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On the construction of measurement channels and quantum tomography

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POVMs possessing densities (A.S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory 1982, 2011)

$(X, \mathcal{B}(X), \mu)$ a measurable space

$$\mathfrak{M} : \mathcal{B}(X) \longrightarrow \mathcal{B}(\mathcal{H})_+$$

$$\mathfrak{M}(\emptyset) = 0, \quad \mathfrak{M}(X) = I$$

$$\mathfrak{M}\Big(\bigcup_j B_j\Big) = \sum_j \mathfrak{M}(B_j)$$

$$B_j \cap B_k = \emptyset, \; j \neq k, \; B_j \in \mathcal{B}(X)$$

$$\mathfrak{M}(B) = \int_B P(x)\, d\mu(x)$$

Suppose $X = \{1, 2, 3, \dots\}$, then we need

$$\sum_{k=1}^{+\infty} P(k)\mu(k) = I$$

If $P(k) = |x_k\rangle\langle x_k|$ and (x_k) is the Riesz basis.

joint with A. D. Baranov.

Theorem 0. $\mathcal{P} = \sum_{k=1}^{+\infty} P(k)\mu(k)$ is a bounded invertible operator for any choice of $|\mu(k)| < C$.

$\mathcal{P} = I \iff (x_k)$ is orthonormal basis.

$\Rightarrow \tilde{X} = X \cup \{0\}$, $P(0) = I - \frac{1}{\|P\|}P$, $\mu(0) = 1$, $\mu(k) = \frac{1}{\|P\|}$

If we need projection-valued density $\Rightarrow$ Frame theory

Measurement channels

$$\Phi : \mathfrak{S}(\mathcal{H}) \longrightarrow \Pi(X)$$

the set of states (positive unit trace operators) — the set of probability distributions on X

$$\pi_\rho(B) \equiv \Phi(\rho)(B) = \operatorname{Tr} \rho\, \mathfrak{M}(B), \quad B \in \mathcal{B}(X)$$

If $\mathfrak{M}(B) = \int_B P(x)\, d\mu(x)$ then

$$\pi_\rho(B) = \int_B p_\rho(x)\, d\mu(x), \text{ where } p_\rho(x) = \operatorname{Tr} \rho P(x)$$

Covariant POVM

$X = \mathcal{Y}$ a locally compact Abelian group

μ the Haar measure

$g \longmapsto U_g$ a projective unitary representation of $\mathcal{Y}$ in $\mathcal{H}$

$$U_g \mathcal{M}(B) U_g^{\dagger} = \mathcal{M}(B+g)$$

$B \in \mathcal{B}(\mathcal{Y})$, $g \in \mathcal{Y}$

Possible construction

$$\mathcal{M}(B) = \int_B U_g P_0 U_g^{+} \, d\mu(g)$$

$B \in \mathcal{B}(\mathcal{G})$

$P(g) = U_g P_0 U_g^{+}$ an operator-valued density

Problem: $\nexists \int U_g P_0 U_g \, d\mu(g)$

in general because μ is not finite

Pontryagin duality (Ann. Math. 35 (1934))

Let G be a locally compact Abelian group with the Haar measure ν and $\hat{G}$ - its dual consisting of characters χ. Then $\exists ! \; \hat{\nu}$ on $\hat{G}$ such that

$$\int_{\hat{G}\times G} \chi(h)\overline{\chi(g)}\,\psi(g)\,d\hat{\nu}(\chi)\,d\nu(g) = \psi(h)$$

$$\psi \in L^1(G, \nu)$$

Projective unitary representation of $\mathcal{G} = \hat{G} \times G$ in $\mathcal{H} = L^2(G, \nu)$

$$[U_{\chi,g}\psi](h) = \chi(h)\psi(h+g)$$

$$\chi \in \hat{G}, \quad g, h \in G, \quad \psi \in \mathcal{H}$$

$$U_{\chi,g} U_{\chi',g'} = \chi'(g) U_{\chi\chi', g+g'}$$

$$U^{+}_{\chi,g} = \chi(g) U_{\overline{\chi}, -g}$$

Theorem 1. The map

$$\mathfrak{S}_1(\mathcal{H}) \ni \rho \longmapsto \operatorname{Tr} \rho U_{x,g}$$

can be extended to isometrical isomorphism

$$\mathfrak{S}_2(\mathcal{H}) \text{ and } L^2(\mathcal{Y}, \hat{\mathcal{Y}} \times V)$$

Subgroups of $\mathcal{Y}$ and their unitary representations

$$G_{\chi,g} \overset{def}{=} \{ (\chi',g') \in \mathcal{Y} : \chi'(g) = \chi(g') \}$$

$G_{\chi,g}$ is a subgroup of $\mathcal{Y}$

$$G_{\chi,g} \ni (\chi',g') \longmapsto [\chi'(g')]^{1/2} U_{\chi',g'}$$

is a unitary representation of $G_{\chi,g}$

Thus,

$$F_\rho(\chi,g,\chi',g') = [\chi'(g')]^{1/2} \operatorname{Tr} \rho U_{\chi',g'}, \quad (\chi',g') \in G_{\chi,g}$$

is a charcteristic function.

Denote $\{\mu_\rho(x,g,x',g'), (x',g')\in G_{x,g}\}$ the set of probability distributions corresponding to $\{F_\rho(x,g,x',g'), (x',g')\in G_{x,g}\}$ by means of the Bochner-Khinchin theorem.

Then, $\{\mu_\rho\}$ is a quantum tomogram and allows to restore ρ due to Theorem 1.

Quantum tomography (G. M. D'Ariano, M. G. A. Paris, V. I. Man'ko, S. Mancini, P. Tombesi, ...)

Covariant measure

$$\mathfrak{M}(B) = \int_B |U_{x,g}\psi_0\rangle\langle U_{x,g}\psi_0|\, d\hat{\nu}(x)\, d\nu(g) \quad (*)$$

$$B \in \mathcal{B}(\mathcal{Y})$$

Theorem 2. $(*)$ correctly determines a covariant POVM.

Naimark dilation (Bull. Acad. Sci. URSS 7 (1943))

$\exists$ isometrical embedding $\mathcal{H} \subset \mathcal{K}$

$$\mathfrak{M}(B) = P_{\mathcal{H}} E(B)\Big|_{\mathcal{H}}$$

$$W_{\chi, g} \stackrel{def}{=} \int \chi(g') \overline{\chi'(g)}\, dE(\chi', g')$$

$\Downarrow$ a set of unitary operators

$$T_{\chi, g} = P_{\mathcal{H}} W_{\chi, g}\Big|_{\mathcal{H}} = \int \chi(g') \overline{\chi'(g)}\, d\mathfrak{M}(\chi', g')$$

$\Downarrow$ a set of contractions

B. Sz-Nagy - C. Foias dilation

(Harmonic Analysis of Operators on Hilbert space, Budapest, 1970)

$$T_{\chi,g}^{\,n} = P_{\mathcal{H}} W_{\chi,g}^{\,n}\Big|_{\mathcal{H}}$$

$$T_{\chi,g} = \int_{\mathcal{G}} \chi(g') \overline{\chi'(g)}\, |U_{\chi',g'}\psi_0\rangle\langle U_{\chi',g'}\psi_0|\, d\hat{\nu}(\chi') d\nu(g)$$

Theorem 3. $\exists$ a measurable function f on $\mathcal{G}$ such that

$$T_{\chi,g} = f(\chi,g)\, U_{\chi,g}$$

Corollary. The measure

$$d\mathfrak{M}(\chi, g) = |U_{\chi,g}\psi_0\rangle\langle U_{\chi,g}\psi_0|\, d\hat{\nu}(\chi)\, d\nu(g)$$

is informationally complete

References

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