New Trends in Mathematical Physics

Steklov Mathematical Institute   Russian Academy of Sciences

Correlation Decay and Markovianity in Open Systems

Marco Merkli

Department of Mathematics and Statistics

Memorial University of Newfoundland

CANADA

MEMORIAL UNIVERSITY

November 7-12, 2022

Open quantum systems

system - bath models

Hamiltonian $$H = H_S + H_B + \lambda I \quad \text{on} \quad \mathcal{H}_S \otimes \mathcal{H}_B$$

$H_S \equiv H_S \otimes 1\!\!1_B$ $\quad$ $H_B \equiv 1\!\!1_S \otimes H_B$ $\quad$ coupling constant $\quad$ coupling operator

- Full density matrix at time t

$$\rho_{SB}(t) = e^{-itH}\,\rho_{SB}(0)\,e^{itH}$$

- Reduced system density matrix

$$\rho_S(t) = \operatorname{tr}_B\,\rho_{SB}(t)$$

- For uncorrelated initial $\rho_{SB}(0) = \rho_S(0) \otimes \rho_B(0)$ define

$$\rho_S(0) \longmapsto \rho_S(t) = V_t\,[\rho_S(0)]$$

- V_t CPTP map but not group $V_t \circ V_s \neq V_{t+s}$

Markovian approximation

$V_t \approx \Phi_t$ CPTP map on system density matrices

$$\Phi_{t+s} = \Phi_t \circ \Phi_s \,, \qquad \Phi_t = e^{t\mathcal{L}}$$

Markovian master equation: $\rho_S^M(t) := \Phi_t\left(\rho_S(0)\right)$

$$\partial_t \rho_S^M(t) = \mathcal{L}\, \rho_S^M(t)$$

Heuristics (ubiquitous in physics literature)

Born approximation: $\rho_{S+B}(0) = \rho_S(0) \otimes \rho_B \longrightarrow \rho_{S+B}(t) \approx \rho_S(t) \otimes \rho_B$

Markov approximation: $\tau_{S\,\mathrm{relax}} \gg \tau_{B\,\mathrm{corr}}$

Rotating wave approximation: $\tau_{S\,\mathrm{relax}} \gg \tau_S$

$\boxed{\text{Master equation}}$

Challenge: Mathematically rigorous derivation

Some advances

Van Hove ultra-weak coupling limit (Van Hove 1955 Davies 1974)

$$\forall a > 0 : \quad \lim_{\lambda \to 0} \sup_{0 \le \lambda^2 t \le a} \left\| V_t - e^{t \mathcal{L}_\lambda} \right\| = 0$$

"Davies generator"

Weak coupling regime

(Jakšić-Pillet 97, Berman-Sigal-Merkli, Könenberg-Merkli, Merkli 22)

$$|\lambda| < \lambda_0 \Rightarrow \quad \sup_{t \ge 0} \left\| V_t - e^{t \mathcal{L}_\lambda} \right\| \le C |\lambda|^2$$

Mathematically rigorous technique: Quantum Resonance Theory

Further advances

- Strong coupling (C. Latune, A. Rivas, J. Thingna, A. Trushechkin ...)
- Non-markovian corrections (B. Vacchini, A. Rivas, I. Sinayskiy, K. Modi ...)
- (Near) degeneracy of energy (A. Trushechkin ...)

Present talk:

What happens for initially correlated SR states?

In ultra-weak coupling regime $\lambda \to 0$, $\tau \equiv \lambda^2 t \leq \tau_0$

↳ (van Hove's limit)

Available online at www.sciencedirect.com

ScienceDirect

Annals of Physics 322 (2007) 631–656

ANNALS of PHYSICS

www.elsevier.com/locate/aop

On the assumption of initial factorization in the master equation for weakly coupled systems I: General framework

S. Tasaki [a], K. Yuasa [b,*], P. Facchi [c,d], G. Kimura [b,1], H. Nakazato [b], I. Ohba [b], S. Pascazio [e,d]

Available online at www.sciencedirect.com

ScienceDirect

Annals of Physics 322 (2007) 657–676

ANNALS of PHYSICS

www.elsevier.com/locate/aop

On the assumption of initial factorization in the master equation for weakly coupled systems II: Solvable models

K. Yuasa [a,*], S. Tasaki [b], P. Facchi [c,d], G. Kimura [a,1], H. Nakazato [a], I. Ohba [a], S. Pascazio [e,d]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_{\mathrm{I}}(\tau) = \mathcal{K}\rho_{\mathrm{I}}(\tau), \qquad \rho_{\mathrm{I}}(0) = \mathcal{P}\rho_0 = \mathrm{tr}_{\mathrm{B}}\{\rho_0\} \otimes \Omega_{\mathrm{B}}. \tag{4.4}$$

That is, even if the initial state ρ_0 is not in a factorized form, but rather there is entanglement, or simply a classical correlation, between system S and reservoir B, all correlations disappear in van Hove's limit and system S behaves as if the total system started from the factorized initial state in (4.4) with a reservoir state Ω_{B} specified below.

For correlated SR initial states $\rho_{SR}(0)$

- Born approximation ✗ $\rho_{SR}(t) \not\approx \rho_S(t) \otimes \rho_R$

- Dynamical map ✗ $\rho_S(0) \overset{V_t}{\not\mapsto} \rho_S(t)$

Heuristic picture we uncover:

- Initial state SR correlations decay due to dispersive R dynamics @ speed indep. of λ

- After decay of correlations, Born and Markov approx. become valid.

Model & Results

N-level system & thermal bose reservoir

$$\mathcal{H} = \mathbb{C}^N \otimes \mathcal{F}\left(L^2(\mathbb{R}^3, d^3k)\right)$$

$$H_\lambda = H_S \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(|k|) + \lambda\, G \otimes \varphi(g)$$

matrix

$$\sum_{j=1}^{N} E_j\, |\phi_j\rangle\langle\phi_j| \qquad \int_{\mathbb{R}^3} |k|\, a^*(k) a(k)\, d^3k \qquad \int_{\mathbb{R}^3} \{ g(k)\, a^*(k) + h.c. \}\, d^3k$$

FORM FACTOR

Assumptions

form factor $g : \mathbb{R}^3 \to \mathbb{C}$

$k = (u, \Sigma)$

A1 Regularity of $g(k)$

- $g(u, \Sigma)$ 4 times continuously differentiable in radial $u \in [0, \infty)$
- UV: u large $\to$ $|g(u,\Sigma)| \leq \frac{C}{u^q}$ $\quad q > 3/2$
- IR: u small $\to$ $|g(u,\Sigma)| \sim u^p$ $\quad p = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

  $p > 3$

A2 Fermi Golden Rule condition

(a) $E_m \neq E_n$: $\quad |\langle \phi_m, G \phi_n \rangle| > 0, \quad \int_{S^2} |g(E_m - E_n, \Sigma)|^2 \, d\Sigma > 0$

(b) The level shift operators have simple eigenvalues

Initial states:

$$\rho_{SR} = \begin{cases} \text{A} \quad \text{uncorrelated} & \rho_S \otimes \omega_{R,\beta} \\ \text{B} \quad \text{correlated} & \sum_\alpha \rho_S \otimes \omega_{R,\beta} \left(K_\alpha^* \cdot K_\alpha\right) \end{cases}$$

ρ_S : arbitrary S density matrix

K_α : Kraus operators

$\omega_{R,\beta}$: R equilibrium state (thermodyn. limit)

Reduced system density matrix $\rho_S(t)$

$$\rho_S(t) = \mathrm{Tr}_R \; \rho_{SR} \left(e^{itH_\lambda} \cdot e^{-itH_\lambda} \right)$$

Observable algebra $\mathcal{O}$ and correlator algebra $\mathcal{C}$

$\mathcal{O}$: $\quad \mathcal{B}(\mathbb{C}^N) \otimes \mathcal{W}(L^2_{obs})$ $\quad$ (Weyl)

$\mathcal{C}$: $\quad \operatorname{span}\{ K_1 \cdots K_n \; : \; n \in \mathbb{N} \}$

$$\begin{cases} K_j \in \mathcal{B}(\mathbb{C}^N) \otimes \mathcal{P}ol(L^2_{cor}) & \text{Polyn. in } a^{\#}(f) \\ K_j \in \exp\left[\mathcal{B}(\mathbb{C}^N) \otimes \mathcal{P}ol(L^2_{cor}) \right] \end{cases}$$

L^2_{obs}, L^2_{cor} : Test function spaces

(3 times cont. diff , UV decay , IR behaviour)

$$\rho_{SR}\left(e^{itH_\lambda} \cdot e^{-itH_\lambda}\right) = \sum_\alpha \rho_S \otimes \omega_{R,\beta}\left(K_\alpha^* \, e^{itH} \cdot e^{-itH} K_\alpha\right)$$

$\rho_S = \mathrm{tr}_R \, \rho_{SR}$ Reduced system initial state

Theorem (SR dynamics for correlated initial states) *There is a constant $\lambda_0 > 0$ such that if $|\lambda| < \lambda_0$, then for all $t \geq 0$, $A \in \mathcal{O}$,*

MARKOV CORREL REM

$$\rho_{SR}\left(e^{itH_\lambda} A e^{-itH_\lambda}\right) = \left(e^{t\mathcal{L}_S(\lambda)} \rho_S \otimes \omega_{R,\beta}\right)(A) + \chi(\lambda, t, A) + R(\lambda, t, A),$$

where $\mathcal{L}_S(\lambda)$ is the Davies generator and the remainder $R(\lambda, t, A)$ satisfies

$$|R(\lambda, t, A)| \leq C(A)|\lambda|^{1/4}.$$

The dispersive term satisfies

$$\chi(\lambda, t, A) = 0 \qquad \text{if} \ \ \rho_{SR} = \rho_S \otimes \omega_{R,\beta}$$

$$\chi(\lambda, t, A_S \otimes \mathbb{1}_R) = 0$$

$$|\chi(\lambda, t, A)| \leq \frac{C(A)}{1 + t^3}$$

Theorem shows:

(1) Born & Markov approx. valid for initial product states

$$\rho^t_{SA}(A) = \left(e^{t\mathcal{L}_{S}(\lambda)}\rho_S \otimes \omega_{R,\beta}\right)(A) + O\left(|\lambda|^{1/4}\right)$$

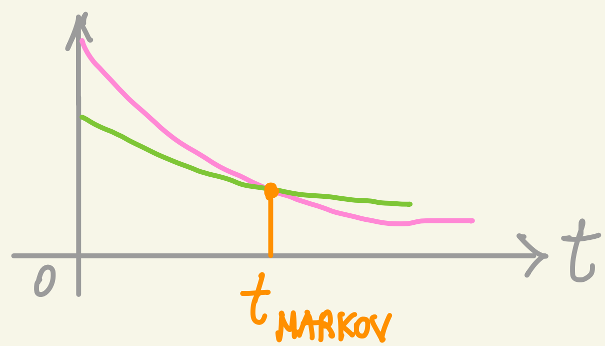
(2) Markov approx. valid for initial correlated states

$$\sup_{t\geq 0} \left\| \rho^t_S - e^{t\mathcal{L}(\lambda)}\rho^0_S \right\| \leq C\,|\lambda|^{1/4}$$

(3) Decay of correlations & emergence of Born-Markov regime $t > t_{MARKOV}$

$$\chi \sim 1/t^3$$

$$e^{t\mathcal{L}(\lambda)} \sim e^{-\lambda^2 t}$$



Strategy of proof

① GNS representation (purification – Araki–Woods)

$$\rho^t_{SR}(A) = \rho_S \otimes \omega_{R,\beta}\left(K^* e^{itH_\lambda} A e^{-itH_\lambda} K\right)$$

(Def) $\downarrow$

$$\rho^t_{SR}(A) = \left\langle \Omega_0 ,\ \pi(K^*)\, e^{itL_\lambda}\, \pi(A)\, e^{-itL_\lambda}\, \pi(K)\, \Omega_0 \right\rangle_{\mathcal{H}}$$

$L_\lambda = L_0 + \lambda I$ Liouville operator

Ω_0 : KMS for $e^{itL_0} \cdot e^{-itL_0}$

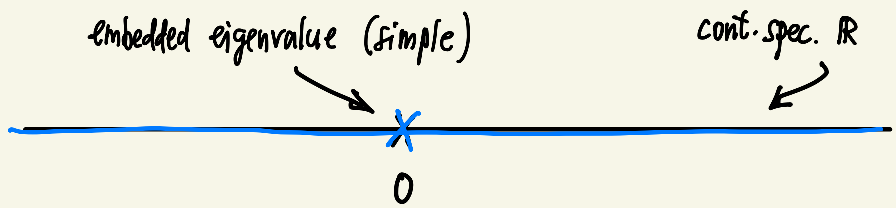
$$= \left\langle \Omega_0 ,\ \pi(K^* K)\, e^{itL_\lambda}\, \pi(A)\, \Omega_0 \right\rangle_{\mathcal{H}} + O(\lambda)$$

$$\equiv \left\langle \Psi ,\ e^{itL_\lambda}\, \Phi \right\rangle_{\mathcal{H}} + O(\lambda).$$

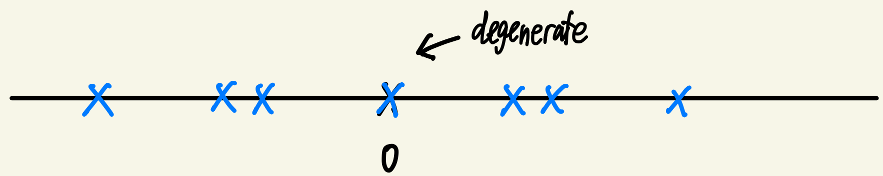
② Analysis of $\langle \Psi, e^{itL_\lambda}\Psi \rangle$: Singular perturbation theory

$$L_\lambda = \underbrace{L_S \otimes \mathbb{1}_R + \mathbb{1}_S \otimes L_R}_{L_0} + \lambda I$$

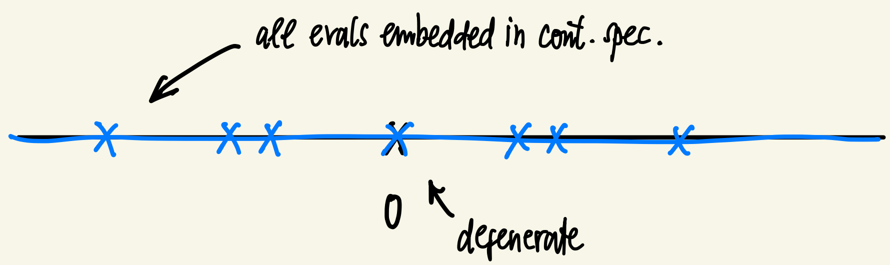
spec (L_R)



spec (L_S)



spec (L_0)



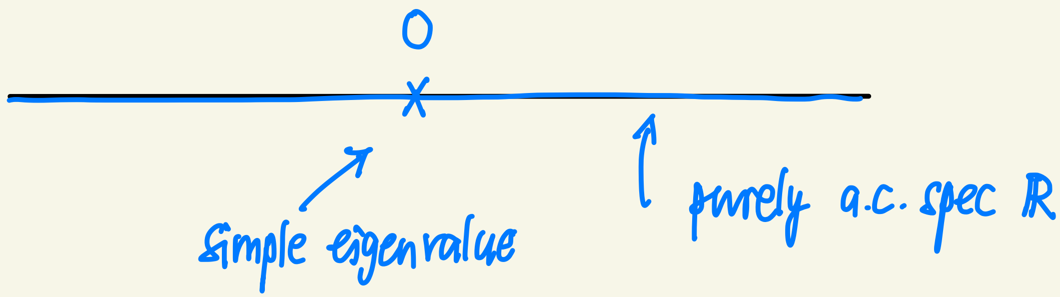
What happens when the interaction is switched on?

- $\lambda \mapsto L_\lambda = L_0 + \lambda I$ is analytic at $\lambda = 0$ 👍
- But unperturbed eigenvalues are not isolated 🤔
  (singular perturbation theory)

General wisdom: embedded eigenvalues may well disappear for $\lambda \neq 0$

"Well known": $\exists\, \lambda_0 > 0$ s.t. if $0 < |\lambda| < \lambda_0$, then

$\mathrm{spec}(L_\lambda) =$



Physical interpretation: the uncoupled SR complex has many stationary states but as soon as coupled, SR has a unique stationary state ('coupled equilibrium').

‼️⁉️

What happens to the unstable eigenvalues?

Where do they go?

Will they still influence the dynamics?

Eigenvalues migrate into complex plane!

They produce metastable dynamics!

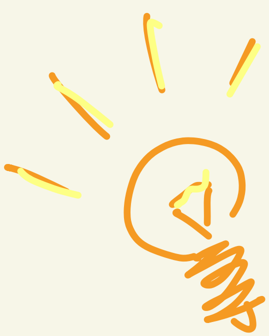
Extension of analytic perturbation theory:

Level shift operator $\quad \Lambda_e := -P_e I P_e^{\perp} (L_0 - e + i0_+)^{-1} P_e^{\perp} I P_e$

describes spectrum of L_λ close to eigenvalue e of L_0

$\Lambda_e \neq \Lambda_e^*$ so eigenvalues of Λ_e are <u>complex</u> ....

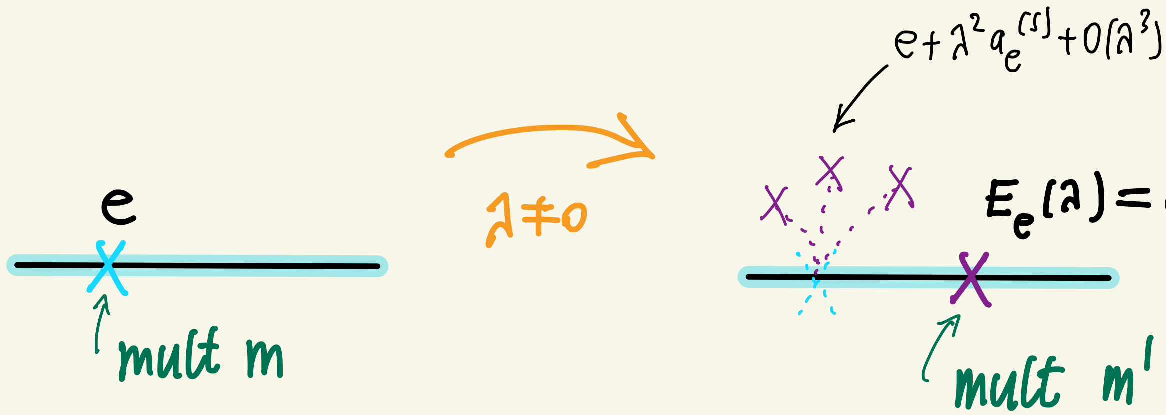
but corrections to spec of L_λ must be <u>real</u> because $L_\lambda = L_\lambda^*$!



Complex eigenvalues of Λ_e $\rightsquigarrow$ unstable e

Real evals of Λ_e $\rightsquigarrow$ 2nd order correction of surviving evals

Λ_e describes fate of embedded e

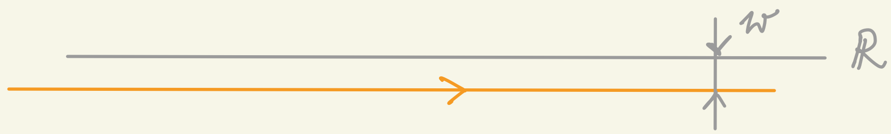


$$E_e(\lambda) = e + \lambda^2 a_e^{(s)} + O(\lambda^3)$$

- Eigenvalues $\leftrightarrow$ bound states
- Unstable eigenvalues $\leftrightarrow$ metastable states ("almost bound")

- By <u>Feshbach map</u>, one can link eigenvalues of Λ_ε to complex poles of resolvent $(L_\lambda - z)^{-1}$
- Then can extract metastability of dynamics using <u>Fourier-Laplace</u> representation

$$e^{itL_\lambda} = \frac{-1}{2\pi i} \int_{\mathbb{R}-iw} e^{itz} (L_\lambda - z)^{-1} dz \qquad (w>0)$$



Details can be found in

Quantum
the open journal for quantum science

Quantum 6, 615 (2022).

Dynamics of Open Quantum Systems I, Oscillation and Decay

Marco Merkli

Quantum 6, 616 (2022).

Dynamics of Open Quantum Systems II, Markovian Approximation

Marco Merkli

резюме

- Finite S coupled to thermal quantum field
- Expansion of propagator $\left\{\begin{array}{l}\text{part on } R \text{ in equil.} \\ \text{part on } R \perp \text{equil}\end{array}\right. + O(\lambda^{1/4})$
- Expansion valid $\forall t$ and for correlated initial states $\Rightarrow$

  - Markovian approx. for <u>S dyn.</u> valid for all times & for correlated in states

  - Born approx. for <u>SR dyn.</u> valid
    - uncorr. in states : $\forall t$
    - corr. in states : after decay of corr.

Talk based on

Ann. Henri Poincaré *Online First*

https://doi.org/10.1007/s00023-022-01226-5

Annales Henri Poincaré

Check for updates

Correlation Decay and Markovianity in Open Systems

Marco Merkli

Dedicated to the Memory of Gennady P. Berman, my Friend and Teacher.

Quantum

the open journal for quantum science

Quantum 6, 615 (2022).

Dynamics of Open Quantum Systems I, Oscillation and Decay

Marco Merkli

Quantum 6, 616 (2022).

Dynamics of Open Quantum Systems II, Markovian Approximation

Marco Merkli

Thank you 🙏