad C = U^* L U \tag{3.3}$$

where U is an arbitrary $n \times n$ unitary matrix. As in the proof of Theorems 2 and 3, the quadratic field

$$X = \sum_{i,j=1}^{n} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right)$$

has vacuum characteristic function, for suitably small $s \in \mathbb{R}$,

$$\langle \Phi, e^{i\,s\,X}\,\Phi \rangle = \prod_{j=1}^{n} f_j(s) \ .$$

The above field X is diagonalizable if and only if A is symmetric, C is Hermitian and $CA = A\overline{C}$. In view of (3.3), these conditions are satisfied if and only if L is real. □

4. MAIN RESULT: A NECESSARY AND SUFFICIENT CONDITION FOR DIAGONALIZABILITY

Theorem 5. *A quadratic field $X = (A, C)$ with A symmetric and C Hermitian is* ***diagonalizable*** *if and only if CA is symmetric, i.e. if and only if $CA = A\overline{C}$. If C is real, this reduces to $[C, A] = 0$.*

Proof. Suppose that the pair (A, C) is diagonalizable. Then $UAU^T = M$ and $UCU^* = L$ where L, M are diagonal and U is unitary. Then $A = U^*M\bar{U}$ and $C = U^*LU$ and we have

$$CA = U^*LUU^*M\bar{U} = U^*LM\bar{U}$$

and, since $(LM)^T = M^T L^T = ML = LM$, we have

$$(CA)^T = (U^*LM\bar{U})^T = U^*LM\bar{U} = CA$$

so CA is symmetric. Conversely, suppose that CA is symmetric and let $\lambda_1, ..., \lambda_d \in \mathbb{R}$, where $d \leq n$, be the distinct eigenvalues of C with multiplicities $n_i, ..., n_d$ respectively, with $n_1 + ... + n_d = n$. Then C can be spectrally decomposed as $C = W^*LW$, where W is a unitary matrix and the diagonal matrix L has the block diagonal form

$$L = \begin{pmatrix} \lambda_1 I_{n_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_d I_{n_d} \end{pmatrix} = \lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_d I_{n_d}$$

where, for $i = 1, ..., d$, I_{n_i} denotes the $n_i \times n_i$ identity matrix. Then

$$\begin{aligned} CA = (CA)^T &\iff CA = A\bar{C} \iff W^*LWA = AW^T L\bar{W} \\ &\iff LWA = WAW^T L\bar{W} \iff LWAW^T = WAW^T L \end{aligned}$$

which, by a Corollary to Sylvester's Theorem

(Note: **Sylvester's theorem** states that if $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$ are given, then the equation $AX - XB = C$ has a unique $n \times m$ matrix solution X, for each $n \times m$ matrix C, if and only if A and B have disjoint spectrums. A **corollary of Sylvester's theorem** is that if $B, C \in M_n(\mathbb{C})$ are block diagonal matrices of the form

$$B = B_1 \oplus \cdots \oplus B_k \ , \ C = C_1 \oplus \cdots \oplus C_k$$

for some $k \leq n$, where the blocks B_i, C_i have disjoint spectrums for each $i = 1, 2, ..., k$, and if $A \in M_n(\mathbb{C})$ satisfies with B, C the intertwining relation $AB = CA$, then A is also of the block diagonal form

$$A = A_1 \oplus \cdots \oplus A_k$$

with $A_i B_i = C_i A_i$ for each $i = 1, 2, ..., k$),

implies that WAW^T is of block diagonal form

$$WAW^T = A_1 \oplus \cdots \oplus A_d$$

with symmetric blocks $A_j \in M_{n_j}(\mathbb{C})$, $j = 1, 2, ..., d$, since A is symmetric. In particular,

$$A = W^T(A_1 \oplus \cdots \oplus A_d)W$$

Moreover, by the **Autonne-Takagi factorization theorem** (note: its use was first proposed by Chebotarev and Teretenkov) , for each $j = 1, 2, ..., d$, there exists a unitary matrix $V_j \in M_{n_j}(\mathbb{C})$ and a nonnegative diagonal matrix $M_j \in M_{n_j}(\mathbb{C})$ such that

$$A_j = V_j M_j V_j^T$$

Letting

$$V = V_1 \oplus \cdots \oplus V_d \ , \ M = M_1 \oplus \cdots \oplus M_d \ , \ U = VW$$

we notice that

$$[V_j, \lambda_j I_{n_j}] = 0$$

for each $j = 1, 2, ..., d$, implies that

$$[V, L] = [V_1 \oplus \cdots \oplus V_d, \lambda_1 I_{n_1} \oplus \cdots \oplus \lambda_d I_{n_d}] = 0$$

which since L is a real diagonal matrix, implies that

$$[V^*, L] = 0$$

as well. Moreover

$$UU^* = VW(VW)^* = W^*V^*VW = W^*W = I$$

so U is unitary, and

$$U^*LU = W^*V^*LVW = W^*LV^*VW = W^*LW = C$$
$$U^T MU = W^T V^T MVW = W^T(A_1 \oplus \cdots \oplus A_d)W = A$$

so the pair (A, C) is diagonalizable. □

5. Example of diagonalizable but not quadratic vacuum factorizable or vacuum decomposable

Important examples of diagonalizable quadratic fields are provided by the n**-dimensional angular momentum fields**,

$$X = \sum_{k,m=1}^{n} w_{km} L_{k,m}$$

where, $w_{km} \in \mathbb{R}$, and the $L_{k,m}$'s are the Hermitian (non linearly independent) generators of the n**-dimensional angular momentum** Lie algebra, introduced by J. D. Louck in the 1960's, with commutation relations

$$[L_{k,m}, L_{k',m'}] = i\,(\delta_{k,k'} L_{m,m'} + \delta_{m,m'} L_{k,k'} - \delta_{k,m'} L_{m,k'} - \delta_{m,k'} L_{k,m'})$$

and skew-symmetry, duality properties

$$L_{k,k} = 0 \quad , \quad L_{k,m} = -L_{m,k} \quad , \quad L_{k,m}^* = L_{k,m} \ .$$

Using multi-index notation

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

a representation on polynomials

$$p(x) = \sum_{\alpha} a_\alpha x^\alpha$$

is provided by the operators

$$L_{k,m} = X_k P_m - X_m P_k$$

where, for $j \in \{1, 2, ..., n\}$,

$$X_j p(x) = x_j p(x) \quad , \quad P_j p(x) = -i\partial_{x_j} p(x)$$

are the classical position and momentum operators, with commutation relations, for $k, m \in \{1, 2, ..., n\}$,

$$[X_k, P_m] = i\delta_{m,k}\mathbf{1} \quad , \quad [X_k, X_m] = [P_k, P_m] = 0$$

and duality relations

$$X_k^* = X_k \quad , \quad P_m^* = P_m \ .$$

Theorem 6. *There exists a Hermitian matrix $C = (c_{km})$, with main diagonal entries equal to zero, for which*

$$X = \sum_{\substack{k,m=1 \\ k \neq m}}^{n} c_{km} a_m^\dagger a_k = a^\dagger C a$$

where, for $k, m \in \{1, 2, ..., n\}$,

$$\left[a_k, a_m^\dagger\right] = \delta_{m,k}\mathbf{1} \quad , \quad [a_k, a_m] = \left[a_k^\dagger, a_m^\dagger\right] = 0 \quad , \quad a_k^* = a_k^\dagger \quad \left(a_m^\dagger\right)^* = a_m$$

i.e, X is a diagonalizable homogeneous quadratic boson field in standard form $X = (\mathbf{0}, C)$.

Proof. Letting, for each $j = 1, 2, ..., n$,

$$X_j = \frac{a_j + a_j^\dagger}{\sqrt{2}} \quad , \quad P_j = \frac{a_j - a_j^\dagger}{\sqrt{2}i}$$

i.e.,

$$a_j = \frac{X_j + iP_j}{\sqrt{2}} \quad , \quad a_j^\dagger = \frac{X_j - iP_j}{\sqrt{2}}$$

it is well-known that, for $k, m \in \{1, 2, ...\}$,

$$\left[a_k, a_m^\dagger\right] = \delta_{m,k}\mathbf{1} \quad , \quad [a_k, a_m] = \left[a_k^\dagger, a_m^\dagger\right] = 0 \quad , \quad a_k^* = a_k^\dagger \quad \left(a_m^\dagger\right)^* = a_m \ .$$

Moreover,

$$L_{k,m} = X_k P_m - X_m P_k = i\left(a_m^\dagger a_k - a_k^\dagger a_m\right) \ .$$

Thus, the field X takes the form

$$X = \sum_{\substack{k,m=1 \\ k>m}}^{n} w_{km} L_{k,m} - \sum_{\substack{k,m=1 \\ k>m}}^{n} w_{mk} L_{k,m} \ .$$

Letting

$$r_{km} = w_{km} - w_{mk} \quad , \quad c_{km} = i r_{km} \quad , \quad c_{km} = -c_{mk}$$

we have

$$X = \sum_{\substack{k,m=1 \\ k \neq m}}^{n} c_{km} a_m^\dagger a_k \ .$$

from which we conclude that X is a homogeneous quadratic boson field in standard form $X = (\mathbf{0}, C)$, where the main diagonal entries of the Hermitian matrix C are equal to zero. By Theorem 5, X is diagonalizable. □

Corollary 1. *The vacuum characteristic function of X is*

$$\left\langle \Phi, e^{i\,s\,X}\, \Phi \right\rangle = 1$$

*i.e., X has a degenerate (i.e. delta function) probability distribution **and is therefore neither quadratic vacuum factorizable nor (vacuum) decomposable**.*

Proof. By Theorem 2, with $M = \mathbf{0}$ and U any unitary matrix that diagonalizes C, in the notation of Theorem 2,

$$X = \sum_{i=1}^{n} l_{ii} b_i^\dagger b_i \ .$$

Therefore, the vacuum characteristic function of X is

$$\left\langle \Phi, e^{i\,s\,X}\, \Phi \right\rangle = \left\langle \Phi, e^{i\,s\,\sum_{j=1}^{n} l_{ii} b_i^\dagger b_i}\, \Phi \right\rangle = \left\langle \Phi, \prod_{j=1}^{n} e^{i\,s\,l_{ii} b_i^\dagger b_i}\, \Phi \right\rangle = \langle \Phi, \Phi \rangle = 1$$

since $e^{i\,s\,l_{ii} b_i^\dagger b_i}\, \Phi = \Phi$, for each j. Therefore, X has a degenerate (delta function) probability distribution. Since

- The product of two characteristic functions is always a characteristic function
- The only characteristic functions whose reciprocals are also characteristic functions belong to degenerate distributions

we see that the n-dimensional angular momentum fields are not quadratic (or any other non-degenerate way) vacuum factorizable and they are not decomposable. □

6. Example of non-diagonalizable but quadratic vacuum factorizable

We consider the non–diagonalizable quadratic field

$$X = \sum_{i,j=1}^{2} \left(A_{ij} a_i^\dagger a_j^\dagger + \bar{A}_{ij} a_i a_j + C_{ij} a_i^\dagger a_j \right) = \left(a_1^\dagger \right)^2 + a_1^2 + \left(a_2^\dagger \right)^2 + a_2^2 + i \left(a_1^\dagger a_2 - a_2^\dagger a_1 \right)$$

corresponding to the matrices

$$A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ C = iS = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \ CA \neq A\overline{C}$$

Theorem 7. *For real t sufficiently close to zero, the vacuum characteristic function of Z is*

$$\langle \Phi, e^{i\,t\,X}\, \Phi \rangle = \langle \Phi, e^{i\,t\,X_1}\, \Phi \rangle \langle \Phi, e^{i\,t\,X_2}\, \Phi \rangle$$

where

$$X_j = \left(a_j^\dagger \right)^2 + a_j^2 \quad , \quad j = 1, 2$$

Proof. We have already seen that, for $j = 1, 2$,

$$\langle \Phi, e^{i\,t\,X_j}\, \Phi \rangle = (\mathrm{sech}\, 2\mathrm{t})^{1/2}$$

Computing

$$\langle \Phi, e^{i\,t\,X}\, \Phi \rangle$$

amounts to computing the exponential of

$$v = \begin{pmatrix} C & 2A \\ -2\bar{A} & -C^T \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & i & 2 & 0 \\ -i & 0 & 0 & 2 \\ \hline -2 & 0 & 0 & i \\ 0 & -2 & -i & 0 \end{array} \right) \ ,$$

which is

$$\begin{aligned} e^{tv} &= \begin{pmatrix} P(t) & Q(t) \\ -R(t) & S(t) \end{pmatrix} \\ &= \left(\begin{array}{cc|cc} \cosh t\, \cos 2t & i\, \cos 2t\, \sinh t & \cosh t\, \sin 2t & i\, \sin 2t\, \sinh t \\ -i\, \cos 2t\, \sinh t & \cos 2t\, \cosh t & -i\, \sin 2t\, \sinh t & \cosh t\, \sin 2t \\ \hline -\cosh t\, \sin 2t & -i\, \sin 2t\, \sinh t & \cos 2t\, \cosh t & i\, \cos 2t\, \sinh t \\ i\, \sin 2t\, \sinh t & -\cosh t\, \sin 2t & -i\, \cos 2t\, \sinh t & \cos 2t\, \cosh t \end{array} \right) \end{aligned}$$

Then,

$$\langle \Phi, e^{i\,t\,X}\, \Phi \rangle = (\det S(it))^{-1/2} = \left(\cosh^2\, 2t \right)^{-1/2} = \mathrm{sech}\, 2\mathrm{t} \ .$$

Therefore

$$\langle \Phi, e^{i\,t\,X}\, \Phi \rangle = \langle \Phi, e^{i\,t\,X_1}\, \Phi \rangle \langle \Phi, e^{i\,t\,X_2}\, \Phi \rangle$$

so X is factorizable. □

7. Commutativity and Diagonalizability

Theorem 8. *Let*

$$X = (A, C) = X_1 + X_2 + X_3$$

where

$$A = (A_{ij}) = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \ , \ C = (C_{ij}) = \begin{pmatrix} k & l \\ \bar{l} & m \end{pmatrix} \ .$$

and

$$X_1 = a\,(a_1^\dagger)^2 + \bar{a}\,a_1^2 + k\,a_1^\dagger a_1$$
$$X_2 = c\,(a_2^\dagger)^2 + \bar{c}\,a_2^2 + m\,a_2^\dagger a_2$$
$$X_3 = 2\,b\,a_1^\dagger\,a_2^\dagger + 2\bar{b}\,a_1 a_2 + l\,a_1^\dagger\,a_2 + \bar{l}\,a_2^\dagger a_1$$

where $a, b, c, l \in \mathbb{C}$, $k, m \in \mathbb{R}$. *If* $[X_1 + X_2, X_3] = 0$ *then* X *is factorizable.*

Proof. Using the multi-dimensional Heisenberg algebra commutation relations

$$[a_i, a_j^\dagger] = \delta_{i,j} \ , \ [a_i^\dagger, a_j^\dagger] = [a_i, a_j] = 0$$

and the linear independence of $a_1^\dagger a_2^\dagger, a_1 a_2, a_1^\dagger a_2$ and $a_1 a_2^\dagger$, we find that $[X_1 + X_2, X_3] = 0$ if and only if

$$(kb - a\bar{l}) + (mb - cl) = 0 \ ,$$
$$(kl - 4a\bar{b}) + (4b\bar{c} - ml) = 0 \ ,$$

Since $[X_1 + X_2, X_3] = 0$ implies the diagonalizability condition

$$(kb - a\bar{l}) + (mb - cl) = 0 \ ,$$

X is diagonalizable, hence X is factorizable. □

8. Example of Decomposable but (generally) non diagonalizable

In an earlier paper, while working on the **renormalization of the higher powers of quantum white noise** (program initiated in the early 1990's by Accardi and Volovich) we considered the **Virasoro fields** defined by:

$$V(w) = w_0 L_0 + \sum_{m=1}^{\infty} (\bar{w}_m L_m + w_m L_{-m})$$

where the w_m's are real numbers and the L_m's are the **Virasoro algebra** generators with commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}\delta_{m+n,0}m(m^2 - 1) \ ; \ m, n \in \mathbb{Z} \tag{8.1}$$

The L_m's can be expressed in terms of the generators of the multi-dimensional Heisenberg algebra for $m \in \mathbb{Z}$ as

$$L_m = \frac{1}{2} \sum_{k \in \mathbb{Z}} : c_{-k}\, c_{k+m} :$$

where, for $k \in \mathbb{Z}$,

$$: c_{-k}\, c_{k+m} := \begin{cases} c_{-k}\, c_{k+m} & \text{if } -k \le k+m \\ c_{k+m}\, c_{-k} & \text{if } k+m < -k \end{cases}$$

and for $k > 0$

$$c_k = \sqrt{k} a_k \qquad ; \qquad c_{-k} = \sqrt{k} a_k^\dagger \ .$$

For $n = 1, 2, ...$ we define the **truncated Virasoro field**

$$Y_n(w) = \beta_1 \mathbf{I} + \sum_{j=1}^{n} \beta_2^j a_j^\dagger + \sum_{j=1}^{n} \beta_3^j a_j + \frac{1}{2} \sum_{j,k=1}^{n} \beta_4^{j,k} a_j^\dagger a_k^\dagger$$
$$+ \frac{1}{2} \sum_{j,k=1}^{n} \beta_5^{j,k} \left(a_j^\dagger a_k + a_k a_j^\dagger \right) + \frac{1}{2} \sum_{j,k=1}^{n} \beta_6^{j,k} a_j a_k \tag{8.2}$$

where $n \ge 1$,

$$\beta_1 = \frac{w_0}{2} \left(\mu^2 - \frac{n(n+1)}{2} \right) \qquad ; \qquad \beta_2^j = \beta_3^j = \mu\, w_j \sqrt{j}$$
$$\beta_4^{j,k} = a_6^{j,k} = \chi_{n]}(j+k) w_{j+k} \sqrt{jk}\, ,\ \beta_5^{j,k} = \chi_{n]}(j+k) w_{|k-j|} \sqrt{jk}$$

where

$$\chi_{n]}(a) = \begin{cases} 1 & \text{if } a \le n \\ \\ 0 & \text{otherwise} \end{cases}$$

and $w_m \in \mathbb{R}$ so that the coefficient $\beta_5^{j,k}$ is common for $a_j^\dagger a_k$ and $a_k a_j^\dagger$.

Theorem 9. *For each $n \ge 1$, the matrix β_5 is real symmetric. With the choice of the sequence $w = (w_m)$ given by $w_m = w \ne 0$ ($m \in \mathbb{N}$) and denoting by $(\lambda_{n,j}/w)^{-1}$, $j \in \{1, \dots, n\}$, the eigenvalues of β_5/w, the vacuum characteristic function of the n–th order Virasoro field $Y_n(w)$ is given by*

$$\langle \Phi, e^{itY_n(w)} \Phi \rangle = \prod_{j=1}^{n} \frac{e^{-\frac{w(n+1)}{4} it}}{(1 - it\lambda_{n,j})^{1/2}}. \tag{8.3}$$

corresponding to the characteristic function of the sum of n independent, **but not identically distributed***, shifted Gamma–random variables*

$$X_{\Gamma_{1/2, 1/(\lambda_{n,j}/w)}} - \frac{w(n+1)}{4} \qquad ; \qquad j \in \{1, \dots, n\} \tag{8.4}$$

where $X_{\Gamma_{1/2, 1/(\lambda_{n,j}/w)}}$ denotes the Gamma–random variable with shape parameter $1/2$ and scale parameter $\lambda_{n,j}/w$ ($j = 1, \dots, n-1$). The joint distribution of the random variables (8.4) is then the non–homogenous product measure on $\mathbb{R}^n$

$$\bigotimes_{j=1}^{n-1} \chi_{[-w(n+1)/4, +\infty)}(x_j) \frac{x_j^{-1/2} e^{-x/(\lambda_{n,j}/w)}}{(\lambda_{n,j}/w)^{1/2} \Gamma(1/2)} dx_j. \tag{8.5}$$

The homogeneous quadratic part of $Y_n(w)$, i.e. for $\mu = 0$, is

$$X_n(w) = \frac{1}{2}\sum_{j,k=1}^{n} \beta_4^{j,k} a_j^\dagger a_k^\dagger + \sum_{j,k=1}^{n} \beta_5^{j,k}\, a_j^\dagger a_k + \frac{1}{2}\sum_{j,k=1}^{n} \beta_6^{j,k} a_j a_k = a^\dagger A a^\dagger + a\bar{A}a + a^\dagger C a \ ,$$

so

$$A = (A_{j,k}) = \left(\frac{1}{2}\beta_4^{j,k}\right) \ , \ C = (C_{j,k}) = \left(\beta_5^{j,k}\right)$$

Since both A, C are real, the condition on the **diagonalizability** of the pair (A, C),

$$CA = A\overline{C} \ ,$$

reduces to the **commutativity** of A and C, i.e. to

$$[A, C] = 0$$

which is equivalent to

$$[\beta_4, \beta_5] = 0 \ .$$

In general, the matrices β_4, β_5 do not commute, so $X_n(w)$ is not diagonalizable. But they do commute, in fact they are equal, so $X_n(w)$ is diagonalizable, when w_m is constant for all m, as in the previous theorem where we obtained the factorizability of the truncated Virasoro field for this choice of coefficients.

9. Example of a decomposable quadratic field that admits a non-quadratic factorization

Consider the quadratic field

$$X = \sum_{i,j=1}^{2}\left(A_{ij}a_i^\dagger a_j^\dagger + \bar{A}_{ij}a_i a_j + C_{ij}a_i^\dagger a_j\right) = \left(a_2^\dagger\right)^2 + a_2^2 + 2\left(a_1^\dagger a_2^\dagger + a_1 a_2\right) + i\left(a_1^\dagger a_2 - a_2^\dagger a_1\right) \ .$$

corresponding to the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad , \quad C = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ ,$$

Then,

$$\left\langle \Phi, e^{i\,t\,X}\,\Phi \right\rangle = \left(\det S(it)\right)^{-1/2} = \operatorname{sech} t \frac{\sqrt{2}}{\sqrt{-3 + 5\cosh\, 2t}}.$$

We know that $\operatorname{sech} t = \sqrt{\operatorname{sech}}\sqrt{\operatorname{sech}}$ is a characteristic function, as the product of characteristic functions of one-dimensional quadratic fields. Since

$$f(t) = \frac{\sqrt{2}}{\sqrt{-3 + 5\cosh\, 2t}} = \mathcal{F}(g(x))$$

where, for $x \in \mathbb{R}$,

$$g(x) = \frac{\sqrt{2}}{\sqrt{5\pi}(1+x^2)} \left((1+ix)F_1 \left(\frac{1-ix}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3-ix}{2}; \frac{3+4i}{5}, \frac{3-4i}{5} \right) \right.$$
$$\left. +(1-ix)F_1 \left(\frac{1+ix}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3+ix}{2}; \frac{3+4i}{5}, \frac{3-4i}{5} \right) \right) \geq 0 \ ,$$

where the *Appell hypergeometric function* F_1 is defined, for $|x| < 1, |y| < 1$, by the infinite series

$$F(a; b_1, b_2; c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}} \frac{x^m y^n}{m!n!} \ ,$$

where, for $z \in \mathbb{R}$, $(z)_n = z(z+1)(z+2)\cdots(z+n-1)$. For other values of x, y it is defined by analytic continuation, it follows that $f(t)$ is also a characteristic function, so X is decomposable. We can show that there is no **one-dimensional** quadratic fields Z with

$$\left\langle \Phi, e^{itZ}\,\Phi \right\rangle = f(t) = \frac{\sqrt{2}}{\sqrt{-3+5\cosh\,2t}}$$

therefore the factorization

$$\left\langle \Phi, e^{itX}\,\Phi \right\rangle = \left(\det S(it)\right)^{-1/2} = \operatorname{sech} t \frac{\sqrt{2}}{\sqrt{-3+5\cosh\,2t}}.$$

is not quadratic, i.e X admits a factorization into factors which are not **all one-dimensional quadratic**.

Remark 1. It is still possible to find homogeneous quadratic fields X and Y such that

$$\operatorname{sech} t \frac{\sqrt{2}}{\sqrt{-3+5\cosh\,2t}} = \left\langle \Phi, e^{itY}\,\Phi \right\rangle \left\langle \Phi, e^{itZ}\,\Phi \right\rangle,$$

where both $\left\langle \Phi, e^{itY}\,\Phi \right\rangle$ and $\left\langle \Phi, e^{itZ}\,\Phi \right\rangle$ are different from $\operatorname{sech} t$. But it can be shown that there are no complex numbers $l_1, l_2, L_1, L_2, K_1, K_2$ such that the equality

$$\operatorname{sech} t \frac{\sqrt{2}}{\sqrt{-3+5\cosh\,2t}} = e^{-\frac{il_1 t}{2}} \sqrt{\frac{\cos L_1}{\cos(iK_1 t + L_1)}}\, e^{-\frac{il_2 t}{2}} \sqrt{\frac{\cos L_2}{\cos(iK_2 t + L_2)}}.$$

holds for arbitrary t. Thus in the vacuum factorization

$$\left\langle \Phi, e^{itX}\,\Phi \right\rangle = \left\langle \Phi, e^{itY}\,\Phi \right\rangle \left\langle \Phi, e^{itZ}\,\Phi \right\rangle.$$

Y and Z cannot both be one-dimensional homogeneous quadratic boson fields.